as of 01/06/19

Alternative Tunings: Theory, Notation and Practice

including the alt-tuner 1.2 manual

by Kite Giedraitis

"God created the harmonic series, all else is the work of humankind"
- after Leopold Kronecker

This book is a work in progress. See www.TallKite.com for the latest draft.

Dedicated to the microtonal community.

Special thanks to Paul Erlich, Maria Olaya, Dave Ryan, Margo Schulter, Cam Taylor, Mike Battaglia, Abhi Timbadia, Praveen Venkataramana, Aaron Wolf and Jason Yerger for their assistance.

Front cover photo by Wolf Peuker.

Table of Contents

Preface....5

Part I – An Introduction to Just Intonation....6
- Chapter 1.1 – A Parable....6
- Chapter 1.2 – Ratios, Cents, Primes, and Limits....7
- Chapter 1.3 – The Harmonic Lattice....16

Part II – JI Color Notation....28
- Chapter 2.1 – Interval Names....28
- Chapter 2.2 – Commas and Wolves....33
- Chapter 2.3 – Note Names and Color Solfege (Da-re-mu)....38
- Chapter 2.4 – JI Chord Names Part I....43
- Chapter 2.5 – Chord Progressions, Scales, Keys and Modulations....50
- Chapter 2.6 – JI Staff Notation....53
- Chapter 2.7 – Chord Voicings....58

Part III – Further Out....65
- Chapter 3.1 – Remoteness Classes....65
- Chapter 3.2 – Extremely Remote Intervals....70
- Chapter 3.3 – Paradoxical Intervals....78
- Chapter 3.4 – Purple Intervals....82
- Chapter 3.5 – The Expanded Harmonic Lattice....85
- Chapter 3.6 – 11-limit and 13-limit Intervals....89
- Chapter 3.7 – Higher Primes: 17, 19, and Beyond....95
- Chapter 3.8 – JI Chord Names Part II....101
- Chapter 3.9 – Using Higher Primes in Chords....107

Part IV – Temperaments....112
- Chapter 4.1 – Basic Tempering....112
- Chapter 4.2 – JI-centric Tempering....117
- Chapter 4.3 – Meantone Temperament....123
- Chapter 4.4 – Other Commas: Rank....128
- Chapter 4.5 – Temperament Names Part I....130
- Chapter 4.6 – More Commas: Periods and Generators....133
- Chapter 4.7 – Adaptive Tuning With Alt-Tuner *....138
- Chapter 4.8 – Temperament Names Part II....146
- Chapter 4.9 – Tuning Innate Comma Chords *....154
- Chapter 4.10 – Constructing MOS scales *....156
- Chapter 4.11 – Case Study: "Central Park West"....157

Part V – Alternative Frameworks....162
- Chapter 5.1 – Frameworks....162
- Chapter 5.2 – The 12-tone Framework....165
- Chapter 5.3 – Pentatonicism....169
- Chapter 5.4 – The 19-tone Framework....175
- Chapter 5.5 – The 22-tone Framework: Ups and Downs....183
- Chapter 5.6 – 22-tone Staff Notation with Ups and Downs....187
- Chapter 5.7 – Other Frameworks: The Scale Tree....198
- Chapter 5.8 – Chord Names and Scale Names with Ups and Downs....220
- Chapter 5.9 – MOS Scales *....231
- Chapter 5.10 – Diatonic Frameworks: 12, 17, 19, 22, 24, 26 and 27 *....233
- Chapter 5.11 – Selected Large Frameworks: 31, 41, 53 and 72 *....241

Chapter 5.12 – Perfect Frameworks: 7, 14, 21, 28 and 35 *....246
Chapter 5.13 – Pentatonic Frameworks: 5, 10, 15, 20, 25 and 30 *....250
Chapter 5.14 – Supersharp Frameworks: 8, 13 and 18....252
Chapter 5.15 – Superflat Frameworks: 9, 11, 13b, 16, 18b and 23 *....256
Chapter 5.16 – Notating Rank-2 Tunings, Part I: Triple Yo *....258
Chapter 5.17 – Notating Rank-2 Tunings, Part II *....263
Chapter 5.18 – Alternate Keyboards, Alternate Notations *....271
Chapter 5.19 – Guitar Frettings: Edos and Udos....277
Chapter 5.20 – Smooth Voice Leading: 3-tone and 4-tone *....286

Glossary/Index....289

Appendix 1 – A Guide to Shorthand Notation....293

Appendix 2 – Possible Extensions to the Notation....295

Appendix 3 – A Guide to Widely-used Microtonal Terms....296

Appendix 4 – Color Notation In Other Languages....299

Appendix 5 – The Ideal Microtonal Notation....302

Appendix 6 – Various Mathematical Formulas and Proofs....303

Part VI – The Alt-tuner Manual – version 1.2....310
Chapter 6.1 – Overview & Setup....310
Chapter 6.2 – Quick Start: Basic Operation....311
Chapter 6.3 – Alt-keyswitcher....316
Chapter 6.4 – Basic Alt-tuner Preference Screens....321
Chapter 6.5 – Multi-keyboard Setups....336
Chapter 6.6 – Recording With Alt-tuner....343
Chapter 6.7 – Advanced Preference Screens....346
Chapter 6.8 – Advanced Topics....357
Chapter 6.9 – Advanced Examples....370
Chapter 6.10 – Hardware & Software Issues....381
Chapter 6.11 – Troubleshooting and Other Considerations....403
Chapter 6.12 – Customizing Alt-tuner....408
Appendix 6.1 – 3000 Ratios....412
Appendix 6.2 – Alt-tuner Flowchart....423
Appendix 6.3 – List of Clickable Items....424

Preface

This book is aimed at the type of musician that has been to or is going to music school. Only a basic understanding of music theory is required: the circle of 5ths, the names of intervals, staff notation, chord names, and roman numeral notation (e.g. I – IV – V).

This book attempts to collect in one place a full exploration of the various appoaches to alternative tunings (microtonal music), by which is meant any tuning which deviates noticably from the standard tuning of 12-tone equal temperament.

Much of this book's focus is on devising a suitable notation for this music. The notation strives to be "backwards compatible", a software developer's term meaning compatible with past versions. Thus the notation is octave-equivalent, heptatonic, and generated by fifths, even if the music being notated isn't. Other criteria are listed in Appendix 5, "The Ideal Microtonal Notation".

The notation is meant to be not just written but also spoken. To be not only functional but also elegant. Microtonal notation exists at the intersection of music, mathematics and language.

What is notation for? That depends a lot on the roles of the performer and the composer. In much folk music, the composer is always the performer, but so are others. In much classical music, the composer is all-powerful, and dictates exactly what the performer plays. Except for cadenzas, no-one in an orchestra would dare change one note of a symphony. In jazz, the performer is expected to compose their own solos on the spot, and often to alter the chords to create their own arrangement. Some avant-guarde compositions are extremely difficult to cover, and there may never be another performer other than the composer. Here are some of the uses of notation:

for the composer/arranger to remember their composition (that sounds great, quick, write it down!)

for the composer to direct the performer (play these specific notes)

for the composer to influence the arranger (chord names which suggest bass lines)

for the composer to influence the improviser (chord names which suggest jazz scales)

for the composer to unwittingly instruct future composers (ooh, that's cool, I'm stealing that!)

for the music teacher to discuss composition

Part I – An Introduction to Just Intonation

Chapter 1.1 – A Parable

Once upon a time there was a king, Duplius, a proud distant man who lived by himself in a large empty castle. He grew quite lonely, until he chanced to meet the lady Tertia. She was noble and calm, and he fell deeply in love with her. They married and filled the castle with children, and they were happy.

But then one day Duplius met the lady Quintia, who was warm and friendly and quick to laugh. He soon fell in love with her. He still loved lady Tertia, so he decided to have them both live in his castle. The king's subjects were shocked, and there was quite some controversy. But he was the king, and Quintia and Tertia got along well, so that was that. Duplius and Quintia wed, she bore him more children, and they were happy.

Both Tertia and Quintia loved to sing and dance. They particularly enjoyed the balls that were held in the king's court. Now the king had a beautiful white dress encrusted with jewels, an ancient family heirloom from a land far away. There was a tradition in the court of a certain dance that always featured the queen wearing that dress. Whenever a ball was held, the two queens would take turns wearing the beautiful white dress and performing the traditional dance. But whereas Tertia was quite tall and slender, Quintia had a more curvaceous and womanly figure. Whenever Tertia wore the dress, she would have the royal seamstress let the hem down and take in the bosom. And Quintia would likewise have it be altered to fit her.

In time the dress began to show wear from the many alterations, and holes appeared in the fabric. Seeing the dress in such disrepair, Duplius forbade either of his wives to ever wear it again. They were very upset and argued with him constantly about the dress and the traditional dance. Every time a ball was held, the king knew no peace.

So the king called on his wisest wizard, Zarlino, to solve the problem. Zarlino thought deeply about the matter until he devised a solution. First he made careful measurements of both woman's figures. Then he had the dress altered one last time to be tight enough for Tertia and short enough for Quintia. He made a tight corset for Quintia to wear, so tight that it took the aid of three handmaidens to put it on. For Tertia he made a type of harness that pulled her shoulders closer to her ankles, causing her to stoop down. With these devices, he managed to fit both women into the dress. But Zarlino's solution was a mixed blessing. Whenever a ball was held, the one wearing the dress could never dance quite as well as before.

Things went on like this for many years, until one day Duplius met the passionate and mercurial lady Septima. He fell in love with her, married her as well, and she too came to live in the castle. Again there was great controversy among the people. At the very next ball, she was to perform the traditional dance in the beautiful white dress. But she was quite tall and busty, with wide hips as well. She had to wear both the corset and the harness to fit into the dress, and she could hardly dance at all. Seeing her discomfort, Duplius resolved to help her.

By then Zarlino had passed away, so Duplius called on the wise woman Midia for aid. Midia thought long and hard about the problem until she finally found a solution. She treated the dress with mysterious potions and powders that transformed the fabric into a flexible material with magical properties. When the right spell was spoken, it could be stretched like taffy or compressed like an accordion to assume any shape, but when the spell wore off, it was firm and unyielding. She presented the magical dress to the three women who were overjoyed. They threw away Zarlino's uncomfortable corsets and harnesses and danced freely and gracefully. And they were happy.

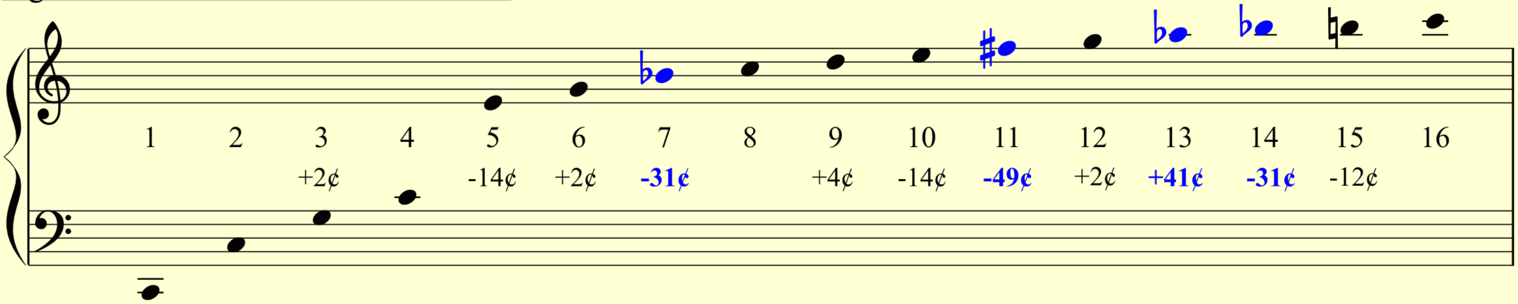
The End

This parable recounts the history of tuning theory and practice for the last eight centuries of Western music. In the next chapter, we'll meet a few of Duplius' wives...

Chapter 1.2 – Ratios, Cents, Primes, and Limits

In the standard tuning, **equal temperament,** or **12-ET**, many intervals are markedly out of tune. They only sound in tune because it's what we're used to hearing; we're conditioned to accept them as in tune. In other words, they're <u>subjectively</u> or <u>culturally</u> in tune. But if you compare **just intonation** chords to 12-ET for any length of time, you will hear a clear difference, based not on cultural conditioning but acoustics. They "beat" less, producing a smoother sound. These intervals are <u>objectively</u> or <u>acoustically</u> in tune. You can verify this with simple experiments using the **harmonic series**. All string and wind instruments (including the human voice) have harmonic overtones which are contained in this series:

<u>Figure 1.2.1 – The harmonic series in C</u>



For more on the harmonic series, see cnx.org/content/m11118/latest. The 1st harmonic is also called the fundamental. Confusingly, sometimes the 2nd harmonic is called the 1st overtone. Some of the notes are not in tune with 12-ET. The 3rd harmonic is 2% of a semitone sharp, and the 5th one is 14% flat. The notes in blue differ significantly from 12-ET.

<u>Acoustic piano experiment</u>: On a piano, play a C below middle C. Next press the key down very slowly, so that it doesn't sound, and hold it down. Now play the C two octaves below middle C fairly loudly (the fundamental in the figure above). Release it, and you will hear the higher C (harmonic #2) ringing out. The lower note contains a harmonic (or overtone) that matches the higher note and resonates with it. Now play the lower note by itself and try to hear the higher note contained in the lower one. It's there, keep listening.

Do all this again with the higher note moved up a 5th to G, making a 12th (an octave and a fifth) with the low note (harmonic #3). Again, the high note will ring out. Play the low note by itself. Can you hear the high note contained in the low note? Now try it with a double octave (#4). Try using a different lower note, playing the same octave, 12th and double octave intervals. Once you train your ears to hear these "built-in notes", a single note becomes an entire chord.

So far, so good. Now play a double octave plus a major 3rd, harmonic #5. The higher overtones are quieter, so you may have to play the lower note harder. You should be able to hear the built-in note as before. Now play the two keys simultaneously. If you listen closely, you'll hear rapid beats, because the 12-ET major 3rd is noticeably sharper. These interference beats are explained here: en.wikipedia.org/wiki/Beat_(acoustics). The 6th overtone is at a double octave plus a 5th. No interference beats. #7 is a double octave plus a minor 7th. The 12-ET minor 7th is much sharper than this harmonic! The next overtones are at the triple octave, the triple octave plus a major 2nd, and the triple octave plus a major 3rd.

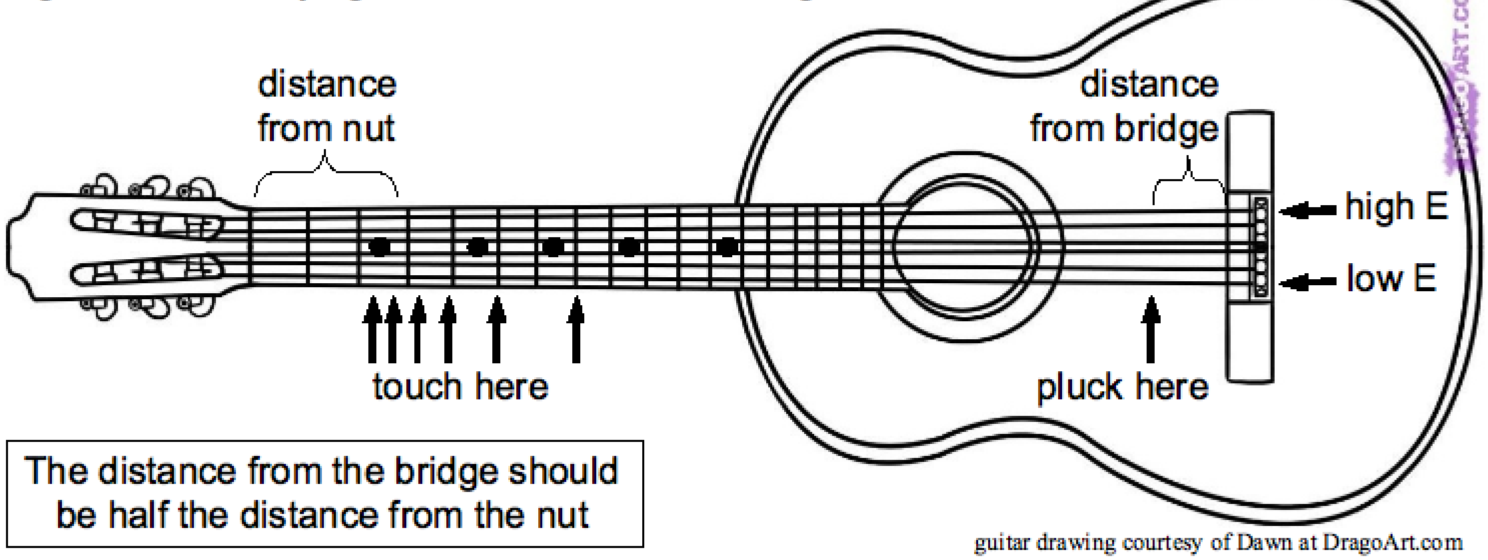
<u>Electronic keyboard experiment</u>: Select a fat but realistic sound, like organ. Play the lower note by itself and try to hear the higher note contained in it. Play the same intervals as before (8ve, 12th, etc.) and listen for interference beats. Avoid sounds with vibrato, everything will beat!

<u>Guitar experiment</u>: Pluck the deepest (thickest) string loudly, then quickly dampen it. The highest (thinnest) two strings, a 12th and a double octave above, will ring out.

Play the harmonic series on the deepest string. The open string is the fundamental. Next touch it lightly at the 12th fret as you pluck it. That's the octave, can you hear it contained in the open string? All string instruments have the same built-in notes as the piano does. Now touch it at the 7th fret for the 3rd harmonic, a 12th. Touch it at the 5th fret for the 4th, a double octave. This one should match the highest string. If not, tune up!

The 5th harmonic is just a little left (flat) of the 4th fret. It's 1/6 of the way from the 4th to the 3rd fret, in other words, 3 ⅚ frets from the nut. Compare this with the note at the 4th fret of the top string. That note should be slightly sharper than the harmonic.

Figure 1.2.1 – Playing the harmonic series on a guitar



The next table shows the location of the first 10 harmonics, and where the matching note is. The higher overtones are harder to hear. You may need to dampen the other strings, and pluck closer to the bridge. A good quality guitar with new strings helps. You may find it easier to hear harmonics with steel strings rather than nylon ones.

Table 1.2.1 – Finding the harmonic series on the lowest (deepest) string of the guitar

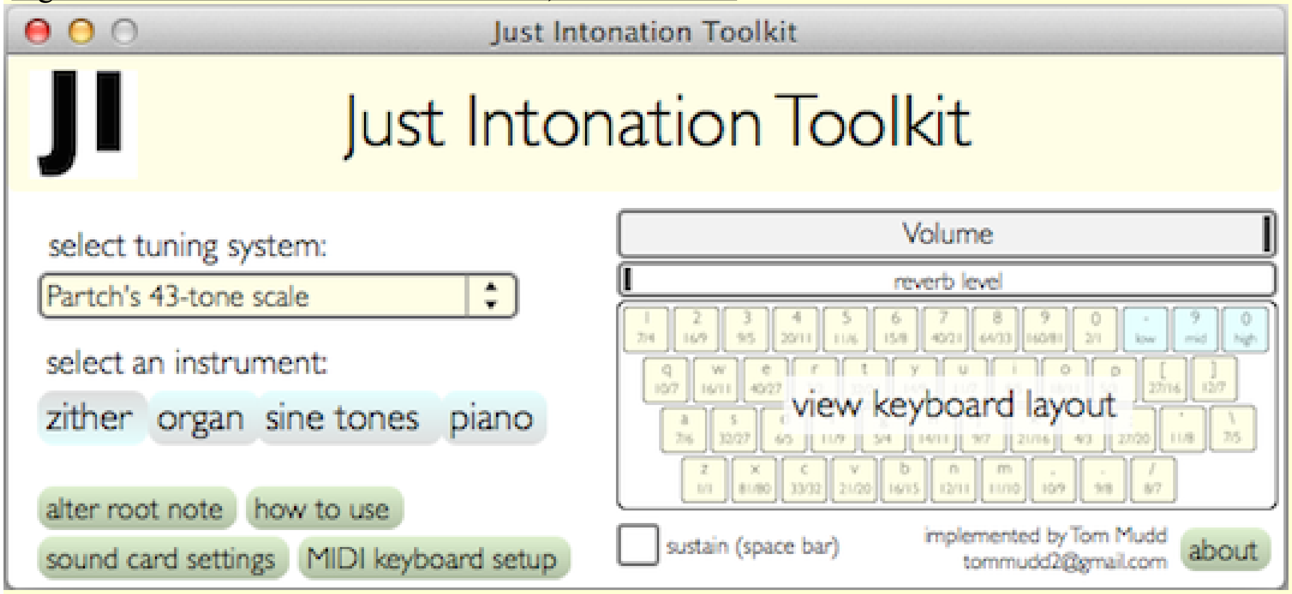
harmonic	**note**	**interval from the fundamental**	**interval from last harmonic**	**frets from the nut**	**note that matches the harmonic**	**deviation from 12-ET**
1	E	unison	——	——	——	0¢
2	E	8ve	8ve	12	4th highest string, 2nd fret	0¢
3	B	8ve + 5th	5th	7	2nd highest string, open	+2¢
4	E	double 8ve	4th	5	highest string, open	0¢
5	G♯	double 8ve + maj 3rd	maj 3rd	3 ⅚	highest string, 4th fret	-14¢
6	B	double 8ve + 5th	min 3rd	3 ⅙	highest string, 7th fret	+2¢
7	D	double 8ve + min 7th	min 3rd	2 ⅔	highest string, 10th fret	-31¢
8	E	triple 8ve	maj 2nd	2 ⅓	highest string, 12th fret	0¢
9	F♯	triple 8ve + maj 2nd	maj 2nd	2	highest string, 14th fret	+4¢
10	G♯	triple 8ve + maj 3rd	maj 2nd	1 ⅚	highest string, 16th fret	-14¢

The intervals between the harmonics, those in the 4th column of the table above, are the essential building blocks of music. The sooner in the series they occur, the more essential they are. The most essential intervals are the first three, the 8ve, the 5th and the 4th. Harmonics #4, #5 and #6 form a major chord. This is where the major chord comes from, and why it sounds so natural. All of Western music is based on the intervals between the first 6 harmonics. For example, the major scale comes from three major chords, rooted on the 8ve, the 5th and the 4th.

Western music is based on a tuning system that approximates the first 6 harmonics fairly well, but not perfectly, and certain higher harmonics very poorly. What would music sound like if it were more acoustically in tune? If one avoids the blue notes in Figure 1.2.1, it sounds familiar, but in my opinion smoother and more relaxed. Using the blue notes adds strange new sounds that seem to me "weirdly natural".

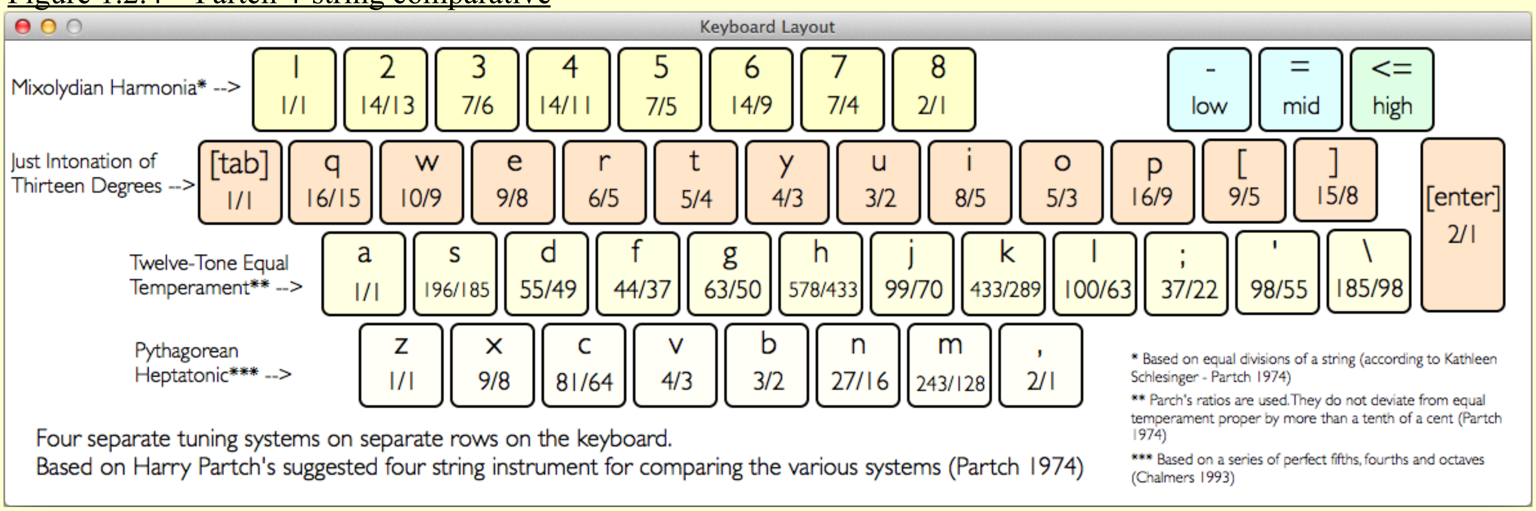
To play music in just intonation, or **JI**, with no interference beats, download Tom Mudd's free Just Intonation Toolkit at TomMudd.co.uk/JustIntonation. This standalone app for Windows and OS X lets you play music with your computer's QWERTY keyboard or with a midi keyboard, and hear the result right away with the built-in sounds.

Figure 1.2.3 – The Just Intonation Toolkit, main window



Run the Toolkit, and select the "Partch 4-string comparative" tuning system and the organ instrument. You should see this window; if not, click on "view keyboard layout" in the main window:

Figure 1.2.4 – Partch 4-string comparative



Pressing the computer's QWERTY keys will make sounds. Switch to the higher octave by clicking the "high" key in the upper right, or pressing <= (PC) or "delete" (mac). Press the A key and the enter key. You'll hear a just octave, with no interference beats. Now press the A key and the K key, for a 5th. This interval is slightly out of tune in 12-ET and will beat slowly. Now press A and U to hear a just, beatless fifth. You may prefer the AK fifth to the AU fifth, as the slow beating gives the sound a little life. Sometimes a little mistuning can be a good thing!

Play the U and K keys one at a time. If you have a very good ear, you can tell which is sharper. (Answers to this and other ear tests are at the end of this section.) Play them together to hear a slightly mistuned unison, which sounds richer than a perfect unison. This is what a chorusing effect does.

Play AGK for a 12-ET major chord with more rapid beats. Compare it to ATU for a just, beatless major chord. This is the "smoother, more relaxed" sound from 4 paragraphs earlier. The ATU chord is exactly the same as the major chord formed by harmonics 4, 5 and 6.

Play the T and G keys one at a time. Can you tell which is sharper? Play them together, and hear the rapid beats, caused by extreme chorusing. RF will sound similar, as will WD. Can you tell which of R and F is sharper? What about W and D? Or E and D?

The space bar functions as a sustain pedal. Hold it down and play these keys one at a time: A, G, K and ' (apostrophe). AGK' is a 12-ET dom7 chord. Release all keys, hold down the space bar, and play A, T, U and 7 one at a time. ATU7 is the just, beatless dom7 chord formed by harmonics 4, 5, 6 and 7. (To play this chord without using the space bar, play

ATB7.) Compare AGK' and ATU7. Listen to the overall sound, not the individual notes. Which sounds smoother? ATU7 is the "weirdly natural" sound.

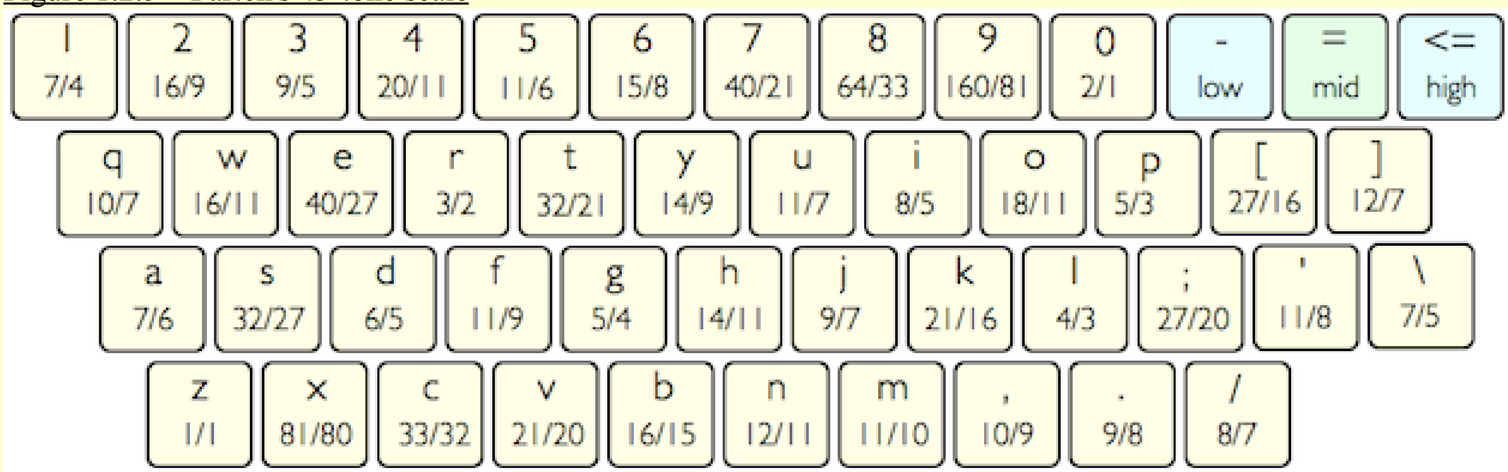
You can add some slow beating to ATU7 by using the slightly-off 12-ET 5th, making ATK7. Which do you prefer?

Play the 7 and ' keys one at a time. Which is sharper? Play them together to hear very rapid beating. Compare the K' minor 3rd with the U7 one. The latter one probably sounds wrong. And yet U7 makes a smoother dom7 chord.

The U7 interval can be played lower down as Y6. Improvise a melody in the key of "Y minor" using this scale: TYUIP8. Now play the same melody using TYU6P8. Notice how the character changes completely. I and 6 are about half a semitone apart. Compare the YI8 minor chord with the Y68 minor chord. You'll have to use the space bar, or else play the 8 note with the enter key. Y68 is formed by harmonics 6, 7 and 9. YI8 is harmonics 10, 12 and 15.

To explore many different JI intervals at once, select the tuning system "Partch's 43-tone scale".

Figure 1.2.5 – Partch's 43-tone scale



Play each note in turn: ZXCVBNM,./, then ASDFGHJKL;'\, then QWERTYUIOP[], then 1234567890. With practice, each note is clearly distinguishable from its neighbors. Play each note over a Z drone: play ZX, ZC, ZV etc. Each interval has its own quality. ZX, ZC, ZE, ZT, Z8 and Z9 are particularly dissonant.

Compare these triads: ZAR, ZSR, ZDR, ZFR, ZGR, ZHR and ZJR. Also these diminished triads: ZD\ and ZA\.

Compare these tetrads: ZARP, ZAR1, ZDR], ZDR3, ZFR5, ZGRP, ZGR1, ZGR6, and ZJR3.

Play this chord progression, using the space bar: ZGR1 then ZA\1 then ZALP.

Play harmonics 1-8: using the space bar, play low Z mid ZG high AGR10 (low, mid and high are the octave keys in the upper right). Play harmonics 2-16: low ZR mid ZGR1 high Z.G'RO160. (The 13th harmonic O is slightly off.)

Answers to the ear tests:

U to K: U is 2% of a semitone sharper than K.

T to G: T is 14% of a semitone flatter than G.

R to F: R is 16% of a semitone sharper than F.

W to D: W is 18% of a semitone flatter than D.

E to D: E is 4% of a semitone sharper than D.

7 to ': 7 is 31% of a semitone flatter than '.

In just intonation, every musical interval is a frequency **ratio**. Octaves sound the way they do because the higher note's frequency is twice the lower one's, making a 2-to-1 ratio, written 2/1, or sometimes 2:1 or 1:2. It doesn't matter much what the exact frequency of the two notes are, it's the ratio between them that we hear and recognize as a musical interval. For example, the A note below middle-C has a frequency of 220 cps (cycles per second, equivalent to hz or hertz) and the A above middle-C is 440 cps. These two notes together will make an interval of an octave. The nearby G notes at 196 cps and 392 cps will also make an octave. So would any two frequencies with that 2-to-1 ratio, like 200 cps and 400 cps, or 240 cps and 480 cps.

If two notes are a fifth apart, the higher one's frequency is one & a half times greater. That's a ratio of 3-to-2, or 3/2. Examples would be 220 cps and 330 cps, or 240 cps and 360 cps. Fourths are 4/3 (e.g. 240 to 320 cps), major thirds are 5/4 (e.g. 240 to 300 cps), minor 3rds are 6/5, etc. The unison is 1/1.

The intervals from the fundamental up to each note in the harmonic series are 1/1, 2/1, 3/1, 4/1, 5/1, etc. Every ratio occurs as an interval between two harmonics. The 5/3 ratio is the interval from harmonic #3 up to harmonic #5. Since that's from the 5th up to the maj 3rd, 5/3 is a major 6th. From harmonic #6 to #10 is also 5/3, since 10/6 reduces to 5/3.

The human ear isn't very good at detecting the actual frequency of a note (except for those few with perfect pitch), but it's very good at detecting frequency ratios, and small deviations from those ratios. With a piano or guitar sound, the ear can easily hear the difference between two notes tuned to 220 and 440 cps vs. two notes tuned to 220 and 441 cps. The 220 cps note is A, and the 440 cps note is A exactly an octave higher. This interval doesn't beat. But in the second interval, the 2nd harmonic of the low A is 440 cps, almost but not quite the same as the 441 cps high A. The ear hears interference beats between the two, and the second interval sounds slightly out of tune.

Likewise, a 220 cps A note and a 331 cps E note will beat. A's 3rd harmonic is 660 cps. E's 2nd harmonic is 662 cps. The two are not quite the same, and will beat. Whereas A-220 and E-330 won't beat. Likewise, A-220 and C♯-276 will beat, but A-220 and C♯-275 won't (ratio 5/4).

Many people working with JI memorize specific ratios for each and every interval. In my experience, knowledge of the harmonic lattice (chapter 1.3), in combination with my color notation (part II), makes this unnecessary and makes JI less mathematical and more intuitive. So don't sweat the math in this chapter! Just get the general concepts, then use your ears.

Every ratio has two numbers, a numerator on top and a denominator on the bottom. The numerator is always bigger, unless it's a descending interval. Those are flipped: since an ascending fourth is 4/3, a descending fourth is 3/4.

The human ear hears pitch logarithmically, not linearly. This means that what we perceive as <u>adding</u> musical intervals is actually <u>multiplying</u> ratios. Adding two intervals together means multiplying the tops and bottoms together. A 5th plus a 4th = 3/2 x 4/3 = 12/6 = 2/1 = an 8ve. To subtract an interval, flip it and multiply. A 5th minus a maj 3rd = 3/2 ÷ 5/4 = 3/2 x 4/5 = 12/10 = 6/5 = a minor 3rd. If this is at all confusing, see these extremely readable pages:

www.MathsIsFun.com/improper-fractions.html
www.MathsIsFun.com/fractions_multiplication.html
www.MathsIsFun.com/fractions_division.html
www.MathsIsFun.com/simplifying-fractions.html

Combining ratios is important because in JI, the exact size of an interval depends on how it's derived. For more on this, see "comma" in the next chapter.

You can see ratios at work on any string instrument. Frequency is inversely related to string length, and frequency ratios are inversely related to string length ratios. If you play an open string on a guitar, and then play at the 12th fret on the same string (actually fretting it, not just touching it to play a harmonic), you've divided the string length in half. Because 1/2 inverted is 2/1, and because the octave has a ratio of 2/1, the 12th fret is an octave. If you play at the 7th fret, the string length is about 2/3 of the open string. 2/3 inverts to 3/2, which makes a fifth. Likewise, the 5th fret shortens the open string to about 3/4, and 4/3 is a fourth.

The wider the interval, the larger the number the ratio evaluates to. The fourth is 4/3 = 1.333, the fifth is 3/2 = 1.5, and the octave is 2/1 = 2. However, ratios are rarely written this way. A much more useful measure of the width or **size** of an interval is **cents**. One cent is a hundredth of an equal-tempered semitone.

The standard tuning system, 12-ET, is called equal temperament because the octave is divided into 12 equally-sized semitones of 100¢ each. 12-ET represents a <u>melodic</u> division of the octave, in which all intervals are essentially a stack of 100¢ semitones, and every interval's cents are always a multiple of 100. On the other hand, JI represents a <u>harmonic</u> division based on simple ratios. For any interval except the octave, these two approaches give slightly different results. For example, the 12-ET major third's cents are a nice round number, 400¢. But the frequency "ratio" is an irrational number, the cube root of 2 (because a 12-ET maj3 is exactly 1/3 of an octave). This comes out to a rather messy number, 1.259921. In contrast, the JI major third has a messy number of cents, 386.313721¢, but the ratio has nice round numbers, 5/4 = 1.25. For every interval except the octave, if the cents are round, the ratio will be messy, and vice versa. With a completely random interval, usually both the cents and the ratio are messy.

The next table shows the differences between JI and 12-ET. For example, a 5/4 happens to be about 14¢ flatter than the 12-ET major third. The JI 4th and 5th are very close to 12-ET.

Table 1.2.2 – A few sample JI intervals

interval	**semitones**	**JI ratio**	**JI cents**	**12-ET equivalent**	**deviation from 12-ET**
perf unison	0	1/1	0¢	0¢	0¢
min 2nd	1	16/15	112¢	100¢	+12¢
maj 2nd	2	10/9	182¢	200¢	-18¢
		9/8	204¢		+4¢
min 3rd	3	6/5	316¢	300¢	+16¢
maj 3rd	4	5/4	386¢	400¢	-14¢
perf 4th	5	4/3	498¢	500¢	-2¢
aug 4th or dim 5th	6	45/32	590¢	600¢	-10¢
		64/45	610¢		+10¢
perf 5th	7	3/2	702¢	700¢	+2¢
min 6th	8	8/5	814¢	800¢	+14¢
maj 6th	9	5/3	884¢	900¢	-16¢
min 7th	10	16/9	996¢	1000¢	-4¢
		9/5	1018¢		+18¢
maj 7th	11	15/8	1088¢	1100¢	-12¢
octave	12	2/1	1200¢	1200¢	0¢

Cents are calculated with logarithms: cents = 1200 x log (ratio) / log (2). The ratio can be calculated as a decimal number from the cents: ratio = 2 ^ (cents / 1200). For example, a 12-ET semitone's "ratio" is 2 ^ (1/12) = the twelfth root of 2 = 1.059463. Again, don't sweat the math! For a more in-depth look at ratios and cents as well as some great insights about JI, I highly recommend reading this page: www.KyleGann.com/tuning.html. Here's a sample paragraph:

> "I've had interesting experiences playing just-intonation music for non-music-major students. Sometimes they will identify an equal-tempered chord as 'happy, upbeat,' and the same chord in just intonation as 'sad, gloomy.' Of course, this is the first time they've ever heard anything but equal temperament, and they're far more familiar with the first sound than the second. But I think they correctly hit on the point that equal temperament chords do have a kind of active buzz to them, a level of harmonic excitement and intensity. By contrast, just-intonation chords are much calmer, more passive; you literally have to slow down to listen to them. (As Terry Riley says, Western music is fast because it's not in tune.) It makes sense that American teenagers would identify tranquil, purely consonant harmony as moody and depressing. Listening from the other side, I've learned to hear equal temperament music as a kind of aural caffeine, overly busy and nervous-making. If you're used to getting that kind of buzz from music, you feel the lack of it as a deprivation when it's not there. But do we need it? Most cultures use music for meditation, and ours may be the only culture that doesn't. With our tuning, we can't."

Small tuning deviations can be thought of as an audio effect like EQ or compression. One could certainly argue that the edginess of major chords in 12-ET is desirable. Like a chorusing effect, that slight dissonance adds fullness and depth. But one can't argue that this musical effect is at all fresh or innovative, not after over a century of 12-ET. It's as if everyone were playing guitar through the exact same effect box, with the exact same settings. In this situation, the most innovative thing one can do is to turn off the effect.

Prime numbers (2, 3, 5, 7, 11, 13, 17, 19, etc.) are the basic building blocks of a number, as explained here: www.MathsIsFun.com/prime-factorization.html. Every number can be factored into primes in one and only one way. Since ratios have two numbers, there are two sets of primes. Prime exponents are a way of counting up these building blocks. For example 10/9 factors into (2·5)/(3·3). It has one two above, two threes below, and one five above. Thus $10/9 = 2^1 \cdot 3^{-2} \cdot 5^1$, written as (1, -2, 1). Another example: 9/8 factors into $(3·3)/(2·2·2) = 2^{-3} \cdot 3^2$. Its monzo is (-3, 2). $7/5 = 2^0 \cdot 3^0 \cdot 5^{-1} \cdot 7^1$ = (0, 0, -1, 1). Every ratio, not counting unreduced ratios like 6/4, can be expressed as a series of exponents in one and only one way. Such a list of prime exponents is called a **monzo**.

Helpful links for the math-challenged among us:

www.MathsIsFun.com/definitions/factor.html
www.MathsIsFun.com/prime-composite-number.html
www.MathIsFun.com/exponent.html
www.MathsIsFun.com/algebra/logarithms.html

But what do ratios sound like? When you listen to music, you're listening to melodies, chords and chord progressions, all of which contain intervals, which are ratios, which have two numbers, each of which can be factored into primes. The prime numbers 2 and 3, which create perfect intervals like octaves, fifths and fourths, are present in the ratios of virtually all music. Primes larger than 2 or 3 give flavor to the music. Amazingly, the ear can recognize these larger prime number building blocks, and whether they are on the top or the bottom of a ratio. Ratios with five on top sound major, and those with five on the bottom sound minor. Seven on the top sounds... different.

Musically, each prime has its own personality. The number 2 creates octaves, which seems to me proud and distant (Duplius). The number 3 creates perfect intervals like the fourth and the fifth, which feel noble and calm (Tertia). The number 5, when on top of the ratio, creates major thirds and sixths that sound warm and friendly to me (Quintia). But watch out, when she's on the bottom, she creates a dark and moody minor key. The number 7 (Septima) on top is even darker and moodier, but when on the bottom, she becomes so bright and intense that she can seem harsh and annoying.

In general, the smaller the two numbers in the ratio are, the more consonant it is. The strongest, most consonant intervals like the octave (2/1), the fifth (3/2) and the twelfth (3/1) contain only very small numbers, three or less. The major 3rd (5/4) and the major 6th (5/3), slightly less consonant, contain numbers of five or less. At the other end of the spectrum, the dissonant major 7th (15/8) and augmented 4th (45/32) contain quite large numbers. I think of the continuum not so much as running consonant to dissonant, but rather useful-but-boring to interesting to obscure-and-boring. Of course, everyone has their own definition of interesting!

So one can actually hear both the "bigness" and the "prime-ness" of a ratio. Every musical piece uses a scale which contains many intervals and many ratios. **Limit** refers to the maximum bigness or prime-ness of all these ratios. Thus increasing 1) the size of the numbers used, called the **odd limit**, or 2) the size of the prime factors used, called the **prime limit**, are both good ways to add interest.

I'm simplifying slightly, odd limit is actually based on the largest number in the ratio after factoring out all the twos. The **integer limit** is based on the largest number used in the ratio, odd or even. The odd limit is more useful musically because factoring out the twos makes the limit octave-equivalent and thus voicing-independent. Inverting an interval or widening it by an octave changes the integer limit, but not the odd limit. For example, the fifth 3/2, the 4th 4/3, and the 11th 8/3 are all different voicings of the same interval, and all three have the same odd limit of 3.

Prime limit is the more musically fundamental limit. If neither prime nor odd is specified, "limit" refers to prime limit. Western music has been steadily evolving towards higher prime limits. Medieval music is 3-limit, i.e. based on intervals whose ratios use 2 and 3. Medieval music is the child of Duplius and Tertia. Beginning with the Renaissance, when Duplius met Quintia, music became 5-limit (using 2, 3 and 5). Jazz and blues use tetrads and "blue notes" which hint at 7-limit or **septimal** music. This progression will become very clear in the next chapter, as each new prime number literally adds a new dimension. For a terrific read on historical tunings, including meantone and other **temperaments**, see www.KyleGann.com/histune.html.

Of course, there are many more primes beyond 7; 11-limit and 13-limit intervals are introduced in chapter 3.6. Another possibility is including larger primes and excluding smaller ones. The JI **subgroup** is a list of the included primes, for example 2.3.7 (Duplius, Tertia and Septima, without Quintia).

In a JI chord, the overtones of one note will generally coincide with those of the others, whereas in 12-ET they are generally out of tune and clash. In a sense, JI is tuned to the harmonic series, with the prime limit specifying how high up the series we go. Loosely speaking, 5-limit JI goes up to the 6th harmonic (just short of the next prime, 7). 7-limit JI goes up to the 10th harmonic, just short of 11.

Within a given prime limit, there are only certain possible odd limits. Possible odd limits in 5-limit JI are 3, 5, 9, 15, 25, 27, 45, 75, etc. (all products of 3 and/or 5). Odd numbers like 7, 11 and 13 are not part of 5-limit, because they are higher primes. Odd numbers like 21 or 33, while not themselves prime, do have factors that are higher primes.

Higher prime limits have more possibilities. Possible odd limits in 7-limit JI are 3, 5, 7, 9, 15, 21, 25, 27, 35, 45, 49, etc. 11-limit JI would add to this list the numbers 11, 33, 55, etc.

Odd limits can be applied to chords as well as ratios (see chapter 2.4). More dissonant chords tend to have a higher odd limit. Prime limit and odd limit are discussed further at the end of chapter 1.3 and the end of chapter 2.2.

The prime number building blocks represented by King Duplius and his wives can be written as ratios in the form 2/1, 3/1, 5/1 and 7/1. They can also be written as octave-reduced ratios, for more compact intervals:

symbol	prime	ratio	interval	cents	cents from 12-ET
Duplius	2	2/1	octave	1200¢	0¢
Tertia	3	3/2	fifth	702¢	+2¢
Quintia	5	5/4	maj 3rd	386¢	-14¢
Septima	7	7/4	min 7th	969¢	-31¢

The first interval, the octave, is called the **period** (or **interval of equivalence**) because the scale is assumed to repeat periodically within it. The other intervals are called **generators** because they create, or generate, all the other intervals within the prime limit. As we'll see in the next chapter, Duplius and Tertia together can create an entire scale.

The period is usually but not always the octave. For example, Bohlen-Pierce tunings exclude the prime 2 and use the subgroup 3.5.7. The period is the twelfth 3/1, and the generators are 5/1 and 7/1 (or 5/3 and 7/3 in reduced form).

Otonal loosely means the primes greater than 2 are on the top (over), and **utonal** has them on the bottom (under). See en.wikipedia.org/wiki/Otonality_and_Utonality. Otonal is considerably more consonant than utonal. Otonal and utonal apply not to intervals but only to three or more notes. I've coined similar terms **over** and **under** that apply only to intervals. If the sign of the last number in a ratio's monzo is positive, it's over; if negative, it's under. If the numerator's prime-limit is higher than the denominator's, it's over. Over intervals are more consonant than similar under ones. Thus a ratio's dissonance depends on both its "bigness" (integer limit or odd limit) and "under-ness".

For 5-limit JI, otonal/over corresponds loosely to major, and utonal/under to minor. However, for septimal ratios, as we'll see in the next chapter, otonal/over = minor and utonal/under = major.

The harmonic series is important, but not inescapable. Many parts of the world use non-harmonic idiophones such as marimbas, kalimbas and gamelans. Furthermore, with synthesizers, one can "detune" the harmonic series and create any harmonic spectrum desired. For example, the first overtone could be not an octave but some sort of ninth. This changes music profoundly – the very rules of consonance are redefined for that sound. William A. Sethares has done important work on the relationship of timbre to consonance, and on determining the best scales for a given timbre and the best timbres for a given scale. Read more here: sethares.engr.wisc.edu/contents.html.

In addition, physical instruments aren't perfectly mathematical, see en.wikipedia.org/wiki/Inharmonicity Also, there are limits on how accurately an instrument can be tuned.

Finally, here's a few somewhat obscure terms from conventional music theory that I'll be using: every interval has a **degree** (3rd, 5th, etc.), also known as number, as well as a **quality** (major, perfect, augmented, etc.). For help with such terms (inversions, chord names, etc.), see en.wikipedia.org/wiki/Interval_(music)

More useful links:
On JI: en.wikipedia.org/wiki/Just_Intonation
On prime limits and odd limits: en.wikipedia.org/wiki/Limit_(music)
On temperaments: en.wikipedia.org/wiki/Musical_temperament

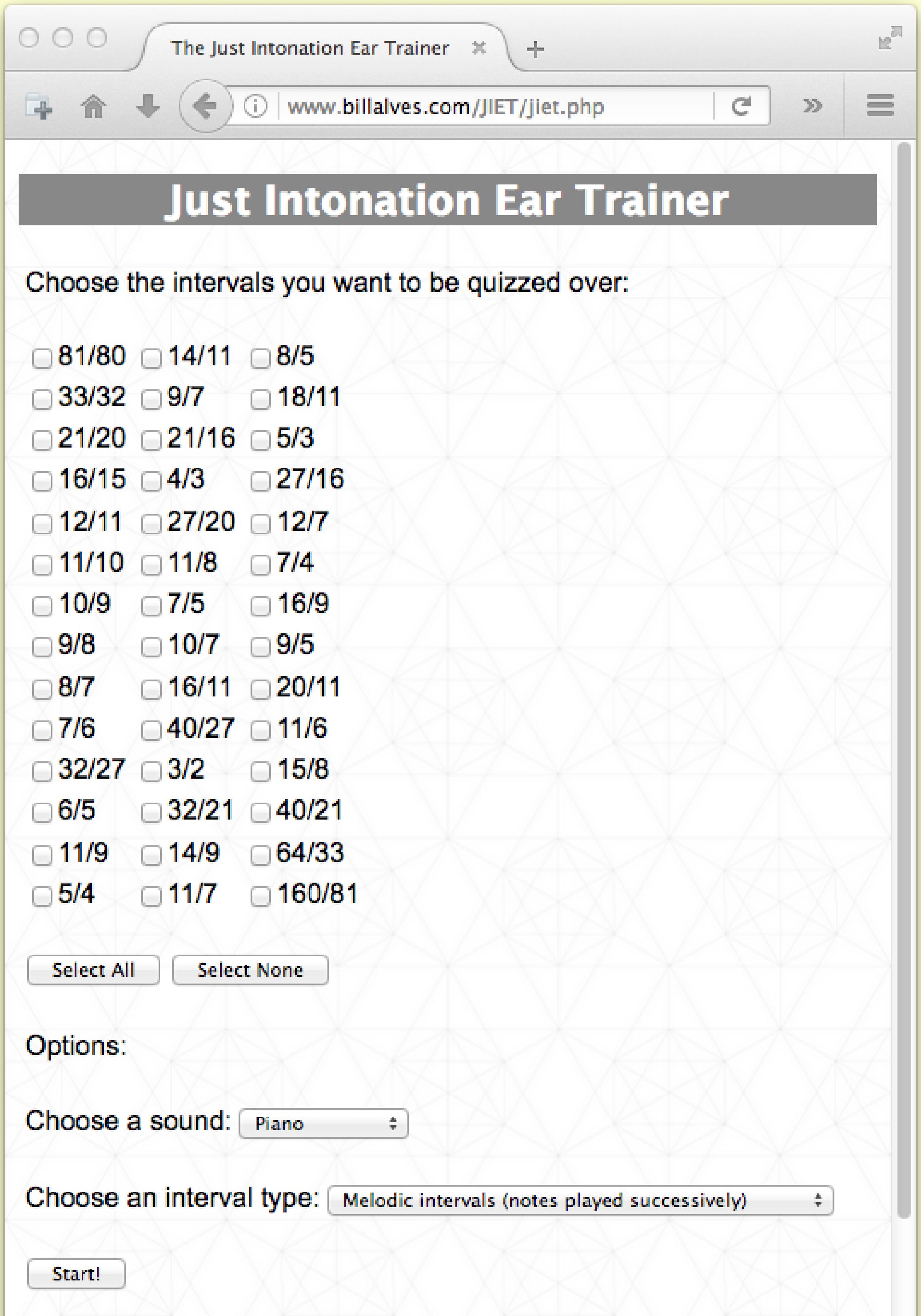
More good reads:
www.KyleGann.com/JIreasons.html
www.dbdoty.com/Words/Primer_2.1.html

soundamerican.org/sa20daviddotymusicalratios.html

There are other approaches besides JI and 12-ET. For example, the octave can be divided into 19 equal steps, known as 19-ET or 19-edo (equal division of an octave, pronounced "EE-doe"). Edos are covered in Part V. The general term for alternative tunings is microtonal or xenharmonic.

In-depth reference sites (again, not just JI):
xen.wiki (a communal wiki that anyone can edit)
tonalsoft.com/enc/encyclopedia.aspx (an encyclopedia of microtonal music theory)
www.maqamworld.com (a site about arabic music)

Online just intonation ear trainer:
www.BillAlves.com/JIET/jiet.php



Chapter 1.3 – The Harmonic Lattice

How does the notion of consonant intervals translate into actual melodies and chords? How might someone create a scale from the two most consonant intervals, the octave and the fifth? Imagine the very first time people sang harmony. Perhaps someone sang a droning low note and someone else joined in a fifth above. Using do-re-mi notation:

Do---------------------So

Then someone else with a high voice might have joined in at the octave:

Do---------------------So---------------Do

Already we have a third interval, the fourth. Then suppose the first person briefly stopped singing and the highest voice shifted from the fourth to the slightly more consonant fifth:

----------------------So---------------------Re

Now they've added a 3rd note to their scale. They've also implied another interval, the major 2nd between the do and the re. Then the first person rejoined and sang the obvious note, an octave below the highest voice:

Re---------------So---------------------Re

This chord is slightly more dissonant than the last one. So perhaps the middle voice moved up to La:

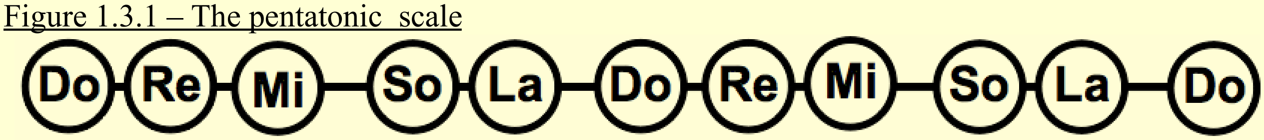
Re---------------------La---------------Re

Now they have a 4-note scale. The highest singer might then go up to a fifth above the middle voice:

Re---------------------La---------------------Mi

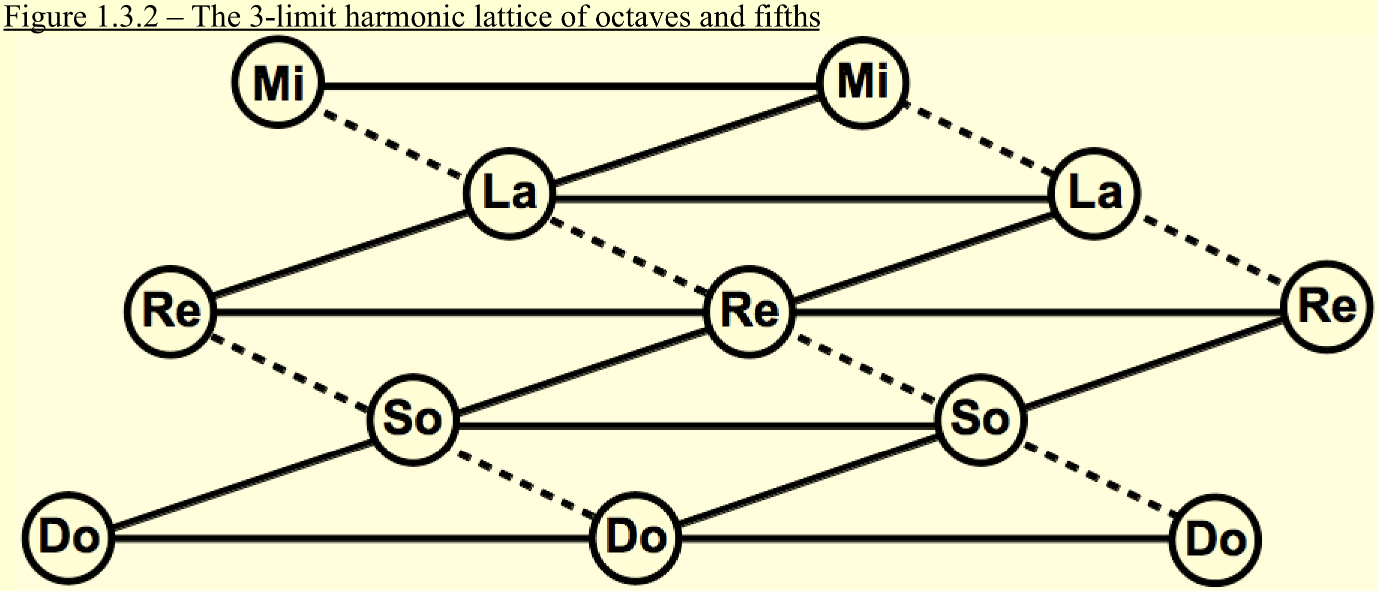
This creates a major 9th between the outer voices, which is more dissonant than the octave, which can then resolve back to Re-La-Re. And harmony is born! Of course, this is pure speculation, no one really knows how harmony began. But this example shows how only two intervals, the period (octave) and the generator (fifth), can create a pentatonic scale:

Figure 1.3.1 – The pentatonic scale



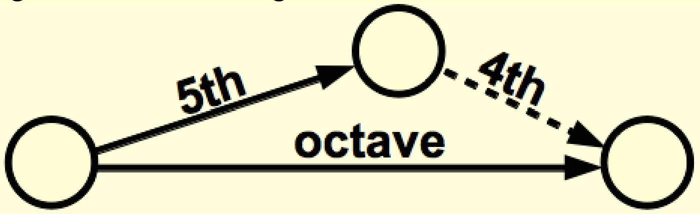
This example also shows how certain notes imply other notes. Every note implies a note an octave away and another one a fifth away. If we draw the octave horizontally and the fifth raised slightly from the horizontal, we can see which other notes each note implies:

Figure 1.3.2 – The 3-limit harmonic lattice of octaves and fifths



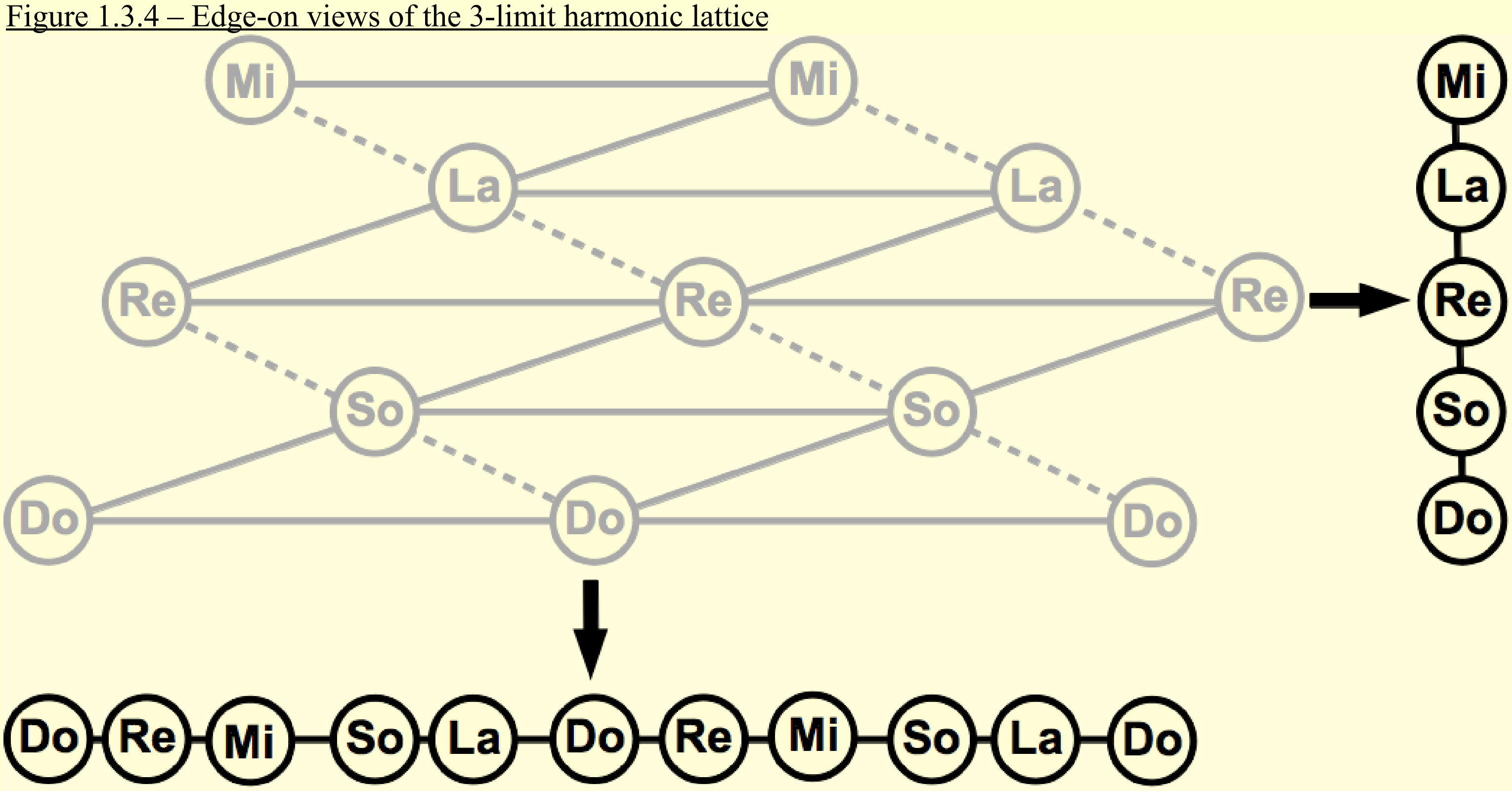
This is an example of a **harmonic lattice**, also known as a chord lattice, or chordal space, or Euler lattice, or ratio space. Lattices are a powerful tool for understanding the relationships between notes. This lattice is made up of nodes (Do, So, etc.) that are connected by two types of **rungs**. The basic principle of lattices is that every rung of a certain type, no matter where it is in the lattice, is always the same exact interval. In other words, every step in a certain direction always covers a certain interval. In this lattice, there are horizontal rungs representing octaves and diagonal rungs representing fifths. Every time you go right horizontally, you go up an octave. Moving diagonally up to the right takes you up a fifth, and moving down to the left takes you down a fifth. Moving up an octave and down a fifth takes you up a fourth. Fourths are represented by dotted lines. The fourth is a secondary rung that is derived from the other two rungs. These three rung types make triangles. The triangle is a little off-kilter because horizontal distance equals pitch distance, and a fifth is larger than a fourth:

Figure 1.3.3 – The rungs of the 3-limit harmonic lattice



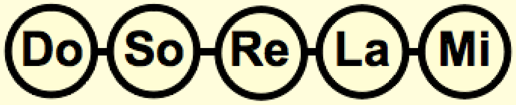
The lattice extends in every direction infinitely. We could add two more rows to get a 7-note scale, or lengthen the rows to get more octaves. The notes are arranged horizontally by pitch and vertically in a chain of fifths. Each row of the lattice is the same note in different octaves. For example, the middle row has all three "Re" notes in it, and no other notes. In the next picture, the righthand column shows which notes are in each row, as if you were looking at the lattice edge-on side-ways. Reading from the bottom up, it makes a chain of fifths. The bottom row shows the lattice edge-on from below.

Figure 1.3.4 – Edge-on views of the 3-limit harmonic lattice



When we analyze scales and chords, we usually take octaves for granted. We don't think of specific notes but rather **pitch classes**. A piano has 88 notes but only 12 pitch classes. The "C" pitch class contains every "C" note, one from each octave. The "A" pitch class contains A-440, A-220, A-110, etc. The octaves in lattices are usually hidden so that an entire pitch class is condensed into one node of the lattice, as in the sideways edge-on view of figure 1.3.4. The octave rung can be thought of as having a length of zero, and each node can be thought of as a miniature stack of octaves, many notes piled one atop the other. Here's that edge-on view rotated to be horizontal:

Figure 1.3.5 – The pentatonic chain of fifths



This chain of fifths is also a harmonic lattice, as is the familiar circle of fifths. But the bottom row of figure 1.3.4 isn't one, because the horizontal steps are different sizes. Some are major 2nds and some are minor 3rds.

Let's extend the chain of fifths to 7 notes, and use standard note names instead of do-re-mi:

Figure 1.3.6 – The 3-limit harmonic lattice with invisible octaves, as notes

F	C	G	D	A	E	B

The letter D stands for all the D notes in every octave. The fifths are often actually fourths or even twelfths. This represents Duplius and Tertia's medieval 3-limit tuning, the **pythagorean** tuning.

If the D in the center is the tonic, we can assign ratios to these notes. The tonic always has the simplest ratio, 1/1.

Figure 1.3.7 – The 3-limit harmonic lattice with invisible octaves, as ratios

32/27	16/9	4/3	1/1	3/2	9/8	27/16

The numbers get bigger, and the ratios more dissonant, as you move out from the tonic. The ratios are octave-reduced. The interval from pitch class D to pitch class A is actually many intervals – D2 to A2 (ascending 5th), D2 to A3 (asc. 12th), D3 to A2 (desc. 4th), etc. These are all different voicings of the same interval. Thus 3/2 stands for 3/1, 3/4, etc. From D to E is two steps of 3/2, making 9/4. We divide 9/4 by 2 to octave-reduce it to a closer interval, 9/8.

In the Renaissance, the lady Quintia transformed music from 3-limit to 5-limit, and the lattice changed from one-dimensional to two-dimensional (not counting the hidden octave dimension, of course).

Figure 1.3.8 – The square 5-limit harmonic lattice

A	E	B	F♯	C♯	G♯	D♯
F	C	G	D	A	E	B
D♭	A♭	E♭	B♭	F	C	G

There are theoretically more rows above and below these. Moving to the right adds a fifth, and moving straight up adds a major 3rd. Mathematically speaking, we're multiplying ratios by 3 or 5 and octave-reducing by dividing by some power of 2. Moving left or down divides by 3 or 5. Some notes appear twice, more on this later. As ratios:

Figure 1.3.9 – The square 5-limit harmonic lattice as ratios

40/27	10/9	5/3	5/4	15/8	45/32	135/128
32/27	16/9	4/3	1/1	3/2	9/8	27/16
256/135	64/45	16/15	8/5	6/5	9/5	27/20

The position of a ratio in the lattice corresponds exactly to its prime number "building blocks" 3 and 5. The horizontal dimension is the "multiply or divide by 3" dimension, or the "3-dimension". The vertical dimension is the 5-dimension. Note that the prime number does not correspond to the interval it creates: multiplying by 3 creates a fifth and multiplying by 5 creates a third. There is a third dimension, the 2-dimension, that is hidden. Again, the numbers get bigger as you move out from the tonic.

At this point, you may want to actually hear the lattice. Choose any letter in Figure 1.3.6 and play that note (in 12-ET) in any octave. Then choose an adjacent letter one step away, and play that note. Continue to "walk" around the lattice in steps, until the lattice is not just letters on the page but sounds in your head. Do the same with Figure 1.3.8. Use only horizontal and vertical steps, not diagonal ones.

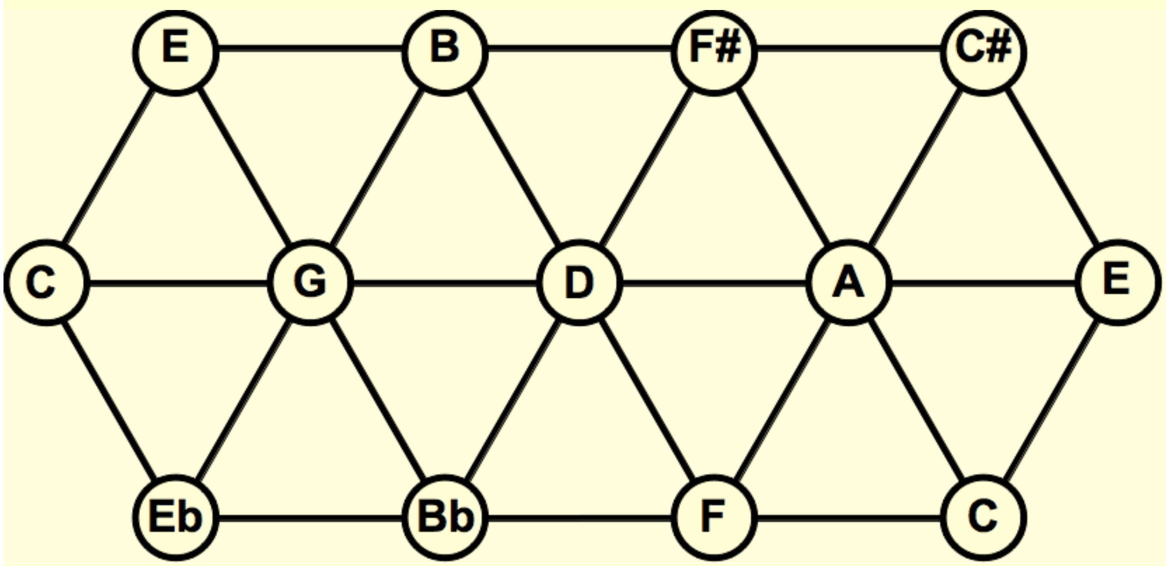
If we shift things a bit and make the lattice triangular, something very useful happens: the closer two notes are in the lattice, the better they sound together. For example, now the maj 6th D – B is a shorter jump than the min 2nd D – E♭:

Figure 1.3.10 – The triangular 5-limit harmonic lattice

	A		E		B		F♯		C♯		G♯	
F		C		G		D		A		E		B
	A♭		E♭		B♭		F		C		G	

Figure 1.3.11 is the same lattice, with shorter rows, and with all the notes connected by rungs, making the triangles clearer. Try walking around this lattice step by step. The rungs create 6 different directions to step in.

Figure 1.3.11 – The 5-limit harmonic lattice



Walk around the lattice again, but this time keep playing the last note along with the next note, so that you're playing two notes at once. Unlike playing random notes, every pair of notes sounds quite pleasant.

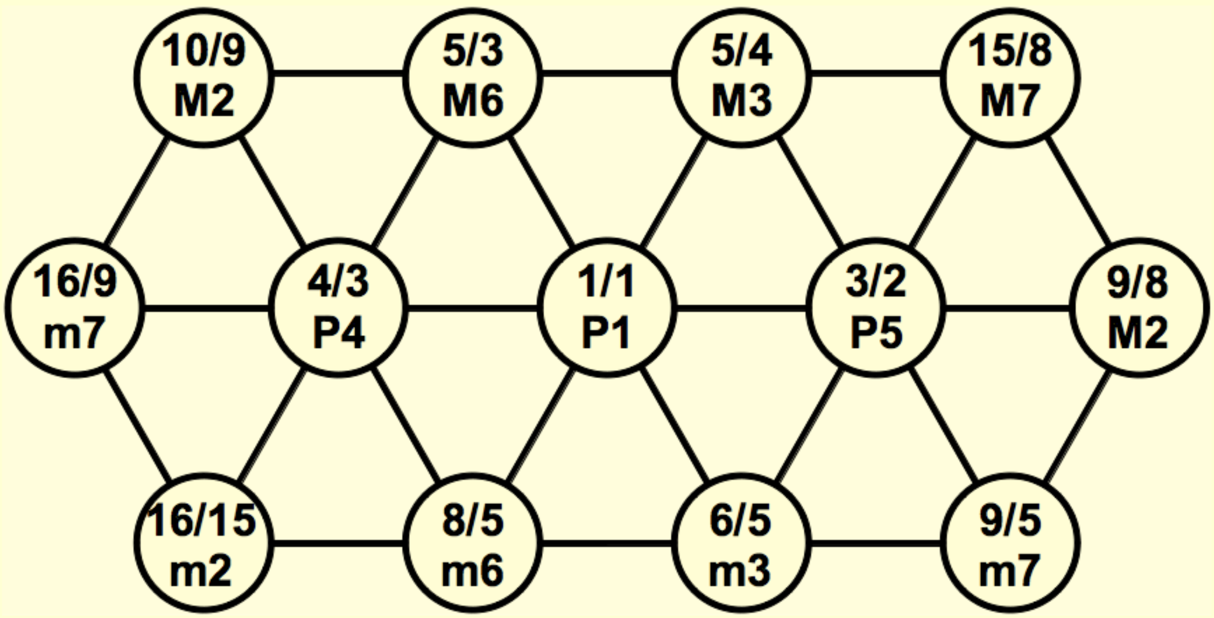
Next, hold the last two notes along with the next, so that you're playing three notes at once. Again, most combinations of notes sound pleasant. If it doesn't, try a different voicing. Every combination of three notes has a certain shape in the lattice. There are eleven possible three-note combinations. Which shapes sound the most consonant to you? Which shapes sound the most interesting?

As noted earlier, every step in the lattice in a certain direction always covers a certain ratio or interval. For example, every rightward step covers 3/2, which is a perfect 5th, and every right-and-up step covers 5/4, which is a major 3rd. Every 5-limit ratio can be expressed as so many rightward steps and so many right-and-up steps. The number of steps is specified by the last two numbers of the ratio's monzo. (The first number is not used because it represents octaves, which are invisible.) For example, 10/9 = (2·5)/(3·3) = (1, -2, 1), so 10/9 is two leftward steps and one right-and-up step. These steps create a vector, and any two notes separated by this vector are also separated by 10/9. From F to G is 10/9, as is A to B. Another example: 6/5 = (2·3)/5 = (1, 1, -1) = one rightward step and one left-and-down step. Every 5-limit ratio is also a vector in the 5-limit lattice.

An important property of all lattices results from this: if three notes line up, the center note is exactly halfway between the other two not only spatially but melodically as well (allowing for octave transpositions if octaves are invisible). For example, on the center row of Figure 1.3.11, G, D and A line up, and D is halfway between A and the G a 7th above. It's also halfway between G and the A a 9th above. In the last walking exercise, if the lattice shape is a straight line, it's possible to voice the chord as three equally-spaced notes. In fact, there will be several such voicings.

This next lattice replaces the notes in Figure 1.3.11 with ratios and intervals. Major intervals are above and on the right, and minor ones are below and on the left. Perfect intervals are in the center.

Figure 1.3.12 – The 5-limit harmonic lattice in relative notation



Intervals add together by moving around the lattice. The vectors add up as one would expect not only spatially but also mathematically and musically. A right-and-up step and a right-and-down step add up to make a rightward step, 5/4 times 6/5 equals 3/2, and a major third plus a minor third add up to make a fifth. The musical interval that corresponds to a 5-limit ratio can always be found by simply adding up each step's musical intervals. Thus 15/8 is a M7 because it is the sum of P5 and M3. If the sum is an octave or larger, octave-reduce it. For example, 9/8 is the sum of P5 and P5, which makes M9, which reduces to M2.

Every triangle is a triad. The triad's root is the leftmost note of the triangle. Upward-pointing triangles (D – F♯ – A) are major (P1 – M3 – P5) and downward ones (D – F – A) are minor (P1 – m3 – P5). Chord progressions can be visualized as jumping from one triangle to another. Try it in Figure 1.3.11 with Pachelbel's Canon: D – A – Bm – F♯m – G – D – G – A.

1/1 can be thought of as the tonic. Chords can be constructed on other roots besides the tonic. The V chord's root is a P5 from the tonic, which is 3/2. The 3rd of the chord is a 5/4 from the root = 3/2 · 5/4 = 15/8 = M7. The 5th of the chord is 3/2 from the root = 3/2 · 3/2 = 9/4, which reduces to 9/8 = M2.

Figure 1.3.12 has two major 2nds, 10/9 and 9/8. They correspond to the two E notes in Figure 1.3.11. They are <u>two different notes</u>, close but not identical. In 12-ET, they are the same. But in JI, they are about a quarter of a semitone apart. This tiny difference is called a **comma**. The ratios 10/9 and 9/8 are numerically very close – 1.111 vs. 1.125. Because 1.125 is the larger number, the righthand E is sharper. The two pitches are so close that they are both called the same name, E, even though they are different ratios. They <u>must</u> have the same name, because 10/9 and 9/8 are both major 2nds.

Figure 1.3.12 has two minor 7ths, 16/9 and 9/5, which correspond to the two C notes in Figure 1.3.11. These two C's are also two different notes a comma apart. This comma has a ratio, the difference between 10/9 and 9/8. This ratio is 9/8 ÷ 10/9 = 9/8 · 9/10 = 81/80. The comma has a vector of three rightwards steps and one right-and-down step, which equals three 5ths plus a minor 3rd. Adding these up and octave reducing, we find that 81/80 is a P1 just like 1/1 is.

You can hear this comma by comparing W and X in Figure 1.2.4, or T and C, or O and N, or] and M. This comma is why it's mathematically impossible to tune a piano so that every note is acoustically in tune with every other note.

Some intervals will inevitably be mistuned by a comma. More on this in chapter 2.2.

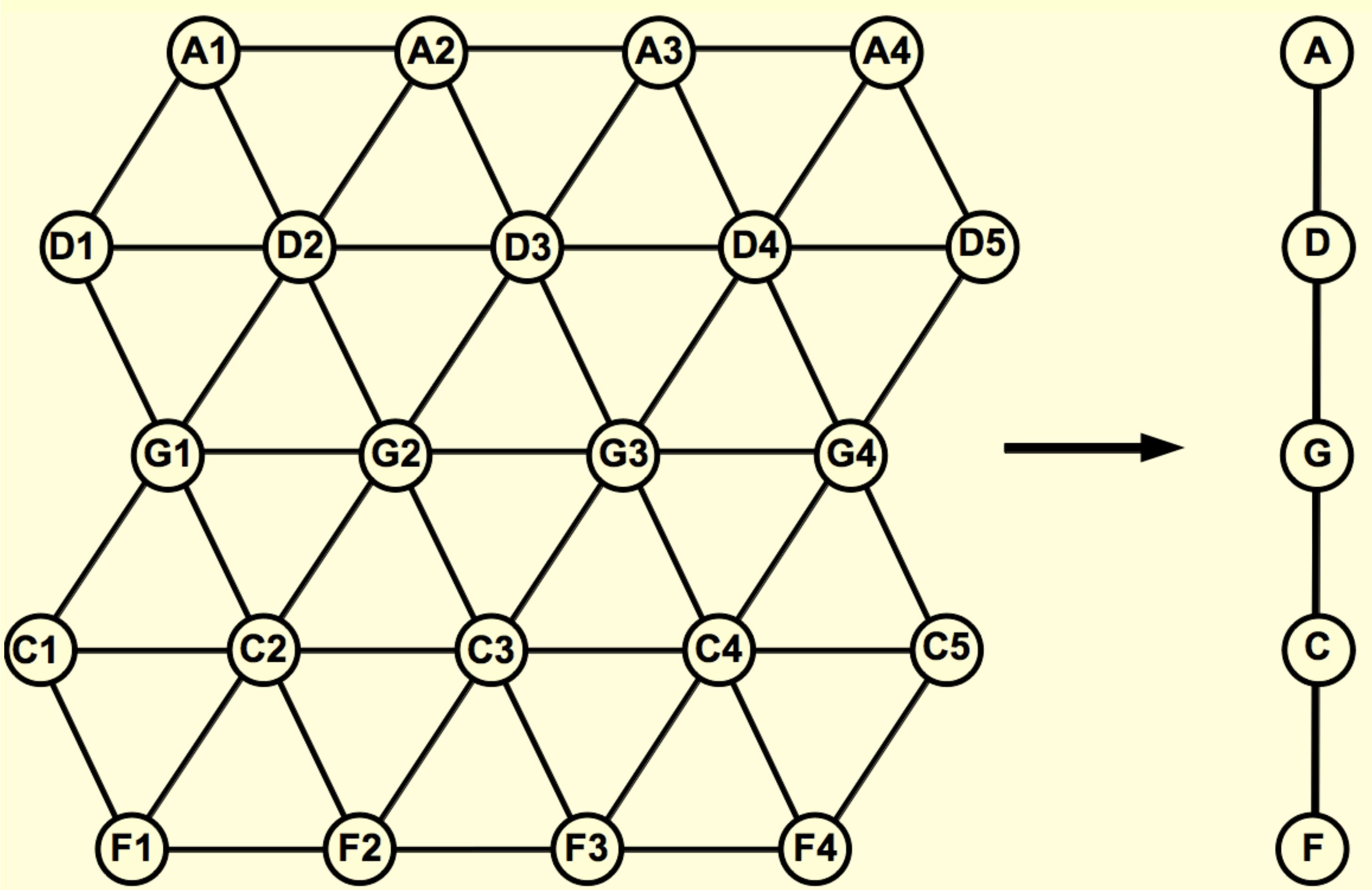
Mathematically, the parable in chapter 1.1 isn't very accurate. The king and his wives represent the prime numbers, but actually it's specific ratios like 10/9 and 9/8 that clash and won't fit into the same dress. These ratios can be thought of as their children. For example, the ratios in the center row all have prime number building blocks 2 and 3, and are the daughters of Duplius and Tertia. Likewise 5/4 is the daughter of Duplius and Quintia. But 5/3 is the daughter of Tertia and Quintia, and 15/8 is the daughter of all three! Prime numbers "reproduce" very differently than humans do!

Just as the white dress was not a problem when Duplius had only one wife, commas are usually not a problem when the lattice has only two dimensions. The third dimension creates many more possible ratios, and many more commas.

Try visualizing this chord progression in Figure 1.3.11: D – G – Em – A. Notice how the last two chords use the two different E notes. If we are constrained to only one E, as on a piano, one of the chords will have an out of tune fifth. It'll be flat by a comma, and it'll sound awful! This is the major disadvantage of just intonation. Other tuning systems, including our current one, 12-ET, alter the notes slightly so that the comma disappears and the two E notes become one, in effect making the harmonic lattice wrap around on itself. But they do so at the cost of mistuning all the chords. The most popular and enduring such tuning was meantone temperament, invented by Gioseffo Zarlino among others in the mid 16th century. Meantone retunes the fifth (Tertia's harness) and/or the major third (Quintia's corset) to force both the 10/9 and the 9/8 to fit onto the same organ key. The exact methods used are covered in chapter 4.3.

All the lattices from figure 1.3.5 up to now have had hidden octaves. Let's see what a lattice with visible octaves would look like. We'll start with a 3-limit lattice of octaves and fifths that runs F – C – G – D – A, somewhat like figure 1.3.4, complete with an edge-on view from the side:

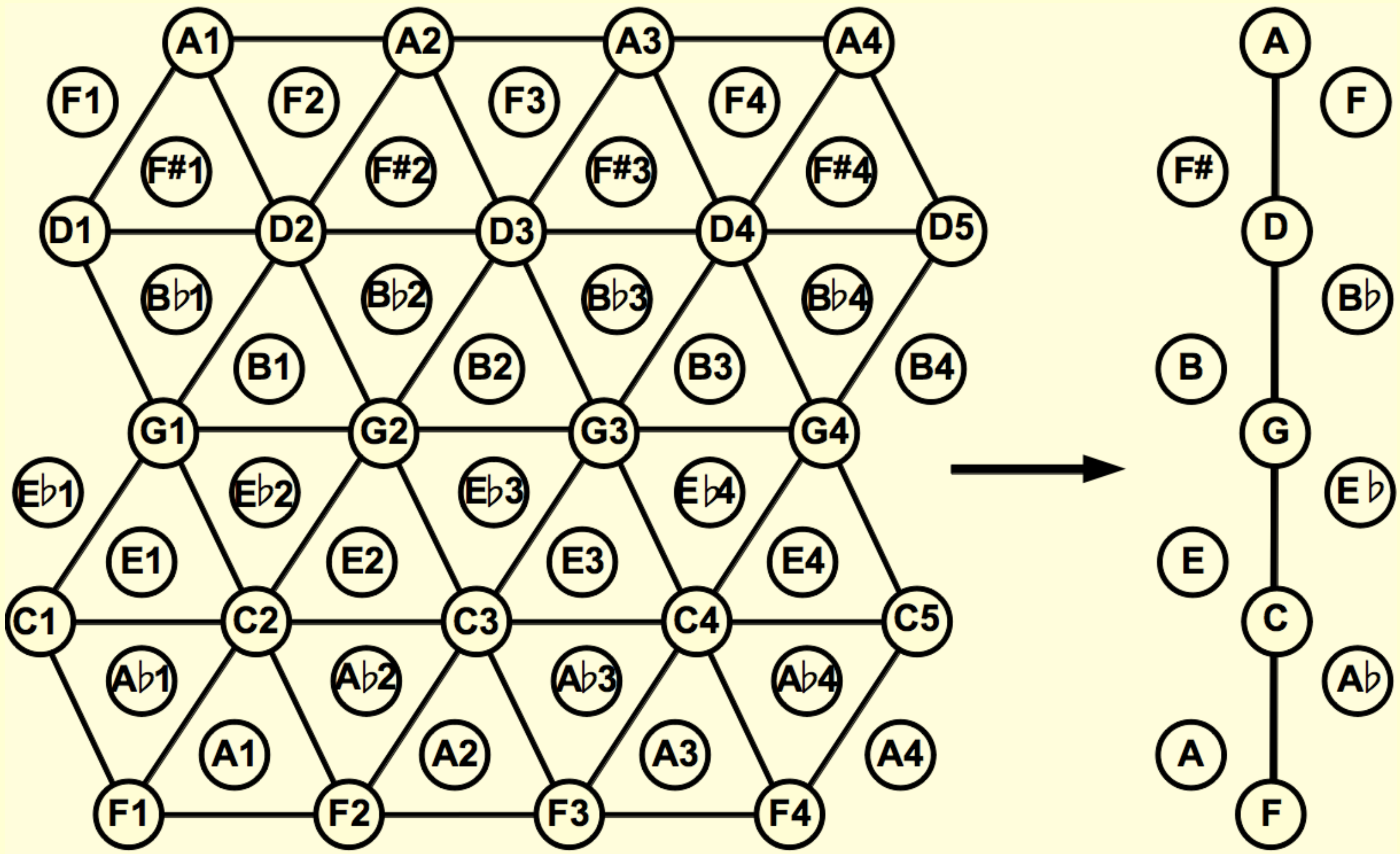
Figure 1.3.13 – The 3-limit harmonic lattice with visible octaves



The numbers indicate what octave each note is in. The triangles are off-kilter so that horizontal distance will equal pitch distance.

Next we add in the major and minor notes. The triangles become tetrahedrons:

Figure 1.3.14 – The 5-limit harmonic lattice with visible octaves

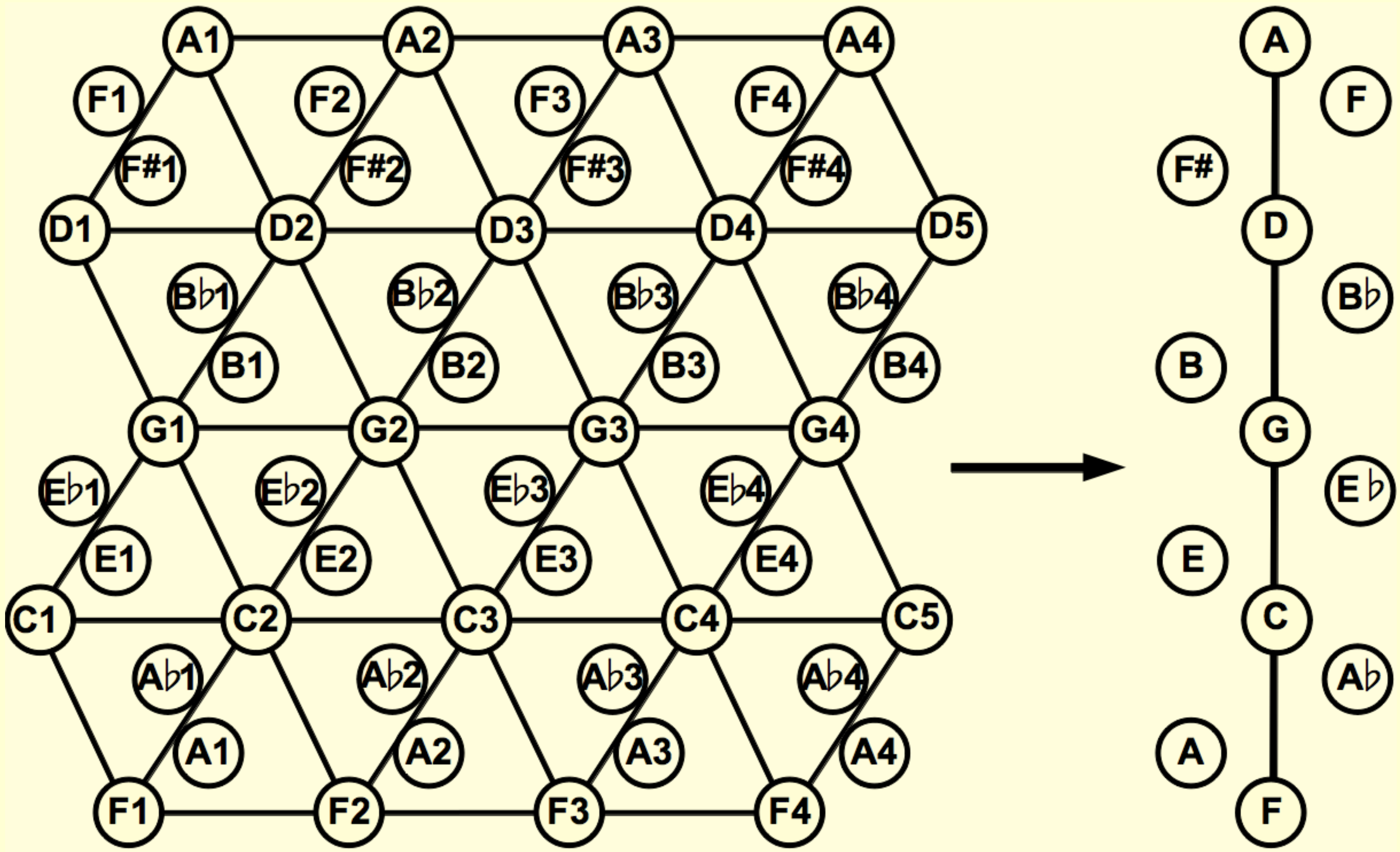


The upward-pointing tetrahedrons are major chords and the downward-pointing ones are minor. The notes inside the upward-pointing triangles are above the page and the ones in the downward triangles are below. The F notes and A notes on the top are different ones than the F and A notes on the bottom.

This lattice shows all three dimensions of the 5-limit lattice. The horizontal 2-dimension is for octaves, the diagonal 3-dimension is for fifths, and the tips of the tetrahedrons are the 5-dimension for major 3rds.

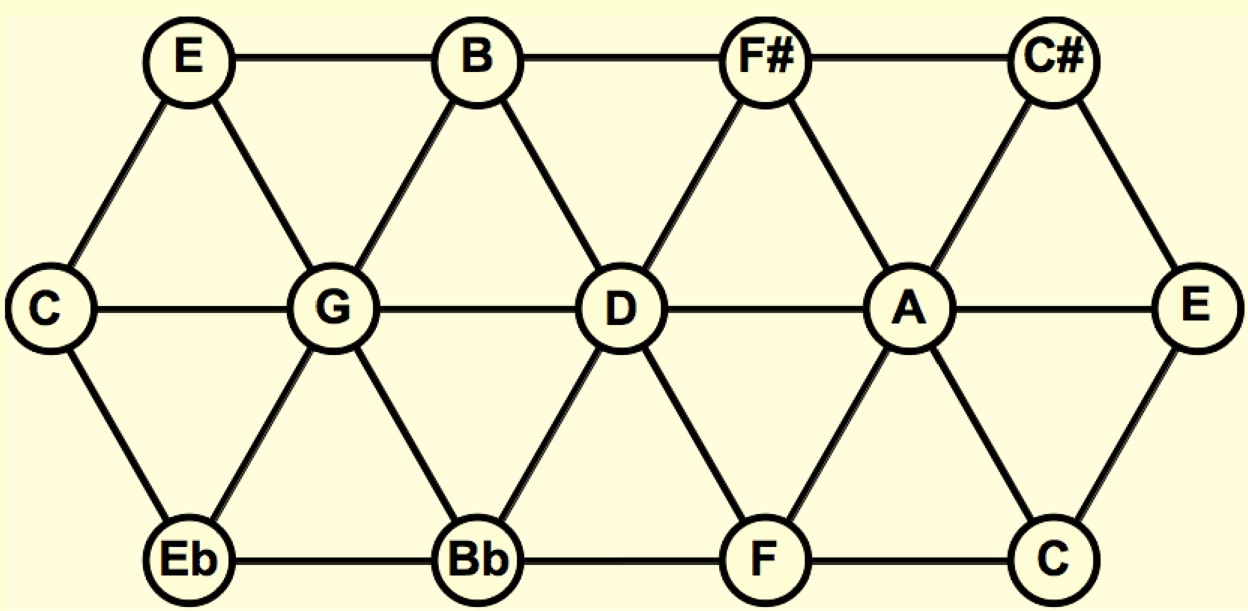
Unfortunately we've lost the correlation between pitch and horizontal distance. We can get it back if we make the tetrahedrons off-kilter as well:

Figure 1.3.15 – The 5-limit harmonic lattice with visible octaves



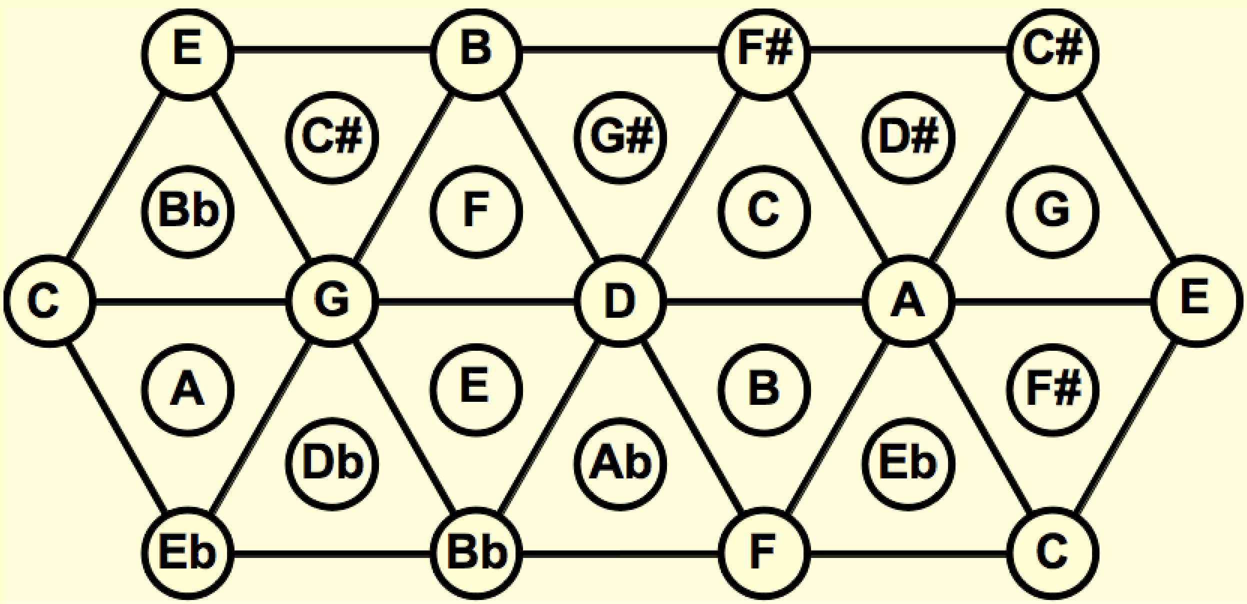
Looking at this lattice, it's easy to see why octaves are normally hidden!

In modern times, the lady Septima transformed music from 5-limit to 7-limit, and the lattice changed from two-dimensional to three-dimensional (not counting the hidden octave dimension). Let's add this third dimension to our lattice. Recall figure 1.3.11, a 5-limit two-dimensional lattice:



The 7-limit adds a third dimension, creating tetrahedrons.

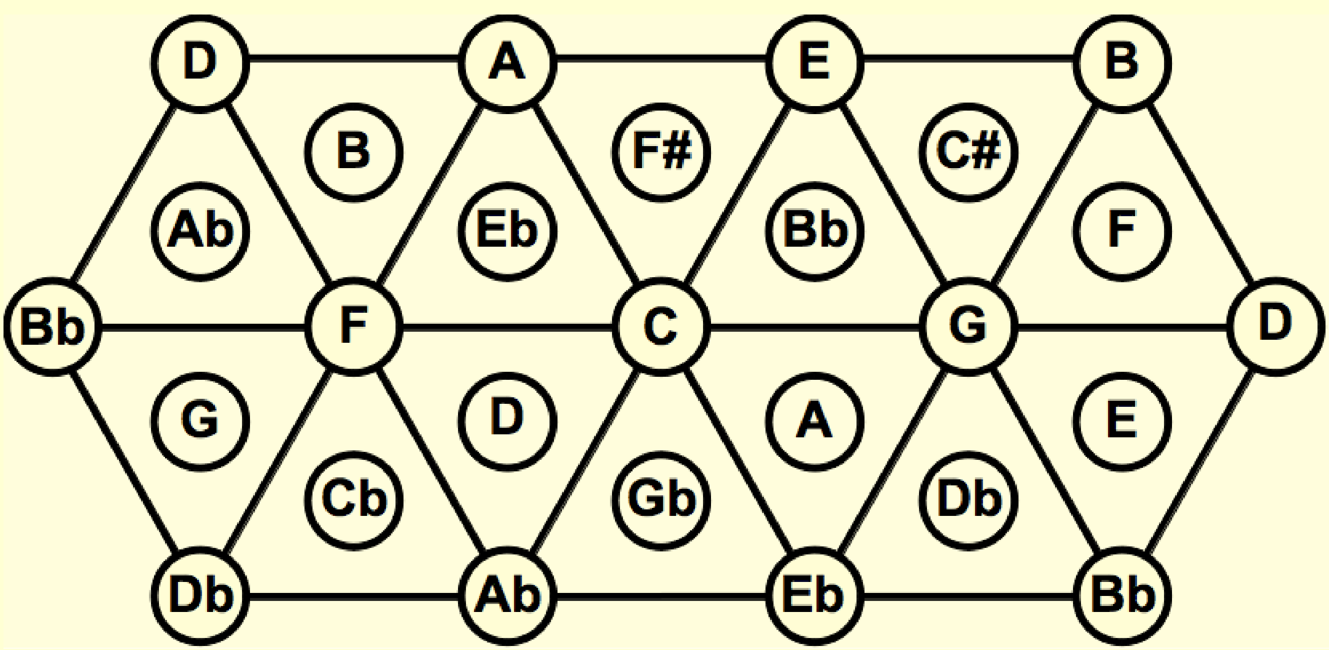
Figure 1.3.16 – The 7-limit harmonic lattice



The F, C and A♭ by the central D are "blue" notes. They lie above the old notes on their own plane, as does every note inside an upward-pointing triangle. The notes inside downward-pointing triangles, like the central E, G♯ and B, lie below. There are two kinds of tetrahedral tetrads, for example D – F♯ – A – C and D – F – A – B. The G on the left and the G on the right are two different notes, again differing by a comma about a quarter of a semitone wide.

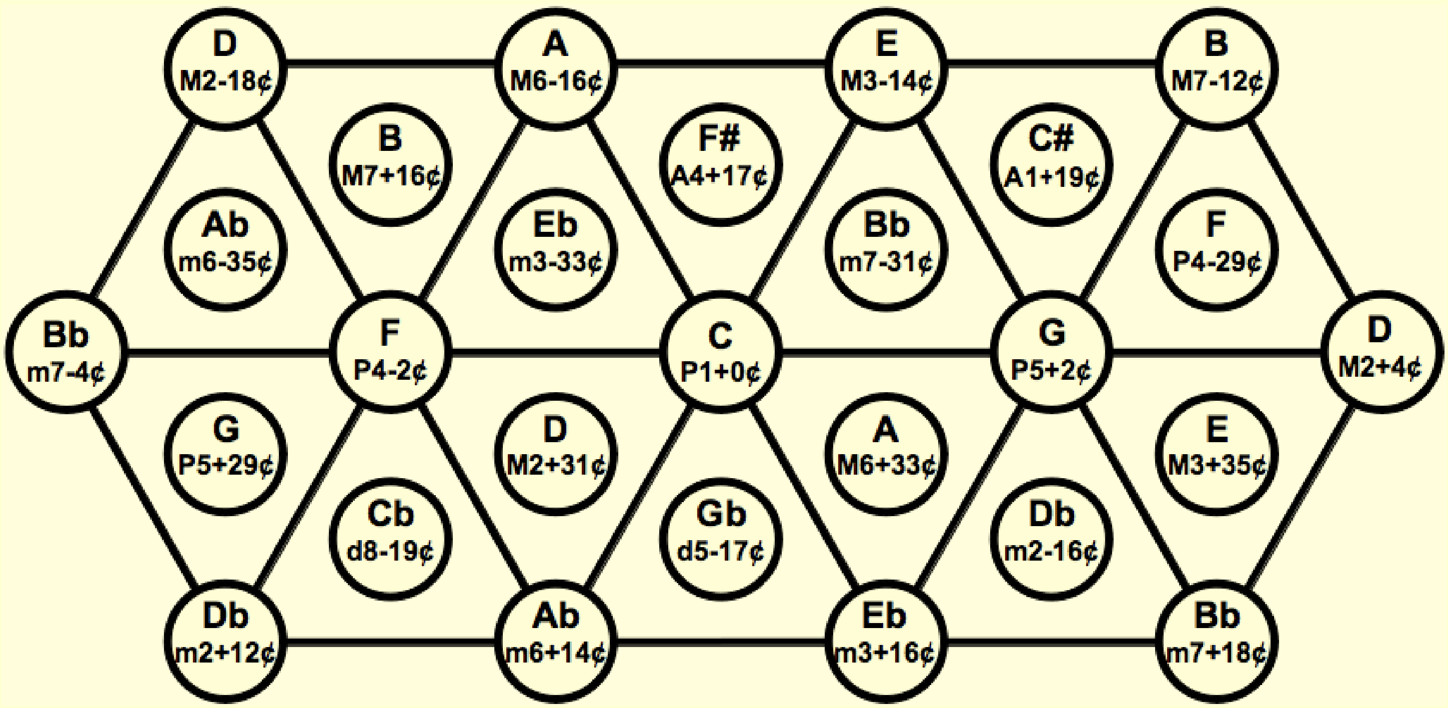
So far the center note has been D. But any note can be in the center, for example C:

Figure 1.3.17 – The 7-limit harmonic lattice in C



Most 5-limit notes are at most about a sixth of a semitone off from the usual 12-ET notes. But some of the new 7-limit notes are a full third of a semitone off. The next lattice shows the exact tuning of each interval expressed relative to 12-ET.

Figure 1.3.18 – The 7-limit harmonic lattice with cents relative to 12-ET

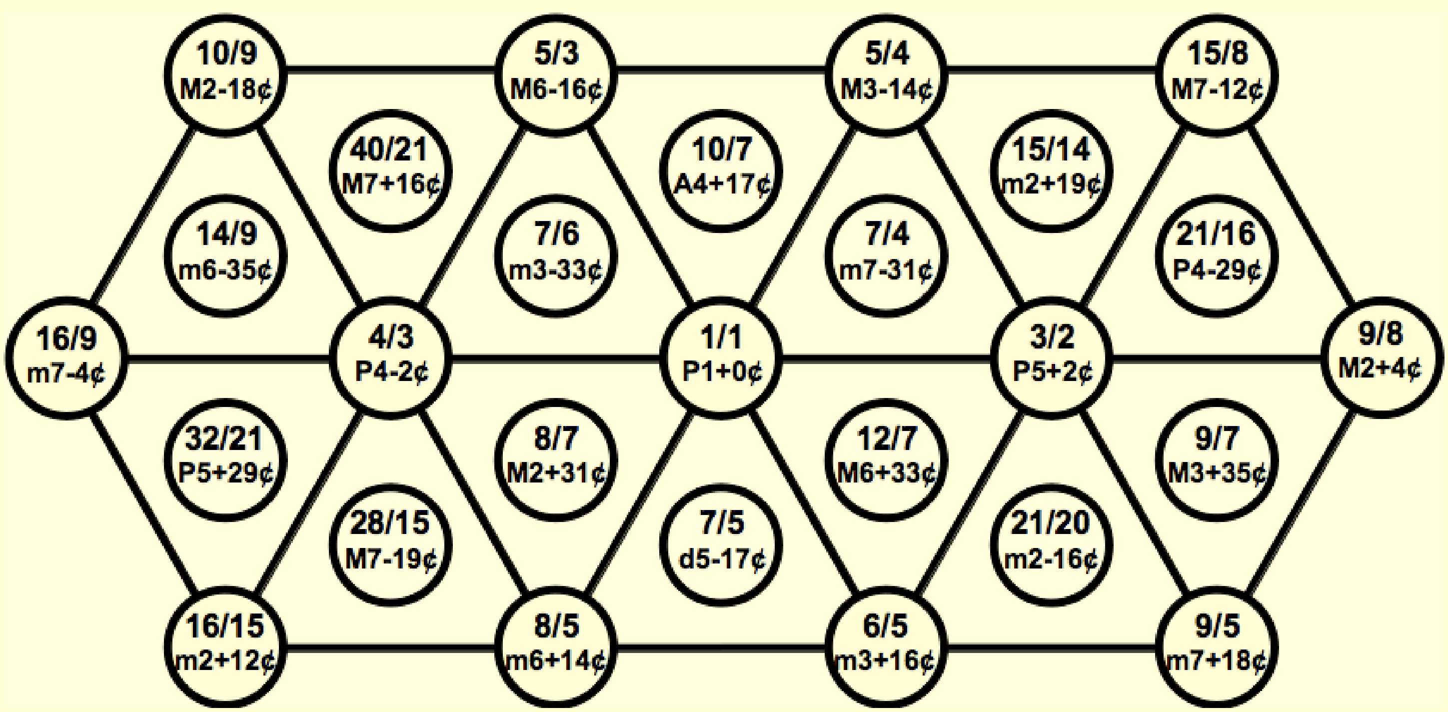


Note the different tuning of the three D notes. Because the JI 5th and the 12-ET 5th are only 2¢ apart, every note on a given row is off from 12-ET by about the same amount. The middle row is fairly close. The top row is a little flat; the A♭ – F row just above the middle row is even flatter. Notes like these are loosely called **subminor**, because they are flatter than minor but sharper than diminished. Their inverses in the G – E row are **supermajor**, sharper than major but flatter than augmented.

As the prime limit rises from 3 to 5 to 7, 12-ET does a progressively worse job of approximating JI intervals. Thus to explore higher prime limits, abandoning 12-ET becomes more necessary. I would argue that the great leap from triadic to tetradic, which started in the early 20th century, has yet to fully materialize (there's still lots of triadic music on the radio) because 12-ET tetrads are so out of tune. Lady Septima has been forced to dance very awkwardly indeed!

The same lattice, this time with ratios instead of notes:

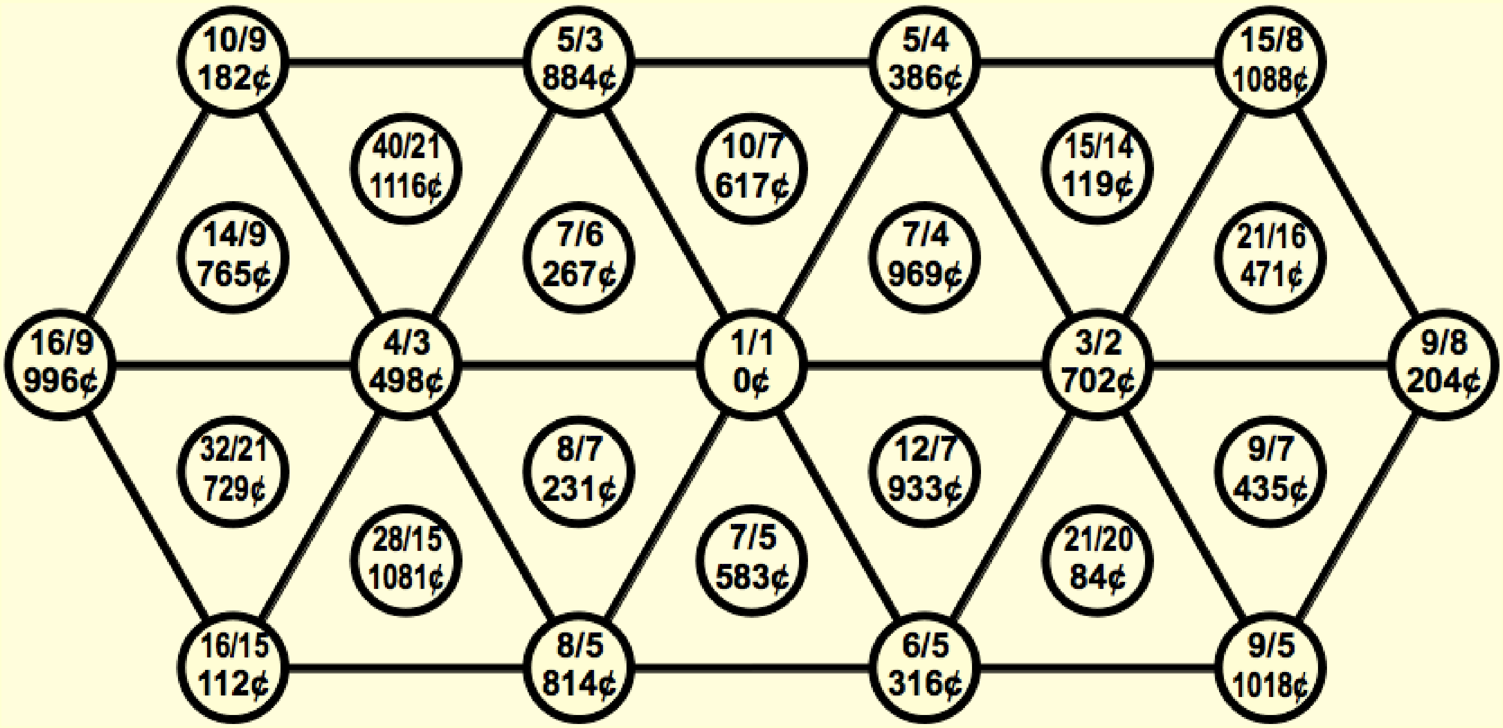
Figure 1.3.19 – The 7-limit harmonic lattice with ratios, relative to 12-ET



There are lots of new ratios here. The simplest and most consonant ones are clustered around the center: 7/6, 7/5 and 7/4. The three different D notes in the last lattice now have ratios: 10/9, 9/8 and 8/7. From 10/9 to 9/8 is 81/80, a 22¢ comma, and from 9/8 to 8/7 is 64/63, a 27¢ comma.

Rather than describe an interval as offset by so many cents from the nearest 12-ET note, we can describe it as offset from the tonic, like so:

Figure 1.3.20 – The 7-limit harmonic lattice with ratios and cents



It's possible to add the octave dimension back in. But for 7-limit, that would make a four-dimensional lattice, which is very hard to visualize.

If your head is spinning from all these numbers, fear not! The next chapter introduces a notation system that replaces the numbers with colors.

In any genre of Western music, some chords are considered stable and others are considered unstable. A stable chord is one consonant enough to resolve to, or to end a piece with. Unstable chords are more dissonant. This dissonance helps drive the cadence from an unstable chord to a stable one, providing a sense of tension and release.

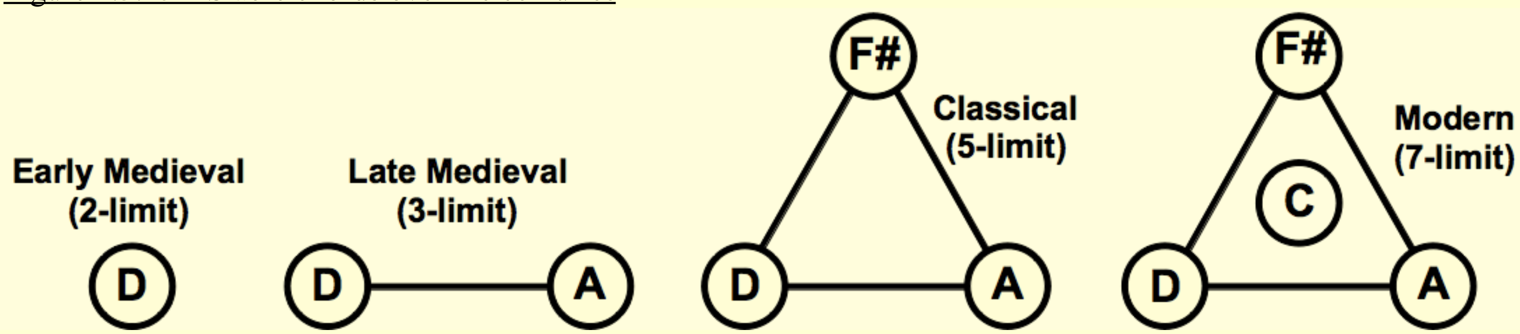
Precisely what is considered stable has changed over time. For example, in the Classical era, both the major triad and the minor triad were deemed stable chords. All other chords were unstable, like augmented and diminished triads, sus4 and sus2 chords, and the various tetrads. In the earlier Baroque era, the minor triad, while not considered entirely dissonant, wasn't thought quite stable enough to end on. A piece in a minor key would often use the Picardy third, in which the final chord of a song in A, for example, would be not A minor but A major. Modern-day examples of songs using the Picardy third are "And I Lover Her" (Beatles) and "Killing Me Softly With His Song" (Roberta Flack).

Even earlier, in late medieval times, all triads were considered unstable. The only stable chord was the open fifth dyad. It was usually voiced root-fifth-octave, a 3-part chord called a trine. Songs would routinely resolve to this chord and end on this chord. To modern Western ears, this third-less ending chord sounds oddly hollow and incomplete. It's too stable! Going back yet further, before the 9th century, songs were sung in unison (Gregorian chants) or in octaves (magadizing). The only consonant interval was the octave, and the only stable "chord" was the one-note monad.

Music since the classical era has continued this trend towards more complex stable chords. In some modern genres, like jazz and blues, the dom7 and min7 tetrads are considered stable. More complex tetrads and pentads supply the necessary dissonance to drive the music. Major and minor triads sound incomplete and are rarely used, just like the medieval open fifth dyad was rarely used in baroque or classical music, and the monad was rarely used in late medieval music. One era's stable consonance becomes the next era's hyperstable incompleteness. (There is one exception to this rule, the power chords played on electric guitars. The heavy distortion used adds higher harmonics that accentuate the out-of-tune-ness of 12-ET triads, necessitating dyads. It could be said that the harmonics themselves add the third to the chord.)

As King Duplius has taken more wives, the stable chord has grown from a single note to a linear dyad to a triangular triad to a tetrahedral tetrad, and Western music has progressed from monadic to dyadic to triadic to tetradic:

Figure 1.3.15 – Stable chords over the centuries



The earliest known example of 5-limit triadic music in the West is "Sumer Is Icumen In", a six-part round from the mid 1200's. The Renaissance began about a century later. It's impossible to prove, but I like to think that the 5-limit triadic revolution in music helped bring about the Renaissance's revolutions in art, science, religion, and philosophy. I also like to think that the coming 7-limit tetradic revolution may help bring about similar revolutions in our time!

The eleven three-note combinations discussed just below Figure 1.3.11 are listed here. The last two chords are difficult to name. The min-maj no5 chord would actually have a different shape in the lattice, since the major 7th would be 15/8, not 48/25.

Table 1.3.1 – Various 5-limit triads

chord type	**example**	**equivalent chord**		**shape**
major	D F♯ A			/\
minor	D F A			\/
sus4	D G A	sus2	G A D	—
aug	F A C♯			/
dim	F♯ A C	dom7 no1	(D) F♯ A C	\
maj7 no3	D A C♯			_/
maj7 no5	D F♯ C♯			/‾
min7 no3	D A C	maj6 no1	(F) A C D	‾\
min7 no5	D F C	maj6 no3	F C D	_
major add♭3 no5	D F F♯			〈
major add♭3 no1	(D) F F♯ A	(min-maj no5)	(F♯ A E♯)	〉

Part II – JI Color Notation

7-limit JI creates a vast sea of intervals. One obstacle to navigating this sea is being able to name these intervals. Color notation provides a concise, logical name for each interval, unique to that interval, that indicates its size and character. It allows us to discuss 7-limit music without using ratios or cents – to talk like a musician, not a mathematician. And yet it's mathematically rigorous and even includes a concise shorthand for equations.

Chapter 2.1 – Interval Names

Every prime greater than 3 gets two color names, one for when over (in the ratio's numerator) and one for when under.

Pythagorean intervals (3-limit) are strong and clear and go with everything, but don't have much character. Let's use **white** to describe them.

5-over intervals like 5/4 and 5/3 are major and warm and sunny and also somewhat bright. Let's use **yellow**.

5-under intervals like 6/5 and 9/5 are minor and more subdued than yellow. Let's use **green**.

7-under (7/6 and 7/4) are subminor, dark and bluesy. Let's use **blue**.

7-under (9/7 and 12/7) is supermajor and dissonant. It reminds me of something inflamed or swollen. Let's use **red**.

However, there's some disadvantages to using specific colors. Those with synesthesia associate certain intervals with certain colors. Furthermore, "blue" already has many musical connotations. To avoid confusion, new words are coined for these colors: white becomes **wa**, yellow becomes **yo**, green becomes **gu** ("goo"), etc. The vowel (-o or -u) indicates over vs. under. The -a in wa indicates all 3-limit ratios, 3-over (3/2, 9/8), 3-under (4/3, 16/9) and neither (1/1, 2/1). These new color names are more concise, and easier for non-English-speakers to learn, spell and pronounce.

The colors have one-letter abbreviations w, y, g, etc. The problem with using blue is that the flat sign is often written with a b. Therefore 7-over is represented by the Spanish and Portuguese word for blue, **azul**, or by the English word **azure**. Its new name is **zo**. (Additional mnemonic: Z looks like 7 with a line on the bottom.) Red becomes **ru**.

Here are all the 7-limit thirds with small-number ratios, in descending order, with a one-letter shorthand:

Table 2.1.1 – The 4-band rainbow of 3rds

9/7	7-under	435¢	supermajor	sharpest	hot	red	ru 3rd	r3
5/4	5-over	386¢	major	sharper	warm	yellow	yo 3rd	y3
6/5	5-under	316¢	minor	flatter	cool	green	gu 3rd	g3
7/6	7-over	267¢	subminor	flattest	cold	blue/azure	zo 3rd	z3

The 3rds make a nice red-yellow-green-blue rainbow. The four bands of the rainbow are about 50-70¢ wide. This rainbow shows up not only for 3rds, but for most degrees of the scale, reminding us of the relative sizes of the intervals, and justifying the somewhat arbitrary color choices.

Just as wa means 3-all or 3-limit, **ya** means 5-all, i.e. 5-limit. **Za** refers to the 2.3.7 subgroup, and **yaza** means 7-limit.

Every yaza interval can be described by these five colors. Here's how it works: any two intervals an octave or a 5th apart are the same color. Thus because 3/2 is wa, so is 9/8. Because 5/3 is yo, so is 10/9. Yo and gu, when added together, cancel out to make wa (y3 + g3 = w5), as do zo and ru (z3 + r2 = w4). Zo and gu combine to make zogu. (Not guzo, because the higher prime always comes first.) For example, z3 +g3 = zg5 = 7/5, the zogu 5th. Likewise ru & yo make ruyo. Intervals add up logically, so 10/7 = 617¢ is a 4th, not a 5th, because 10/7 = 9/7 · 10/9 = r3 + y2 = ry4. Zogu and ruyo also cancel out: zg5 + ry4 = w8.

In the harmonic lattice, each row is a different color. Those without synesthesia might use actual colors:

Figure 2.1.1 – The yaza harmonic lattice with colors, qualities, degrees and cents

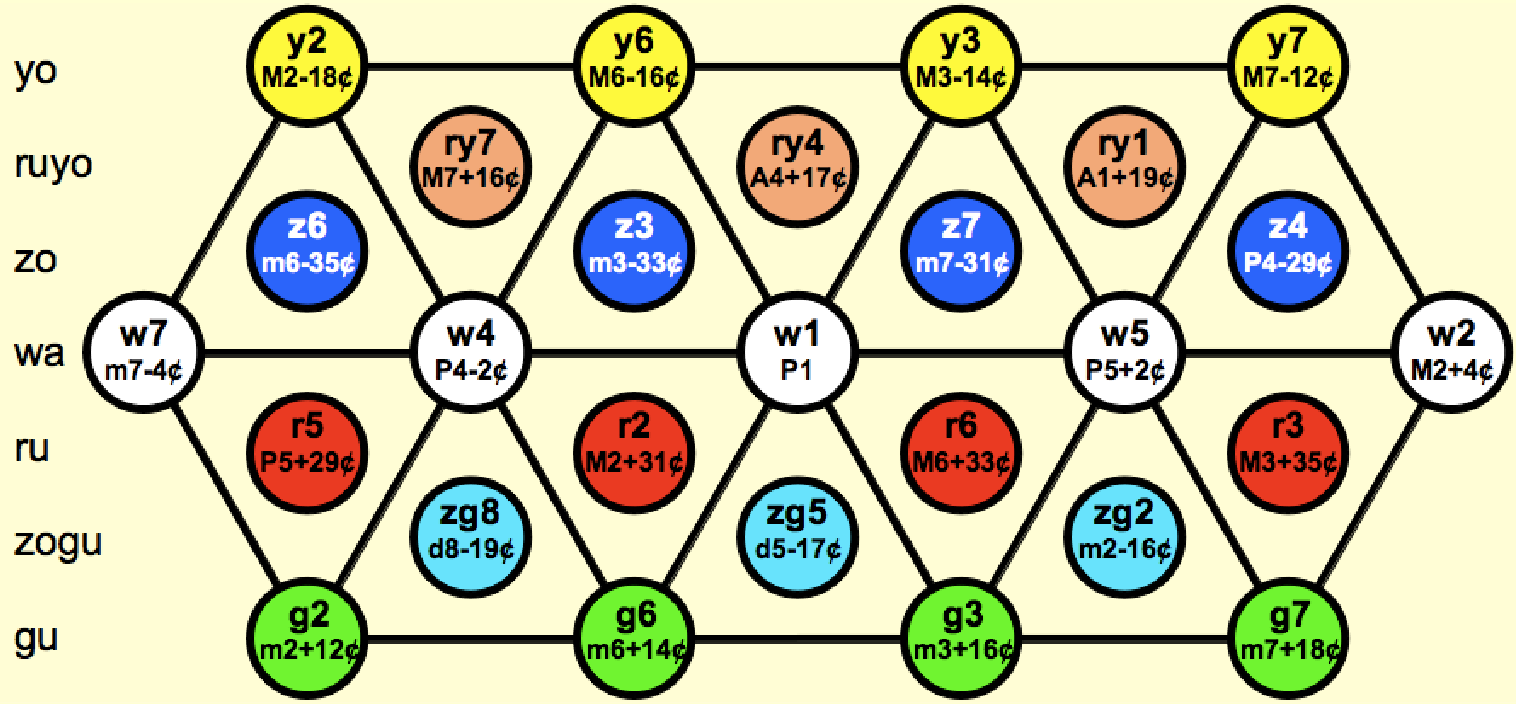


Table 2.1.2 – Approximate deviation from 12-ET for each color

ru	29-35¢ sharp
ruyo	16-19¢ sharp
gu	12-18¢ sharp
wa	0-4¢ sharp or flat
yo	12-18¢ flat
zogu	16-19¢ flat
zo	29-35¢ flat

The next table is (more or less) the 9-odd-limit ratios, along with their counterparts up and down a 5th. The horizontal lines group the intervals by scale degree into six rainbows, each with five bands. The rainbow of 4ths overlaps the rainbow of 5ths. The six rainbows are different, but they have a similar structure:

rainbow of 2nds	zogu – gu – yo – wa – ru
rainbow of 3rds	zo – wa – gu – yo – ru
rainbow of 4ths	zo – wa – gu – yo – ruyo
rainbow of 5ths	zogu – gu – yo – wa – ru
rainbow of 6ths	zo – gu – yo – wa – ru
rainbow of 7ths	zo – wa – gu – yo – ruyo

Table 2.1.3 – Color notation of yaza JI intervals

ratio	cents	keyspan in semitones	deviation from 12-ET	quality & degree	color & degree	shorthand notation	
1/1	0¢	0	0¢	perf unison	wa unison	w1	(or more simply, unison)
21/20	84¢	1	-16¢	min 2nd	zogu 2nd	zg2	
16/15	112¢	1	+12¢	min 2nd	gu 2nd	g2	
10/9	182¢	2	-18¢	maj 2nd	yo 2nd	y2	
9/8	204¢	2	+4¢	maj 2nd	wa 2nd	w2	
8/7	231¢	2	+31¢	maj 2nd	ru 2nd	r2	
7/6	267¢	3	-33¢	min 3rd	zo 3rd	z3	
32/27	294¢	3	-6¢	min 3rd	wa 3rd	w3	
6/5	316¢	3	+16¢	min 3rd	gu 3rd	g3	
5/4	386¢	4	-14¢	maj 3rd	yo 3rd	y3	
9/7	435¢	4	+35¢	maj 3rd	ru 3rd	r3	
21/16	471¢	5	-29¢	perf 4th	zo 4th	z4	
4/3	498¢	5	-2¢	perf 4th	wa 4th	w4	(or simply 4th)
27/20	520¢	5	+20¢	perf 4th	gu 4th	g4	
7/5	583¢	6	-17¢	dim 5th	zogu 5th	zg5	
45/32	590¢	6	-10¢	aug 4th	yo 4th	y4	
64/45	610¢	6	+10¢	dim 5th	gu 5th	g5	
10/7	617¢	6	+17¢	aug 4th	ruyo 4th	ry4	
40/27	680¢	7	-20¢	perf 5th	yo 5th	y5	
3/2	702¢	7	+2¢	perf 5th	wa 5th	w5	(or simply 5th)
32/21	729¢	7	+29¢	perf 5th	ru 5th	r5	
14/9	765¢	8	-35¢	min 6th	zo 6th	z6	
8/5	814¢	8	+14¢	min 6th	gu 6th	g6	
5/3	884¢	9	-16¢	maj 6th	yo 6th	y6	
27/16	906¢	9	+6¢	maj 6th	wa 6th	w6	
12/7	933¢	9	+33¢	maj 6th	ru 6th	r6	
7/4	969¢	10	-31¢	min 7th	zo 7th	z7	
16/9	996¢	10	-4¢	min 7th	wa 7th	w7	
9/5	1018¢	10	+18¢	min 7th	gu 7th	g7	
15/8	1088¢	11	-12¢	maj 7th	yo 7th	y7	
40/21	1116¢	11	+16¢	maj 7th	ruyo 7th	ry7	
2/1	1200¢	12	0¢	octave	wa octave	w8	(or simply octave)

There's no need to memorize this table, because every interval's name can be deduced from its ratio, and vice versa.

Two handy terms are **fourthward** and **fifthward**, abbreviated **4thwd** and **5thwd**, which means leftward or rightward on the harmonic lattice. (One could use subdominantward and dominantward, but I prefer to use shorter, simpler words whenever possible.) If the 3 exponent of the monzo is positive, it's 4thwd, if negative, 5thwd.

Table 2.1.3 has six rainbows of five bands each. Looking farther fourthward and fifthward yields 6-band rainbows. They all follow the same general form, shown here high to low:

Table 2.1.4 – The 6-band rainbow

color	cents from 12-ET	quality
ru	+27¢ to +39¢	mostly major, with aug 4th & perf 5th
ruyo or 4thwd wa	+2¢ to +21¢	"
yo	-22¢ to -10¢	"
	-------	
gu	+10¢ to +22¢	mostly minor, with perf 4th & dim 5th
zogu or 5thwd wa	-21¢ to -2¢	"
zo	-39¢ to -27¢	"

Note the large gap between yo and gu. The missing band will be filled in in chapters 3.4 and 3.6.

I play yaza JI music on a retuned conventional keyboard. I like knowing that if I spread my hand to a fifth, I'll play something that sounds more like a fifth than a fourth or a sixth. This means having a consistent method of determining the **keyspan** (width in semitones) of an interval. As we'll see in Part V, the keyspan of an interval is determined by its degree (3rd, 5th, etc.) and its quality (major, perfect, augmented, etc.). Therefore each yaza interval has both degree and quality, e.g. 7/6 is a minor third. As an added benefit, this approach allows the use of standard staff notation. Those who play fretless instruments, densely fretted guitars, array keyboards, or other instruments with more than 12 tones per octave will find the keyspan concept less useful.

Interval quality is redundant (if a third is yo, it must be major), it's not unique (there are other major thirds available), and its main purpose is to indicate keyspan (all major thirds are 4 semitones wide on a standard keyboard). Subminor and supermajor are not needed to determine keyspan, and are cumbersome, so they are not used.

It may seem odd to see the dissonant yo 5th 40/27 called a perfect 5th. Of course, it's not <u>the</u> perfect 5th = 3/2, but it's played as one on a retuned keyboard and written as one in staff notation. Perfect refers only to keyspan; it merely means not augmented or diminished. In conventional music theory, "perfect" implies consonance. While perfect wa intervals are highly consonant, non-wa perfect intervals are generally quite dissonant.

The harmonic lattice has an invisible rung of length zero, the octave rung. Occasionally the octave rung needs consideration, for example when stretching octaves in alt-tuner. 2-limit ratios, which are ratios with no factors other than 2, are **clear**, abbreviated **ca** or c. Technically, 1/1 and 2/1 should be called c1 and c8, but for simplicity's sake they're called w1 and w8.

Table 2.1.3 has six rainbows. There is a seventh rainbow, the rainbow of octaves. It overlaps the rainbow of 7ths and the rainbow of 9ths, and its colors run out of order: zogu – zo – yo – wa – gu – ru – ruyo.

Table 2.1.5 – The Rainbow of Octaves

28/15	1081¢	dim 8ve	zogu octave	zg8
63/32	1173¢	perf 8ve	zo octave	z8
160/81	1178¢	perf 8ve	yo octave	y8
2/1	1200¢	perf 8ve	wa octave	w8
81/40	1222¢	perf 8ve	gu octave	g8
128/63	1227¢	perf 8ve	ru octave	r8
15/7	1319¢	aug 8ve	ruyo octave	ry8

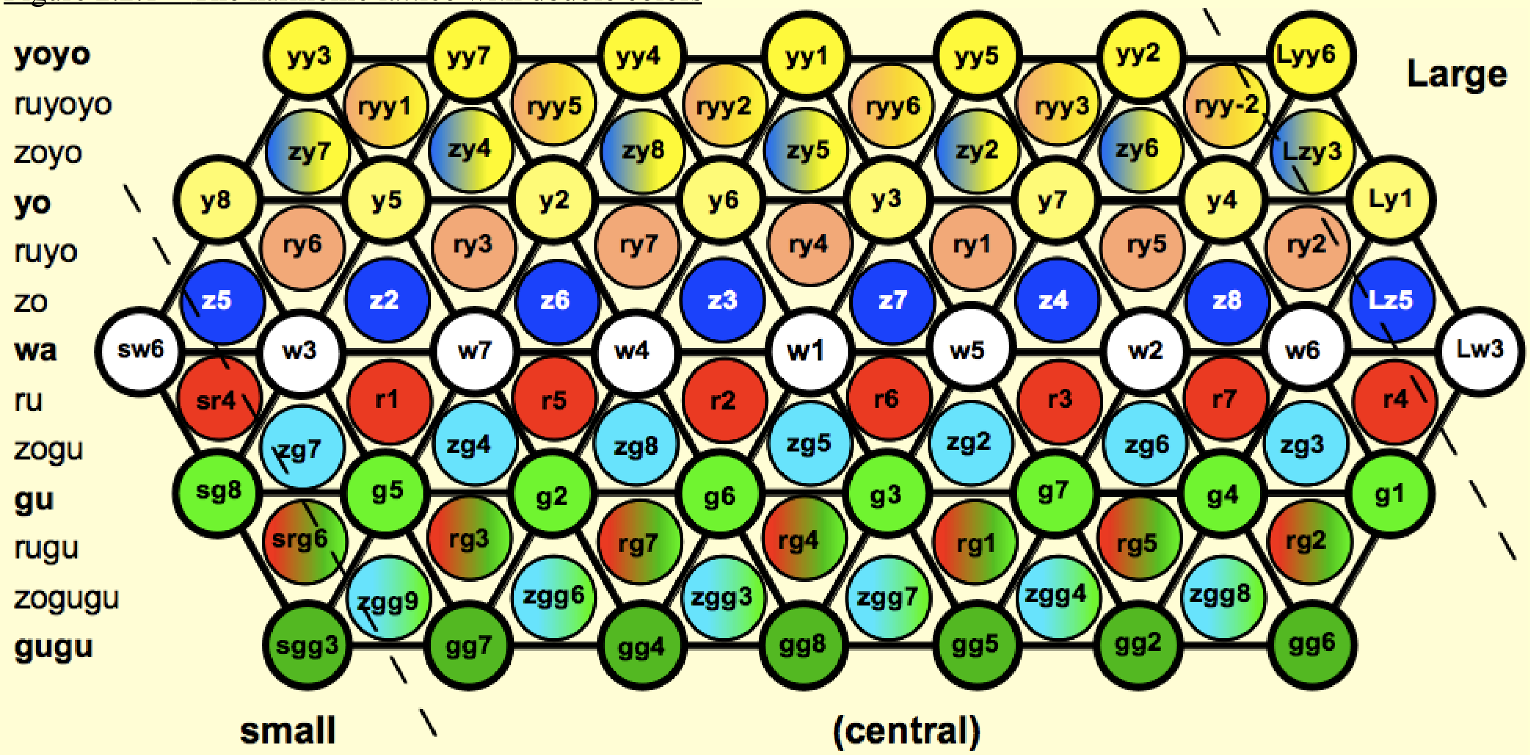
This curious rainbow is the result of taking two rainbows, sz8 – zg8 – sg8 – y8 – w8 – r8 and z8 – w8 – g8 – Ly8 – ry8 – Lr8, and discarding the large and small ratios. The most consonant one after the wa octave is probably the ruyo octave, which sounds like a minor 9th. Again, note the difference between perfect and perfect wa.

Chapter 2.2 – Commas and Wolves

Let's expand our lattice a bit. The augmented 5th 25/16 = yy5 is **double** yo, or yoyo for short. The diminished 5th is gugu, 36/25 = gg5. Likewise ratios using 49 and its multiples are zozo (zz) or ruru (rr). Ruyo and yo make ruyoyo (ryy). Zozo and gugu make zozogugu, or double zogu (zzgg).

There is no double wa; remote wa intervals are **large** or **small**. 32/27 = w3 is a wa 3rd and 81/64 = Lw3 is a large wa 3rd. It's inverse 128/81 = sw6 is a small wa sixth. Large and small are also used for other colors, see chapter 3.2. They're also used to describe scale steps, see chapter 4.1. **Central** means neither large nor small.

Figure 2.2.1 – The harmonic lattice with double colors



Compound colors such as zogu represent ratios with both 5 and 7 factors. **Primary** colors such as wa and yoyo have either 5 or 7, or neither.

The most important of these far-flung intervals are the smallest ones, the commas. Examples, sorted small to large:

Table 2.2.1 – Commas

ratio	cents	quality & degree	short name	full name
225/224	7.7¢	desc dim 2nd	ryy-2	the ruyoyo **minicomma**
81/80	22¢	perf unison	g1	the gu comma
$3^{12} / 2^{19}$	23¢	desc dim 2nd	LLw-2	the wa comma
64/63	27¢	perf unison	r1	the ru comma
50/49	35¢	desc dim 2nd	rryy-2	the double ruyo comma
49/48	36¢	min 2nd	zz2	the zozo comma
36/35	49¢	perf unison	rg1	the rugu comma

LLw-2, ryy-2 and rryy-2 have a degree of minus 2 because they are actually descending 2nds. For more on these **negative 2nds**, see chapter 3.3, “Paradoxical Intervals”.

A comma is defined as any interval smaller than 50¢, and a minicomma is any comma smaller than 10¢. Most minicommas are quite **remote** (many steps away on the harmonic lattice, see chapter 3.1). But the ruyoyo minicomma shows up quite often in yaza JI as the difference between two intervals. For example y4 = 590¢ is a minicomma sharper than zg5 = 583¢. The zg5 is said to be a **miniflat** y4, and y4 is a **minisharp** zg5. In the harmonic minor scale, the interval from g6 to y7 is a minisharp z3.

With remote ratios, color notation can become cumbersome, and ratios are of course extremely cumbersome. One might resort to monzos. Recall that the monzo of a JI ratio $2^a \cdot 3^b \cdot 5^c \cdot 7^d$ is (a, b, c, d). The monzo of the wa comma is (-19, 12). An alternative is to use a **reduced monzo**, which expresses the ratio as the sum or difference of these octave-reduced lattice rungs:

prime	ratio	interval	cents	symbol	colors
2	2/1	octave	1200¢	Duplius	clear
3	3/2	fifth	702¢	Tertia	wa
5	5/4	maj 3rd	386¢	Quintia	yo & gu
7	7/4	min 7th	969¢	Septima	zo & ru

The standard monzo uses non-octave-reduced rungs 3/1, 5/1, 7/1, etc. This notation has the advantage of simplifying the conversion of a ratio into component rungs. One can look at 15/8 and quickly calculate that it contains a 3-rung, a 5-rung, and three descending 2-rungs, and is thus written (-3, 1, 1). But it has the disadvantage of creating very large intervals which are rather musician-unfriendly:

prime	ratio	interval	cents
2	2/1	octave	1200¢
3	3/1	twelfth	1902¢
5	5/1	maj 17th	2786¢
7	7/1	min 21st	3369¢

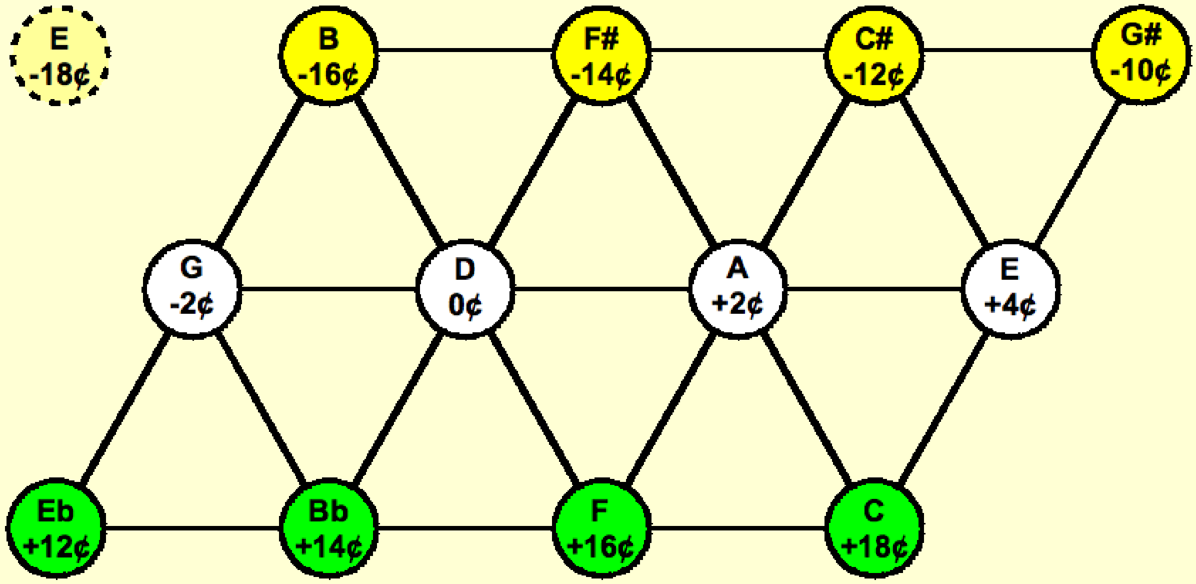
Musicians generally don't think of a major 7th = 15/8 as the sum of a twelfth and a seventeenth, minus three octaves. They are far more likely to think of it as the sum of a fifth and a major third. Indeed, when you ask a musician what the 3rd of the V chord is, he or she will do this very calculation. The reduced monzo is written with curly brackets: 15/8 = {0, 1, 1}. Reduced monzos are more usable by musicians, if less usable by mathematicians. Alt-tuner uses reduced monzos on the modulate and linkages screens.

Commas and minicommas are unavoidable. If you tune a keyboard to a JI tuning, you'll always get what's known as **wolf** intervals somewhere among the 12 keys. This name originally referred to a certain sharp fifth in meantone, see chapter 4.3. It's about 35¢ wider than a usual fifth, and makes interference beats which sound like the howling of wolves. The term has come to mean any interval that sounds not merely dissonant but out of tune. The spectrum of consonance discussed in chapter 1.3, just before Figure 1.3.10, becomes hyperstable – stable – unstable – wolf.

The question of exactly what qualifies as a wolf is quite subjective, and it also depends on the culturally accepted prime-limit. But for JI ratios we can attempt an objective acoustic definition. A ratio is a wolf ratio if adding or subtracting a comma would significantly simplify the ratio (lower the odd limit) without exceeding the accepted prime limit. For example, the yo fifth y5 = 40/27 is a wolf because y5 + g1 = w5 = 3/2, and 3/2 is much simpler than 40/27. Likewise, the large wa third Lw3 = 81/64 is a wolf because Lw3 - g1 = y3 = 5/4. If a ratio is within a mere minicomma of a simpler one, that may not be dissonant enough to qualify as a wolf. For example, zg4 = 56/45 = 379¢ is a miniflat yo 3rd, and sounds passable to me. However, ryy7 = 225/112 = 1208¢ is a minisharp octave that doesn't.

Below is a ya keyboard tuning (ignore the extra E note for now). The wolves mostly arise when you try to go beyond the edge of the lattice. For example, to go up a fifth from C, you would normally go one step to the right, but there isn't any G note there. So you must use a different G, which is tuned differently. The fifths C – G, E – B, and G♯ – E♭ are all wolves. So are major thirds like F♯ – B♭ and major ninths like F – G. There are only 12 triangles in the diagram, therefore half of the possible 24 major and minor triads are out of tune. Minor thirds like E – G are technically wolves, but in fact a wa minor third = 32/27 = 294¢ is hardly more dissonant than a gu minor third = 6/5 = 316¢.

Figure 2.2.2 – A ya JI keyboard tuning, with an extra note

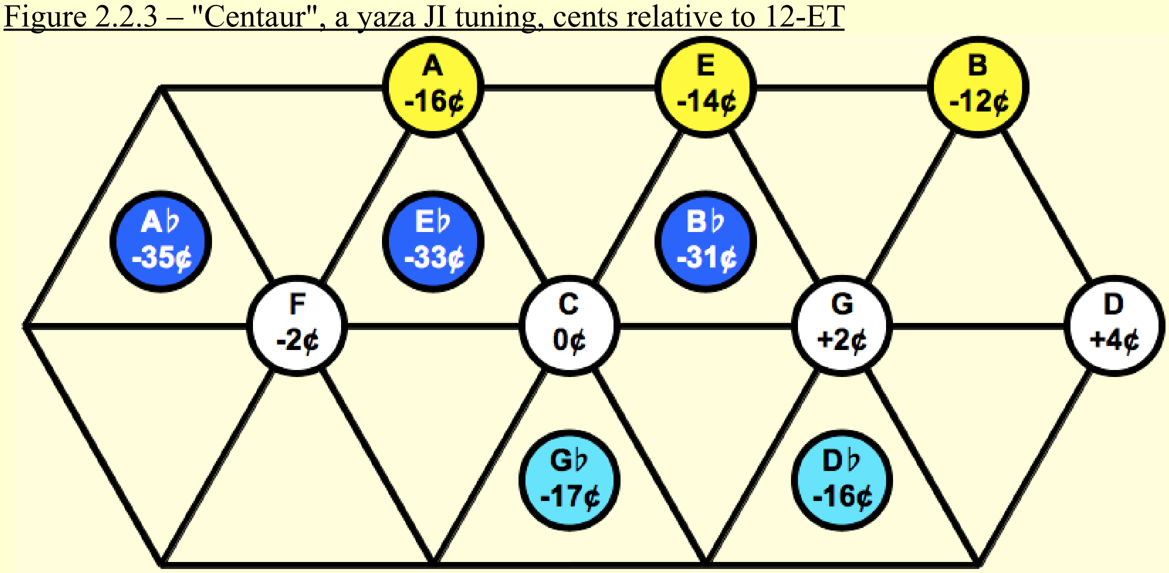


The gu comma is responsible for many tuning problems. For example, the strings of a guitar are tuned E – A – D – G – B – E. If you try to tune a guitar using only harmonics, using the just-flat-of-the-4th-fret harmonic to tune the G string to the B string, it will never come out right. If you start from the lowest string and work your way up, the top string will be out of tune with the bottom string. It's easy to see why from the lattice. From the wa E we move left to A, D, and G, then up to B, then left again to the yo E, the extra note in the picture. We expect the top and bottom strings to make a double octave. But the double octave is flat by a gu comma, which sounds out of tune. Instead of the simple ratio 4/1, we get the very complex 320/81.

Another example: A violin is tuned in fifths, G – D – A – E. Along with the open G string, play an E note on the D string. Now play the same E note along with the open A string. You'll need to sharpen it slightly to bring it in tune. Again, the lattice shows us why. Our open strings are the four wa notes. The wa G sounds best with the yo E. But the wa A sounds best with the wa E, which is a gu comma sharper.

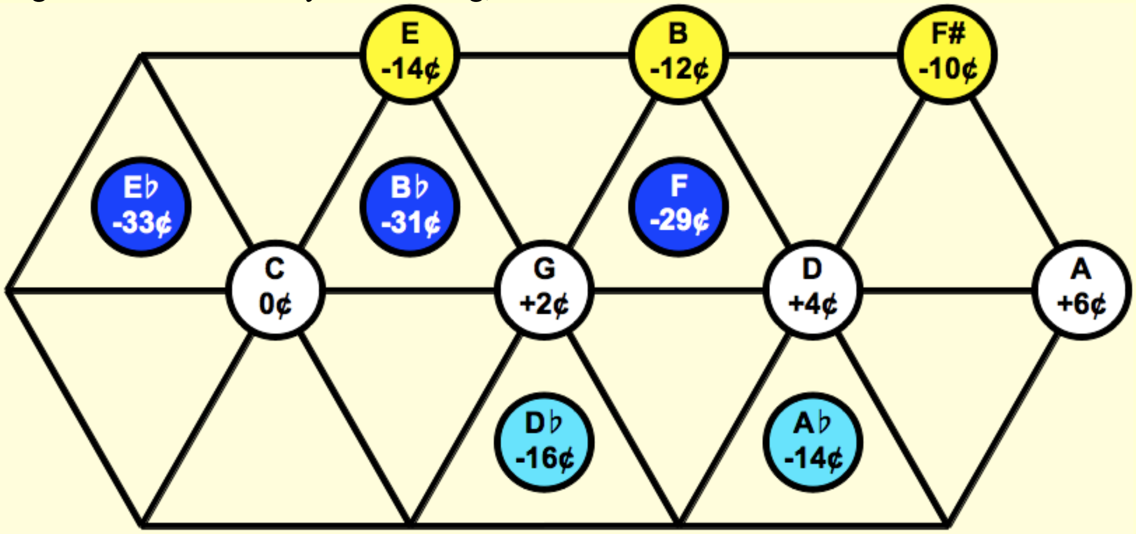
In general, the number of wolf fifths in a 12-note tuning equals the number of breaks in the circle of fifths, which equals the number of rows in the lattice, which equals the number of colors in that tuning. Figure 2.2.2 has three colors and thus three wolf fifths. In yaza JI, there are more colors, hence even more wolf fifths and even fewer triangles, as in the "Centaur" tuning:

Figure 2.2.3 – "Centaur", a yaza JI tuning, cents relative to 12-ET



Most of the tetrads are out of tune, such as F6 and G♭7. One can get around this limitation by using custom guitars, keyboards, etc. with more than 12 notes per octave. Or one can just avoid the out of tune chords. And of course, one can temper the scale, making Duplius' wives wear their corsets and harnesses. Another approach is to retune the keyboard on the fly with alt-tuner, switching mid-song to a nearby tuning like this one:

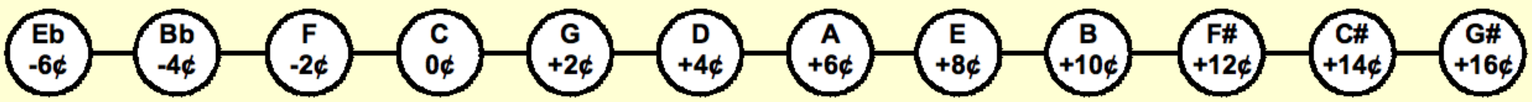
Figure 2.2.4 – Another yaza JI tuning, cents relative to 12-ET



This alternate tuning supplies many of the chords that were missing, as well as chords like Gmaj7 and G9.

Earlier I said that Lw3 = 81/64 is a wolf because a slight tuning adjustment creates the much simpler y3 = 5/4 ratio. This only applies to modern ears that accept ya intervals as a consonance. Recall the discussion of medieval music in chapter 1.3. The y3 was considered so dissonant as to be a wolf, and the Lw3 was merely an unstable interval. The triad was a dissonance that resolved to the stable open fifth chord.

Figure 2.2.5 – The 3-limit JI tuning, cents relative to 12-ET



Played in 5-limit JI or even 12-ET, this hardly sounds like a convincing cadence. But back then instruments were tuned to the 3-limit, all wa notes. The major third was tuned as a large wa third = 81/64 = 408¢, which sounds considerably more tense than the 5-limit yo third = 5/4 = 386¢. A typical medieval cadence would be a tense G – B resolving to a relaxed F – C, or perhaps G – B – E resolving to F – C – F. The latter cadence uses a wa sixth = 27/16 = 906¢, not the smoother yo sixth = 5/3 = 884¢. The dissonance of the unstable thirds and sixths was one of the driving forces of medieval cadences. Performing this music with yo thirds and sixths instead of wa ones would rob it of its power. For more on this subject, see Margo Schulter's writings at www.medieval.org/emfaq/harmony.

There is a parallel with classical music and yaza music. Although the yaza dom7 chord w1 – y3 – w5 – z7 sounds acoustically more consonant than the ya equivalents using g7 or w7, I wouldn't recommend performing a Bach piece using such a dom7 chord. Bach expected his dom7 chords to be an unstable dissonance, and used that dissonance intentionally to create harmonic contrast.

It just so happens that each era's preferred keyboard tunings contain very close approximations of the consonances of future eras. For example, in the medieval 3-limit tuning, the diminished 4th B – E♭ is 384¢, only 2¢ narrower than 5/4. And in the renaissance/baroque quarter-comma meantone tuning, the augmented 6th F – D♯ is 966¢, only 3¢ narrower than 7/4. As we'll see in chapter 3.6, yaza ratios like 60/49 are only 3¢ wider than 11-limit intervals like 11/9.

The question arises, why doesn't the 400¢ major 3rd of 12-ET sound wolfish or at least unstable to Western ears? It's very close to the ratio 63/50 = 400.11¢, certainly a wolf ratio by our definition. The answer is that although it's not objectively (acoustically) in tune, because the overtones clash, it's subjectively (culturally) in tune, because we have grown used to it. I've spent a lot of time listening to chords with just major thirds played on a realistic piano sound. On switching back to 12-ET, I'm always shocked at how jangly and "wrong" the 400¢ major third sounds.

For the rest of the book, I'll be assuming familiarity with and acceptance of low-odd-limit yaza intervals, and using the objective definition of wolf as an interval within a comma of a simpler (and presumably more singable) yaza ratio. By this definition, chords that use the ru 3rd 9/7, although quite jarring at first, are not wolfy, merely unstable. But like the minor chord in renaissance times, they are utonal chords considered dissonant but firmly implied by consonant otonal chords, and thus one that will slowly and inevitably come to be accepted as a consonance.

Chapter 2.3 – Note Names and Color Solfege (Da-re-mu)

The preceding is all **relative notation**; no actual frequency or pitch is specified. **Absolute notation** requires the use of standard note names A, B, C♯, etc. A note is named by its color and its letter name: gu D, wa E, etc. Of course, individual notes don't have color, only intervals have color. But we can assign colors to the notes relative to the tonic, which is always wa. Thus in F, the yo 3rd is yo A, or yA. The gu 3rd is gu A-flat, gA♭. There is no gu A or yo A♭ in the key of F (unless you count large and small notes, see chapter 3.2). If the color is omitted, the note is assumed to be wa.

For example, "Fur Elise" in E might be intoned like so (upper-case G is a note, lower-case g is a color):

B – yA♯ – B – yA♯ – B – F♯ – A – gG – E

Note color is not the same as interval color. The first melodic step from wa B down to yo A♯ is a gu interval, even though neither of the notes is gu.

Fixed-do countries like France and Spain use Do for C, Re for D, Mi for E, etc. "Fur Elise" could be written Ti – yLa♯ – Ti – yLa♯ – Ti – wFa♯ – La – gSo – Mi, or perhaps T – yL♯ – T – yL♯ – T – wF♯ – L – gS – M.

In both fixed-do and movable-do countries, solfege (do-re-mi) can be adapted to include color notation by assigning the five basic vowels, a, e, i, o and u, to the five basic colors. They are ordered by vowel pitch, with i–e–a–o–u meaning ru–yo–wa–gu–zo. Then simply combine the consonant of the note with the vowel of the color:

ru 3rd = <u>M</u>i + <u>i</u> = Mi = "mee"
yo 3rd = <u>M</u>i + <u>e</u> = Me = "meh"
wa 3rd = <u>M</u>i + <u>a</u> = Ma = "ma"
gu 3rd = <u>M</u>i + <u>o</u> = Mo = "mow" (-o doesn't mean over)
zo 3rd = <u>M</u>i + <u>u</u> = Mu = "moo" (-u doesn't mean under)

Compound colors like zogu and ruyo use the vowels of both component colors, with the za color coming last. The zogu 5th = Sol + gu + zo = "s" + "oh" + "oo" = Sou. Similarly the ruyo 4th is Fei = "feh-ee". An "h" or "w" can optionally be inserted between any two adjacent vowels, so the ruyo 4th can also be sung as "fehi" or "fewi".

Table 2.3.1 – Color Solfege (da-re-mu)

	1sn	2nd	3rd	4th	5th	6th	7th
wa	Da	Ra	Ma	Fa	Sa	La	Ta
yo	Di	Ri	Mi	Fi	Si	Li	Ti
gu	Do	Ro	Mo	Fo	So	Lo	To
zo	Du	Ru	Mu	Fu	Su	Lu	Tu
ru	De	Re	Me	Fe	Se	Le	Te
zogu	Dou	Rou	Mou	Fou	Sou	Lou	Tou
ruyo	Die	Rie	Mie	Fie	Sie	Lie	Tie

The color solfege system, whether movable or fixed, is called **da-re-mu**. "Fur Elise" in Mi in fixed da-re-mu is sung Ta – Li – Ta – Li – Ta – Fa – La – So – Ma. In standard fixed-do solfege, sharps and flats are not sung, and both A and A♯ are sung as La. But in fixed da-re-mu, the vowels serve to differentiate sharp, flat and natural: in the key of Mi, La = A and Le = A♯. In movable da-re-mu, the tonic is always Da, and "Fur Elise" would be Sa – Fi – Sa – Fi – Sa – Ra – Fa – Mo – Da.

The major scale on Do using the wa 2nd: Da – Ra – Me – Fa – Sa – Le – Te – Da. If using the yo 2nd, substitute Re for Ra. Thus the V chord is Sa – Te – Ra and the ii chord is Re – Fa – Le. The ii chord wouldn't be Ra – Fa – Le, because Ra – Le is a wolf fifth. To avoid wolves, the vowels of the root and fifth of a chord must always match, except for diminished chords. If the Re chord were major, it would use a yoyo Fa = "f" + "eh" + "eh" = Fe'e, pronounced "fehe" or "fewe".

The occasional large or small ratio is indicated by adding "ba" (b for big) or "pa" (p for petite) before the syllable. The large wa 3rd is baMa and the small wa 6th is paLa. The first syllable is unaccented and sung as quickly as possible with either a schwa vowel or no vowel, so paLa becomes "pla".

Movable da-re-mu can be used to name not only notes in the scale but also intervals between notes. Earlier in the chapter, I said "the step from wa B down to yo A♯ is a gu interval." Using movable da-re-mu, one would say "the step from Sa down to Fe is a Ro." Thus every ratio has a da-re-mu name, even those not usually used in scales, such as commas. For example, the gu comma is Do, the ru comma is Du, and the wa comma is babaTa.

Any solmization system with unique consonants (such as Indian sargam, Byzantine ni-pa-vu-ga-di-ke-zo, etc.) can be treated similarly with vowel substitutions. The major scale in color sargam ("sa-re-gu"), using the wa 2nd: Sa – Ra – Ge – Ma – Pa – De – Ne – Sa. The 22 shrutis, using "la" for large and "ta" (tiny) for small: Sa taRa Ro Re Ra Ga Go Ge laGa Ma Mo Me laMa Pa taDa Do De Da Na No Ne laNa Sa.

In **harmonic da-re-mu**, or "ultra-movable" da-re-mu, the current chord's root is always Da, and each note is named according to its role not in the scale, but in the chord. It's useful for singers wanting to identify the root and their note's relationship to it, to describe the "feel" of the note. For example, regardless of the key, in a C major chord, C is Da, E is Me, and G is Sa. Each note's name changes when the chord changes, so if the next chord is Am, C changes from Da to Mo. Here's "Happy Birthday" written out in both da-re-mu and harmonic da-re-mu:

Sa Sa Le Sa Da Te	(I) Sa Sa Le Sa Da (V) Me
Sa Sa La Sa Ra Da	Da Da Ra Da Sa (I) Da
Sa Sa Sa Me Da Te Le	Sa Sa Sa Me Da (IV) Fe Me
Fa Fa Me Da Ra Da	Da Da (I) Me Da (V) Sa (I) Da
Sa Le Sa Tu	Sa Le Sa Tu

Harmonic da-re-mu illustrates nicely the dissonance of the 2nd to last note in the 3rd line. In standard da-re-mu, it's Te = y7 = 15/8. But in relation to the IV chord, it's Fe = y4 = 45/32. Most of the time, harmonic da-re-mu syllables will come from the central hexagon of the lattice, plus a few ratios to the right. In ya JI, these are Da, Ra, Me, Mo, Fa, Sa, Le, Te and To. In this song, both Fe and Tu stand out as exceptions.

As we'll see in the next chapter, the root of the chord is sometimes ambiguous. However, as noted at the end of chapter 2.5, even the tonic can sometimes be ambiguous. Since tonic ambiguity doesn't stop us from using standard movable-do solfege, root ambiguity shouldn't stop us from using harmonic da-re-mu.

Given a key and a ratio, say, 15/14 = 114¢ in F, exactly which note name do we use, and which keyboard key do we assign it to? First find the color & degree: 15/14 = 3/2 · 5/4 ÷ 7/4 = w5 + y3 - z7 = ry1. It's an F, but is it F natural or F sharp? In conventional terms, w5 + y3 - z7 = perf5 + maj3 - min7 = maj7 - min7 = aug1, so it's augmented, and thus a ruyo F♯.

We can combine color, quality (major vs. minor) and degree into one term: 15/14 = ryA1. For unfamiliar ratios, it can be helpful to include the quality. But remember that the quality is not an independent variable; "ryA1" has meaning, but "ryP1" is meaningless.

Another example: 81/64 = 408¢ in A is 81/64 = 9/8 · 9/8 = wM2 + wM2 = LwM3 = large wa major 3rd = wa C♯. One wouldn't say "large wa C#" for the same reason one wouldn't say "major wa C#". Like quality, the **magnitude** (large vs. small vs. central) is used only in relative notation, never in absolute notation. (Magnitude is different from **size**, defined in chapter 1.2 as an interval's width in cents.)

To convert any ratio to a color interval: Find the monzo by prime factorization. To find the color, combine all the appropriate colors for each prime, higher primes first. To find the degree, first find the stepspan, which is the dot product of the monzo with (7, 11, 16, 20). Then add 1, or subtract 1 if the stepspan is negative. To find the magnitude, add up all the monzo exponents except the first one, divide by 7, and round off. Combine the magnitude, color and degree to make the color name.

Example: ratio = 63/40, monzo = (-3, 2, -1, 1), color = zg, stepspan = -21+22-16+20 = 5, degree = 5+1 = 6, magnitude = round [(2-1+1)/7] = round (2/7) = 0, interval = zg6.

To convert any color interval to a ratio: Let S be the stepspan of the interval, S = degree - sign (degree). Let M be the magnitude of the color name, with L = 1, LL = 2, etc. Small is negative and central is zero. Let the monzo be |a b c d e...>. The colors directly give you all the monzo entries except a and b. Let X = the dot product of |0 0 c d e...> with the 7edo edomapping. Then b = (2S - 2X + 3) mod 7 + 7M - 3, and a = (S - X - 11b) / 7. Convert the monzo to a ratio.

Example: interval = sgg2, S = 2-1 = 1, M = -1, monzo = |a b -2>, X = <7 11 16| dot |0 0 -2> = -32, b = (2-(-64)+3) mod 7 + 7(-1) - 3 = 6-7-3 = -4, a = (1-(-32)-(-44))/7 = 77/7 = 11, monzo = |11 -4 -1>, ratio = 2048/2025.

In absolute notation, both the magnitude and quality are indicated indirectly by sharps and flats. In A, C♯ is a major 3rd and C is a minor 3rd. Wa C♯ is a large wa 3rd, and wa C is a central wa 3rd. The magnitude depends on the tonic, and wC♯ is large in the key of A but not in the key of B. It also depends on the color, a yo C♯ isn't large in either key.

A few paragraphs ago, I said that the quality can optionally be included with the color and degree. You may wonder, why not have the quality be a mandatory feature of relative notation, and let the magnitude be optional? Why not say wM3 instead of Lw3? Three reasons: the first is that the most-discussed intervals are the central ones, thus we can omit the magnitude most of the time. But the quality would have to be included most of the time, and the notation would become more cluttered. Another reason is that the magnitude of a ratio is always known (see chapter 3.2), but with a higher prime limit, like 11-limit, sometimes the keyspan can't be specified exactly (see chapter 3.6). Since the quality reflects the keyspan, that means that sometimes the quality can't be specified. Finally, as we'll see in Part V, the quality of an interval depends on the sizing framework. In a 19-tone framework, the zo third is diminished, not minor.

Color notation tells you the exact interval between any two notes. Let's start with comparing two notes that differ only by color. The gu C is exactly a gu comma g1 = 22¢ sharper than the wa C. For any note in absolute notation (G, F♯, B♭, etc.), the gu "version" is always g1 sharper than the corresponding wa version, regardless of the key. Likewise the yo version is g1 flatter than the wa version. Similarly, the ru version of any note is always a ru comma r1 = 27¢ sharper than the wa version, and the zo version is always r1 flatter.

The ru comma and the gu comma are sufficient to compare any two colors in yaza JI. For example to compare a zo F to a gu F, add a ru comma to change the zo to wa, and add a gu comma to change wa to gu: zF + r1 + g1 = gF. Because r1 + g1 = 49¢, these two colors are fairly far apart.

Next let's compare two notes that differ by color and accidental. Adding a sharp to a note (or removing a flat) increases the interval by what is conventionally known as an augmented prime, an augmented unison, or a chromatic semitone (as opposed to a diatonic semitone, which is a minor second). I find all these terms awkward, and "augmented prime" is especially confusing since it doesn't refer to prime numbers. If we agree to always call a diatonic semitone a minor second, the term "semitone" can always be taken to mean chromatic semitone.

Adding a sharp *without changing the color* increases the interval by the large wa semitone Lw1 = 114¢. To compare gu C to yo C♯, first change natural to sharp by adding a large wa semitone, then change gu to yo by subtracting two gu commas. gC + Lw1 - g1 - g1 = yC♯. The difference is the yoyo semitone yy1 = 71¢.

Comparing two versions of the same note using relative notation requires taking into account the quality. For any given degree and quality, the yo version will always be a gu comma narrower than the wa version. For example, y2 = w2 - g1 because both w2 and y2 are major. If the qualities are different, a large wa semitone must be used to make the qualities the same. To compare w7 (a m7) and y7 (a M7), first add Lw1 to make the wa note major, then subtract g1 to make it yo: yM7 = wm7 + Lw1 - g, a difference of Lw1 - g1 = a large yo semitone Ly1 = 135/128 = 92¢.

To compare two different notes like C and D, or C♯ and D♭, first add or subtract a wa interval to make the notes match, then add or subtract gu and ru commas to make the color match. For example, the difference between gu C and zo D is a major 2nd. Add a wa major 2nd w2 = 9/8 to gu C to make it gu D, then subtract a gu comma and a ru comma to make it zo. The difference is w2 - g1 - r1 = zy2 = 204¢ - 22¢ - 27¢ = 155¢.

The next two (printable) pages show large lattices that contains every key. They extend about as far as possible without encountering triple-sharps and triple-flats. Start by picking an instance of your keynote somewhere in the middle of the chart. Use the notes on the lines, not inside the triangles. For example, for E, use the third row from the top, 6th note from the left. Think of that note as wa, and that whole row as wa. Assign colors to nearby rows based on that. You can see all possible scales and chord progressions using that E note. Comma pumps will take you to a different E note. Try to visualize the "Fur Elise" melody!

Figure 2.3.1 – The ya lattice

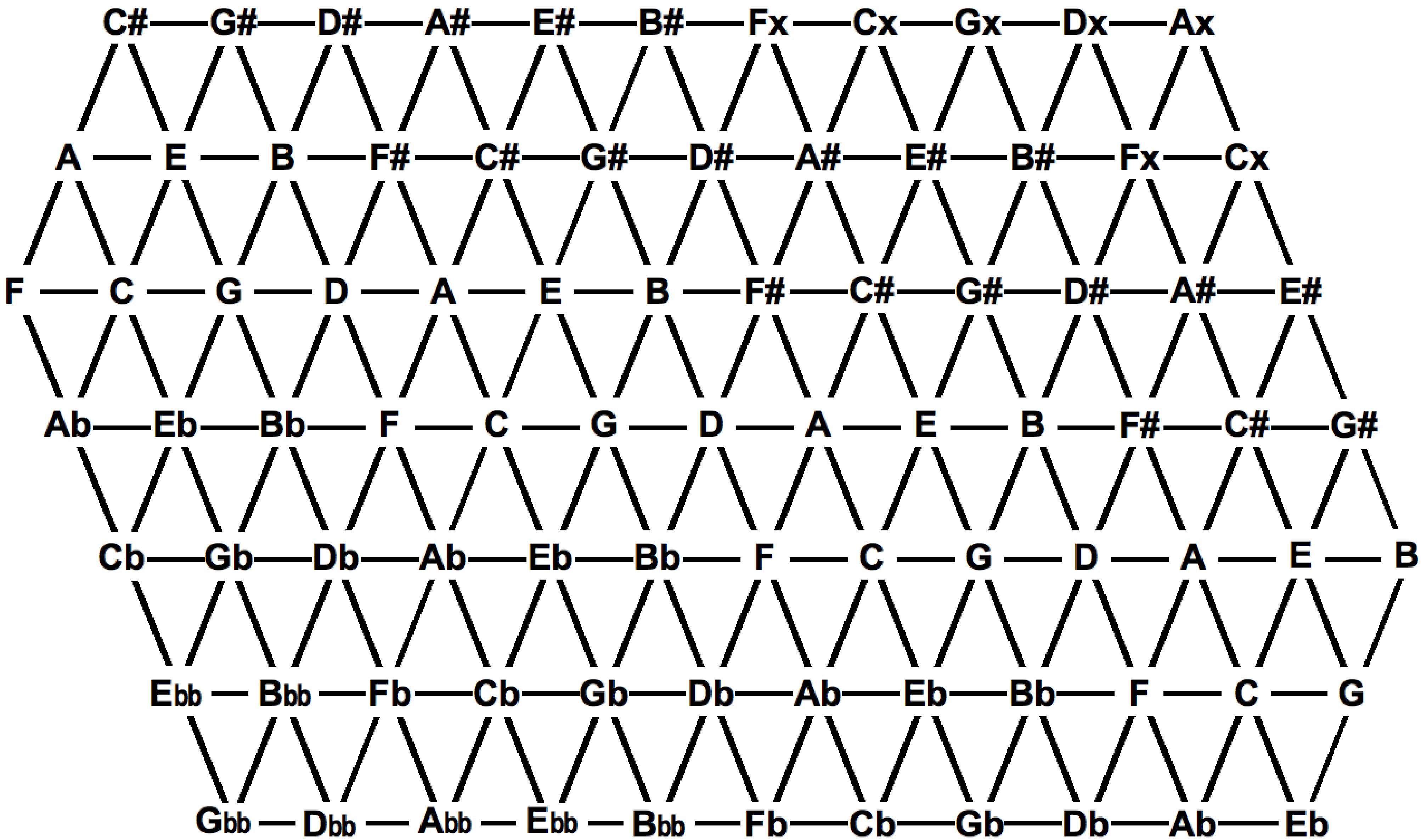
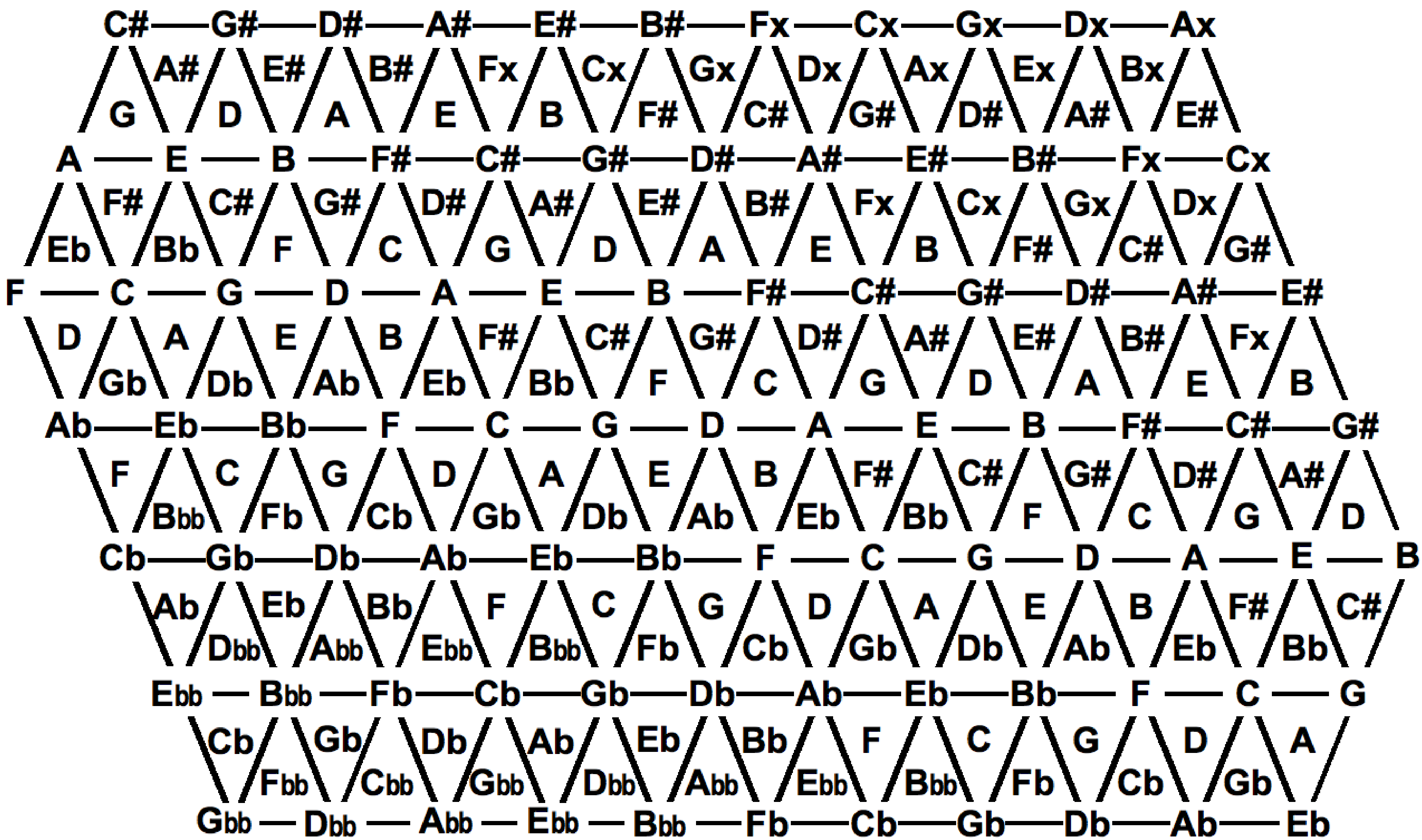


Figure 2.3.2 – The yaza note lattice



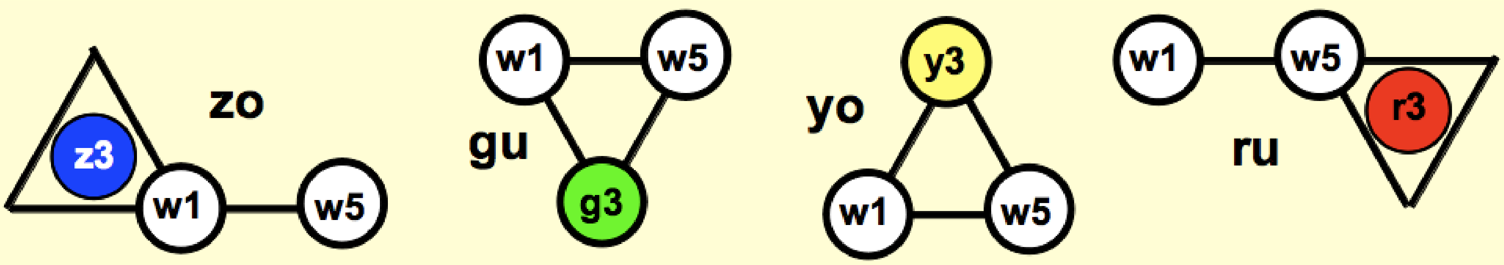
Chapter 2.4 – JI Chord Names Part I

Yaza JI offers a bewildering variety of chords with a huge range of consonance and dissonance. Color notation gives us clear, concise names for them. Basic names are given here, chapter 3.8 "JI Chord Names Part II" has more. See also chapter 3.9. You can use the Just Intonation Toolkit set to 43 tones (see Figure 1.2.5) to play these chords.

A triad is named after the color of its 3rd. The 5th is assumed to be wa. There are four main triads, shown here in close position with both written names and spoken names. The roots are wa in these examples; the next chapter discusses root colors.

Table 2.4.1 – Triads

chord name	chord quality	chord structure		example in C			JI Toolkit keys to press
zo chord	minor chord	1, z3, 5	1/1 – 7/6 – 3/2	Cz	"C zo"	wC, zE♭, wG	ZAR or LY0 or .R1
gu chord	minor chord	1, g3, 5	1/1 – 6/5 – 3/2	Cg	"C gu"	wC, gE♭, wG	ZDR or LI0 or .R3
yo chord	major chord	1, y3, 5	1/1 – 5/4 – 3/2	Cy	"C yo"	wC, yE, wG	ZGR or LP0 or .R6
ru chord	major chord	1, r3, 5	1/1 – 9/7 – 3/2	Cr	"C ru"	wC, rE, wG	ZJR or L]0 or /L2



"Yo C" is a note, whereas "C yo" is a chord. Chords can be referred to by structure, e.g., yo chords or z chords. The chord quality (major, dom7, etc.) is analogous to interval quality, in that it's redundant (if it's yo, it must be major), it's not unique (there are other major triads available, like ru), and its main purpose is to indicate keyspan (both yo and ru triads will in close root position have two intervals of 4 and 3 semitones each).

Like ratios, a chord's dissonance mostly comes from the "underness" and "bigness" of its component intervals. Underness comes from the colors used. Gu and ru are under, zo and yo are over. The bigness is the chord's odd limit, which is the maximum odd limit for all the ratios between every note in the chord. For example, the zo chord has an odd limit of 9 because the interval from 7/6 to 3/2 is 9/7. Neither the underness nor the odd limit depend on the voicing of the chord, but as we'll see in chapter 2.7, voicing also affects dissonance.

Table 2.4.2 – Various triads containing dissonant wolves

chord name	chord quality	chord structure		example in C		JI Toolkit
wa chord	minor chord	1, w3, 5	1/1 – 32/27 – 3/2	Cw	wC, wE♭, wG	ZSR or .R2
large wa chord	major chord	1, Lw3, 5	1/1 – 81/64 – 3/2	CLw	wC, wE, wG	SR2 or L[0
wa yo-5 chord	minor chord	1, w3, y5	1/1 – 32/27 – 40/27	Cw(y5)	wC, wE♭, yG	ZSE or ,R2
yo yo-5 chord	major chord	1, y3, y5	1/1 – 5/4 – 40/27	Cy(y5)	wC, yE, yG	ZGE or ,R6
wa ru-5 chord	minor chord	1, w3, r5	1/1 – 32/27 – 32/21	Cw(r5)	wC, wE♭, rG	ZST or /R2
ru ru-5 chord	major chord	1, r3, r5	1/1 – 9/7 – 32/21	Cr(r5)	wC, rE, rG	ZJT or AR2

All these chords have a fairly large odd limit, as seen by the size of the numbers in their ratios. The first two have wa 3rds, the most common wa wolves. The next two have yo 5ths, the most common ya wolf. The last two have ru 5ths, the most common za wolf. A chord is a **wolf chord** if it contains a wolf interval like Lw3 or y5 or r5.

Diminished triads are named after the color of the third and the fifth. The yaza ones are surprisingly consonant.

Table 2.4.3 – Diminished triads

chord name	**chord quality**	**chord structure**		**example in C**		**JI Toolkit**
gu gugu-5 chord	diminished	1, g3, gg5	1/1 – 6/5 – 36/25	Cg(gg5)	wC, gE♭, ggG♭	,LI or GR3
gu zogu-5 chord	diminished	1, g3, zg5	1/1 – 6/5 – 7/5	Cg(zg5)	wC, gE♭, zgG♭	ZD\ or VR3
zo zogu-5 chord	diminished	1, z3, zg5	1/1 – 7/6 – 7/5	Cz(zg5)	wC, zE♭, zgG♭	ZA\ or VR1
yo ruyo-4 no 5 chord	maj diminished	1, y3, ry4	1/1 – 5/4 – 10/7	Cy,ry4no5	wC, yE, ryF♯	ZGQ or LP7
ru ruyo-4 no 5 chord	maj diminished	1, r3, ry4	1/1 – 9/7 – 10/7	Cr,ry4no5	wC, rE, ryF♯	ZJQ or L]7

Alterations are always enclosed in parentheses, and additions never are. Cg,zg5 would be a "C-gu add zogu-five" chord which has both w5 and zg5. Sus chords are sometimes an exception, see chapter 3.8.

Chords can be classified by the number of colors they contain (including wa) as a rough measure of their complexity. For example, the triads in table 2.4.1 are **bicolored**, but the diminished triads in table 2.4.3 are all **tricolored**.

Augmented and dim7 chords have no obvious yaza tuning, and always have a high odd limit. Such hard to tune chords are called **innate comma chords**, because the comma is unavoidable. This term applies to chord qualities, not specific chords. The large wa chord is a wolf chord, but not an innate comma chord. More on this at the end of the chapter.

Tetrads: We assume a wa 5th. If the 6th/7th is the same color as the 3rd, the chord is named analogous to CM6 or Cm7, with a color replacing "M" or "m". Otherwise the 6th/7th is an added note. Here are my favorite tetrads:

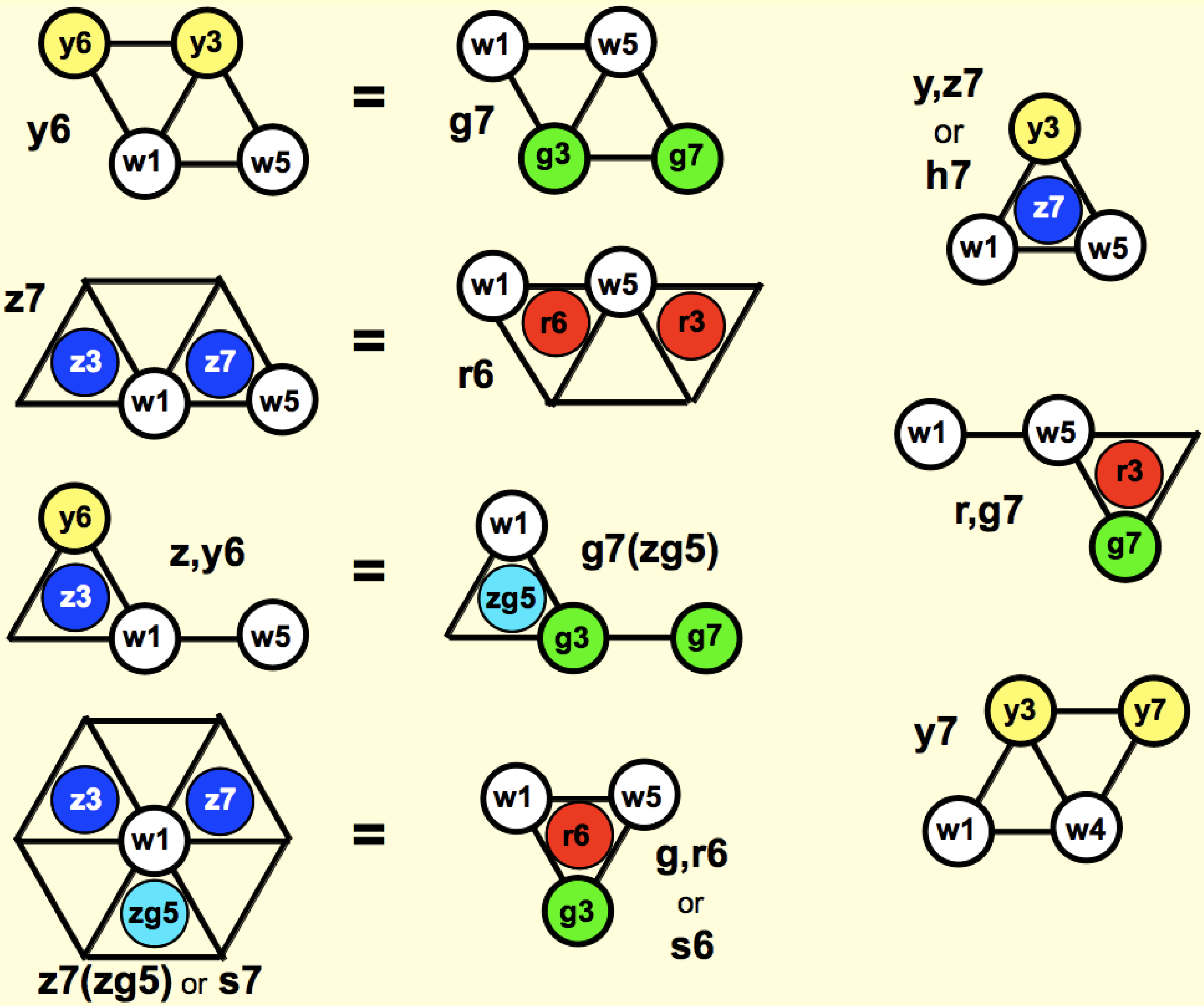
Table 2.4.4 – Examples of tetrads with a low odd limit

yo-6 chord	maj6	1, y3, 5, y6	Cy6	wC, yE, wG, yA	ZGRP or .GR6	a homonym of yAg7
gu-7 chord	min7	1, g3, 5, g7	Cg7	wC, gE♭, wG, gB♭	ZDR3 or .;R3	a homonym of gE♭y6
zo-7 chord	min7	1, z3, 5, z7	Cz7	wC, zE♭, wG, zB♭	ZAR1 or .KR1	a homonym of zE♭r6
ru-6 chord	maj6	1, r3, 5, r6	Cr6	wC, rE, wG, rA	ZJR] or Z/L]	a homonym of rAz7
yo zo-7 chord or har-7 chord	dom7	1, y3, 5, z7	Cy,z7 or Ch7	wC, yE, wG, zB♭	ZGR1 or .KR6	harmonic-series chord
ru gu-7 chord	dom7	1, r3, 5, g7	Cr,g7	wC, rE, wG, gB♭	ZJR3 or ZDL]	
zo yo-6 chord	min6	1, z3, 5, y6	Cz,y6	wC, zE♭, wG, yA	ZARP or .GR1	a homonym of yAg7(zg5)
gu-7 zogu-5	half-dim	1, g3, zg5, g7	Cg7(zg5)	wC, gE♭, zgG♭, gB♭	ZD\3 or V;R3	a homonym of gE♭z,y6
zo-7 zogu-5 or sub-7 chord	half-dim	1, z3, zg5, z7	Cz7(zg5) or Cs7	wC, zE♭, zgG♭, zB♭	ZA\1 or VKR1	subharmonic-series chord, a homonym of zE♭g,r6
gu ru-6 chord or sub-6 chord	min6	1, g3, 5, r6	Cg,r6 or Cs6	wC, gE♭, wG, rA	ZDR] or .JR3	subharmonic-series chord, a homonym of rAz7(zg5)
yo-7 chord	maj7	1, y3, 5, y7	Cy7	wC, yE, wG, yB	ZGR6 or ZGLP	

The y,z7, z7(zg5) and g,r6 chords have alternate names, because they follow the harmonic or subharmonic series. The **subharmonic series** is the harmonic series inverted: C5 – C4 – F3 – C3 – gA♭2 – F2 – rD2 – C2... These types of chords are covered fully in chapter 3.8, "JI Chord Names Part II".

Because Amin7 and Cmaj6 have the same notes, the min7 chord and the maj6 chord are said to be **homonyns** of each other (a conventional music theory term). This concept is extended to just intonation for two chords containing the same ratios, and hence having the same lattice shape.

The next diagram shows all the tetrads in the last table, indicating homonym pairs with an equal sign:

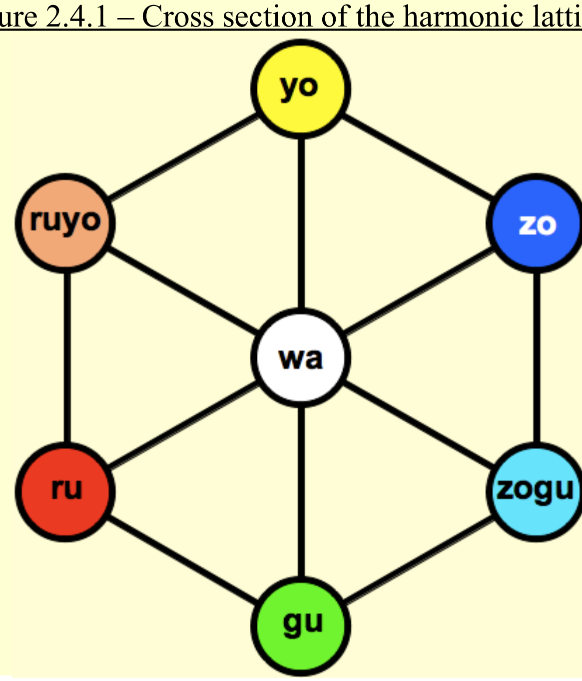


To my ears, the y7 chord, with an odd limit of 15, is considerably more consonant than the s6 chord, odd limit 7. The s6 chord has smaller numbers, but the gu and ru make it far more under and hence more dissonant.

The over colors yo and zo go together, as do the under colors, gu and ru. Imagine the harmonic lattice rotated so that you're looking at the rows edge-on, as in Figure 2.4.1; you can see which colors go with which.

Neighboring colors, colors connected by a line, go together. Mixing non-neighboring colors makes dissonant intervals containing large numbers like 25, 35 and 49, as in the dim7 chords in the next table. These chords are wolves, but not all dissonant chords are wolf chords. Innate comma chords like the aug chord, although dissonant, aren't wolf chords, because there isn't a more consonant chord within a comma of it.

Figure 2.4.1 – Cross section of the harmonic lattice



Diminished seventh chords are innate comma chords with a high odd limit. They're mostly quadricolored and hence non-neighboring.

Table 2.4.5 – Examples of diminished seventh chords (actually minor-6 flat-five chords)

sub-6 zogu-5 chord	min6(♭5)	1, g3, zg5, r6	Cs6(zg5)	wC, gE♭, zgG♭, rA	ZD\] or VJR3	odd limit = 49
zo yo-6 zogu-5 chord	min6(♭5)	1, z3, zg5, y6	Cz,y6(zg5)	wC, zE♭, zgG♭, yA	ZA\P or VGR1	odd limit = 25
wa yo-6 gu-5 chord	min6(♭5)	1, w3, g5, y6	Cw,y6(g5)	wC, wE♭, gG♭, yA	.LI6 or R2BG	odd limit = 75
gu yo-6 gugu-5 chord	min6(♭5)	1, g3, gg5, y6	Cg,y6(gg5)	wC, gE♭, ggG♭, yA	(*not present*)	odd limit = 125

Dominant seventh chords: Two common ya tunings of the dom7 chord have a high odd limit, creating the unstable but not wolfy chords discussed near Figure 2.2.5:

Table 2.4.6 – Ya dom7 chords with a high odd limit

yo gu-7 chord	dom7	1, y3, 5, g7	Cy,g7	wC, yE, wG, gB♭	ZGR3 or .;R6	odd limit = 25
yo wa-7 chord	dom7	1, y3, 5, w7	Cy,w7	wC, yE, wG, wB♭	ZGR2 or .LR6	odd limit = 45

The yo gu-7 chord runs 1/1 – 5/4 – 3/2 – 9/5. The high odd limit isn't obvious when the chord is written this way. It comes from the interval from 5/4 up to 9/5, which is 36/25 = gg5. Likewise yo wa-7 has 5/4 to 16/9 = 64/45 = g5.

Many interesting chords are subsets of tetrads. The simplest ones contain only a fifth:

Table 2.4.7 – "Five" chords (dyads)

5 chord	five chord	1, 5	C5	wC, wG	ZR or .R	power chord
zogu-5 chord	dim five chord	1, zg5	C(zg5)	wC, zgG♭	Z\ or VR	a type of 5 chord

Table 2.4.8 – Chords without a 3rd, but with a 6th or 7th

5 yo-6 chord	maj6, no 3	1, 5, y6	C5y6	wC, wG, yA	ZRP or .GR	
5 ru-6 chord	maj6, no 3	1, 5, r6	C5r6	wC, wG, rA	ZR] or .JR	
5 zo-7 chord	dom7, no 3	1, 5, z7	C5z7	wC, wG, zB♭	ZR1 or .KR	could be written Cz7no3
5 gu-7 chord	dom7, no 3	1, 5, g7	C5g7	wC, wG, gB♭	ZR3 or .;R	
zogu-5 zo-7	half-dim, no 3	1, zg5, z7	C(zg5)z7	wC, zgG♭, zB♭	Z\1 or VKR	homonym of zgG♭y,ry4no5
zogu-5 gu-7	half-dim, no 3	1, zg5, g7	C(zg5)g7	wC, zgG♭, gB♭	Z\3 or V;R	homonym of zgG♭r,ry4no5

Table 2.4.9 – Some fifth-less chords

zo-7 no-5 chord	min7, no 5	1, z3, z7	Cz7no5	wC, zE♭, zB♭	ZA1 or KR1	
gu-7 no-5 chord	min7, no 5	1, g3, g7	Cg7no5	wC, gE♭, gB♭	ZD3 or ;R3	
sub-6 no-5 chord	min6, no 5	1, g3, r6	Cs6no5	wC, gE♭, rA	ZD] or JR3	homonym of rAz(zg5)
zo yo-6 no-5 chord	min6, no 5	1, z3, y6	Cz,y6no5	wC, zE♭, yA	ZAP or GR1	homonym of yAg(zg5)

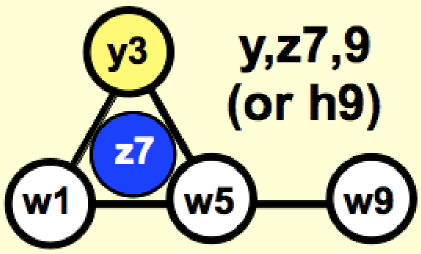
In pentads, a 9th implies a 7th, and an 11th implies a 9th. The 9th is assumed to be wa. In an 11th chord, the 11th is assumed to be the same color as the 7th. But in an add-11 chord, it's assumed to be wa.

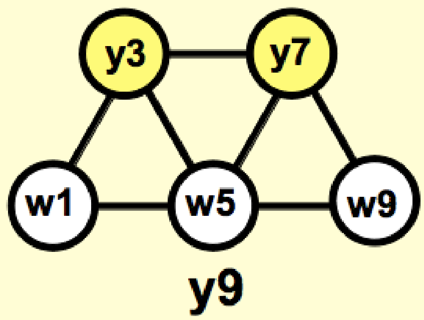
From the discussion of stable and unstable chords near Figure 1.3.15, one might expect pentads to be fairly dissonant. But a wa 9th goes very well with many chords with a major 3rd, and a wa 11th goes well with many minor-3rd chords.

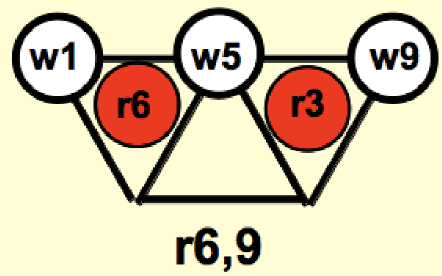
Table 2.4.10 – Chords with 9ths and/or 11ths (in JI Toolkit, use the space bar and the low/high keys to play 9ths/11ths)

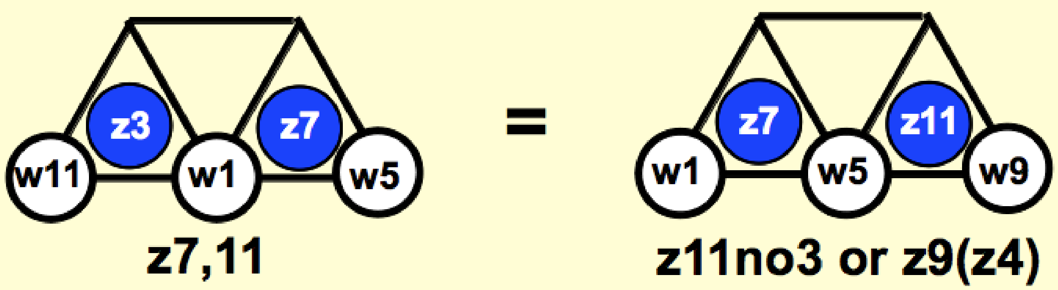
yo add 9 chord	add 9	1, y3, 5, w9	Cy,9	wC, yE, wG, wD	ZGR.	
yo zo-7 9 chord or har-9 chord	9 chord	1, y3, 5, z7, w9	Cy,z7,9 or Ch9	wC, yE, wG, zB♭, wD	ZGR1.	harmonic-series chord
yo-9 chord	maj9	1, y3, 5, y7, w9	Cy9	wC, yE, wG, yB, wD	ZGR6.	
yo-6 9 chord	maj6 + 9	1, y3, 5, y6, w9	Cy6,9	wC, yE, wG, yA, wD	ZGRP.	has a wolf 4th
ru-6 9 chord	maj6 + 9	1, r3, 5, r6, w9	Cr6,9	wC, rE, wG, rA, wD	ZJR].	has a wolf 4th
zo-7 11 chord	min7 add11	1, z3, 5, z7, w11	Cz7,11	wC, zE♭, wG, zB♭, wF	ZAR1L	has a wolf 5th
zo-11 no-3 chord or zo-9 zo-4	11, no 3 (or 9sus4)	1, 5, z7, w9, z11	Cz11no3 or Cz9(z4)	wC, wG, zB♭, wD, zF	ZR1.K	homonym of Gz7,11 has a wolf 4th
gu ru-6 11 chord or sub-6 11 chord	min6 add11	1, g3, 5, r6, w11	Cg,r6,11 or Cs6,11	wC, gE♭, wG, rA, wF	ZDR]L	subharmonic-series chord
ru gu-6 9 chord or sub-9 chord	9 chord	1, r3, 5, g7, w9	Cr,g7,9 or Cs9	wC, rE, wG, gB♭, wD	ZJR3.	homonym of Gs6,11

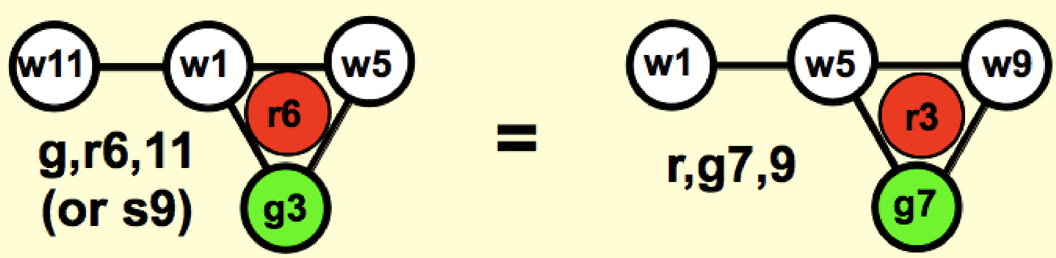
Note the alternate names for the harmonic-series and subharmonic-series chords.











Many pentad chord types (e.g. min7add11) are innate comma chords. The problem with tuning a guitar as described in chapter 2.2 is that the open strings EADGBE form an innate comma pentad. This pentad has many homonyms: G6/9, D6/9sus4, A9sus4, Emin7add11, and even Bmin7add11♯5.

Chords can be written as an **extended ratio** by multiplying every ratio by the denominators:

yo chord = 1 – y3 – 5 = 1/1 : 5/4 : 3/2 = 4:5:6	(multiplying all the ratios by 4)
gu chord = 1 – g3 – 5 = 1/1 : 6/5 : 3/2 = 10:12:15	(multiplying all the ratios by 5 · 2 = 10)
zo chord = 1 – z3 – 5 = 1/1 : 7/6 : 3/2 = 6:7:9	(multiplying all the ratios by 6)

In that last example, we multiplied by 6, not 12, because our goal is to get rid of the denominators, and 6 is all we need to do that. If we had used 12, we would have gotten 12:14:18, which is unnecessarily complicated. Each numerator actually gets multiplied by the least common multiple of all the denominators.

To find the component ratios from the extended ratio, divide each number by the first number and simplify. For example, 4:5:6 becomes (4:5:6)/4 which becomes 4/4 – 5/4 – 6/4 which becomes 1/1 – 5/4 – 3/2.

The extended ratio can sometimes be simplified by inverting it. For example, the ru chord = 1/1 : 9/7 : 3/2 = 14:18:21. It can be written instead as 9/(9:7:6), smaller numbers. The individual ratios are found by pairing each denominator with the numerator, then simplifying. 9/(9:7:6) = 9/9 – 9/7 – 9/6 = 1/1 – 9/7 – 3/2. The inverted extended ratio can be found by inverting each ratio and finding the extended ratio as usual. This extended ratio becomes the denominator, and its first number becomes the numerator. For example, 1/1 : 9/7 : 3/2 becomes 1/1 : 7/9 : 2/3 becomes 9:7:6 becomes 9/(9:7:6).

Many microtonalists refer to a JI chord in any voicing by the extended ratio it has when it's in close position, like so:

<u>triads</u>:	y:	4:5:6
	g:	10:12:15 = 6/(6:5:4)
	z:	6:7:9
	r:	14:18:21 = 9/(9:7:6)
	g(zg5):	5:6:7
	z(zg5):	30:35:42 = 7/(7:6:5)
	sus2:	8:9:12
	sus4:	6:8:9
<u>tetrads</u>:	h7:	4:5:6:7
	y,9:	4:5:6:9
	z,y6:	6:7:9:10
	y6:	12:15:18:20
	g7:	10:12:15:18
	z7:	12:14:18:21
	r6:	14:18:21:24
	s6:	70:84:105:120 = 12/(12:10:8:7)
	r,g7:	70:90:105:126 = 9/(9:7:6:5)
	g7(zg5):	5:6:7:9
	s7:	60:70:84:105 = 7/(7:6:5:4)
	y7:	8:10:12:15
<u>pentads</u>:	h9:	4:5:6:7:9
	s6,11:	210:252:315:360:560 = 12/(12:10:9:8:7)
	s9:	140:180:210:252:315 = 9/(9:7:6:5:4)
	y9:	32:40:48:60:72915 9
	y6,9:	12:15:18:20:27
	r6,9:	28:36:42:48:63 = 36/(36:28:24:21:16)
	z7,11:	12:14:18:21:32
	z11no3:	8:12:14:18:21

When the numbers get above 15, extended ratios become difficult to convert intoru gu-7 component ratios, and should generally be avoided. However, they are useful for concisely showing the exact chord voicing. For example, 4:5:6 can be voiced as 2:3:5 or 3:4:5:6 or 4:5:6:8. See chapter 2.7, "Chord Voicings".

There are many more chords than chord qualities. The next table shows all the chords we've seen, plus a few more, organized by quality. Wolf chords are in (parentheses). Innate comma chord types are in [brackets].

Table 2.4.11 – Chords grouped by quality

major chord	yo chord, ru chord, (large wa chord, yo yo-5 chord, ru ru-5 chord)
minor chord	gu chord, zo chord, (wa chord, wa yo-5 chord, wa ru-5 chord)
sus4 chord	4 chord, (zo-4 chord)
diminished	gu zogu-5, zo zogu-5, (gu gugu-5)
maj diminished	yo ruyo-4 no-5, ru ruyo-4 no-5
maj7	yo-7, ru-7
dom7	yo zo-7, ru gu-7, (yo gu-7, yo wa-7)
maj6	yo-6, ru-6
min7	gu-7, zo-7
min6	zo yo-6, sub-6
min7(♭5)	gu-7 zogu-5, sub-7
[dim7]	(sub-6 zogu-5, zo yo-6 zogu-5, wa yo-6 gu-5, gu yo-6 gugu-5, plus homonyms of these)
add 9	yo add 9, ru add 9
9 chord	har-9, sub-9
maj9	yo-9, ru-9
[maj6 add 9]	(yo-6 9, ru-6 9)
[min7 add 11]	(zo-7 11, zo-7 zo-11, gu-7 11, gu-7 gu-11)
[11 chord, no 3]	(zo-11 no-3, gu-11 no-3)
min6 add 11	sub-6 11, zo yo-6 11
dom7, no 3	5 zo-7, 5 gu-7
maj6, no 3	5 ru-6, 5 yo-6
min7(♭5), no 3	zogu-5 zo-7, zogu-5 gu-7
min7, no 5	zo-7 no 5, gu-7 no 5
min6, no 5	zo yo-6 no 5, sub-6 no 5

In chapter 2.2, I said that the definition of a wolf interval depends on the culturally accepted prime limit. The definition of wolf chords and innate comma chord types also depends on the prime limit. The table above assumes yaza. But in 3-limit JI, w3 and Lw3 aren't wolf 3rds, and the w and Lw chords, although unstable, are not wolf chords. The only wolf intervals are far-flung ones that span at least 7 fifths. Thus the only innate comma chord types in 3-limit are those that can't be constructed from the 7 natural notes. For example, the aug chord, the min-maj chord and the dim7 chord.

Innate comma chords can be tuneable in higher prime limits. For example, 11-limit provides a JI tuning for the aug chord, and 17-limit tunes the dim7 chord. See chapter 3.9.

In chapter 2.7, we'll see how certain voicings can tame an innate comma. Even so, when composing in JI, I find myself avoiding innate comma chords. When working in a temperament, they are more useable. This illustrates Kyle Gann's dictum that "Because it determines what sounds good, tuning has a pervasive influence on compositional tendencies" (www.kylegann.com/histune.html).

Chapter 2.5 – Chord Progressions, Scales, Keys and Modulations

In chord progressions, the root of the chord is indicated by a color before a note name or scale degree. In absolute notation, if the chord root is wa, the color can be omitted, e.g. Cy means wCy. In relative notation, this only applies to the perfect scale degrees I, IV and V, thus Iy means wIy. Imperfect scale degrees II, III, VI and VII require colors: wIIIy not IIIy. This avoids the root of the III chord changing from M3 in conventional notation to m3 in color notation.

For example, C – Am – F – G7 becomes Cy – yAg – Fy – Gh7, or "C yo, yo-A gu, F yo, G har-seven". In relative notation, I – VIm – IV – V7 becomes Iy – yVIg – IVy – Vh7, or "one yo, yo-six gu, four yo, five har-seven". Roman numerals are always upper-case ("VIm" not "vi") to avoid confusion with the down symbol "v", see chapters 5.5 and 5.8. This progression uses wa and yo roots. Root colors are a big part of the feel of a chord progression. Every progression must have at least one wa root, since the tonic is always wa.

In conventional notation, the notes of a chord (e.g. E minor) are determined by the chord's root (E) and the chord quality (minor). The chord quality is a recipe for constructing the chord. Each component of the chord is always a specific interval from the root (unison, min 3rd, perf 5th). To find a chord's notes, add each interval to the root to get a new note. E + m3 = G and E + P5 = B, so E minor is E, G and B.

Color notation works the same way. For any chord (e.g. yEg), the chord structure (g) is the recipe. Add the intervals (w1, g3, w5) to the root (yE) to get the notes: yE + g3 = wG and yE + w5 = yB, so yEg is yE, wG and yB.

More examples from pop music, all in D:

Table 2.5.1 – Examples of chord progressions

Iy – yIIy – IVy – Iy	Dy – yEy – Gy – Dy	“You Won't See Me” verse (The Beatles)
Iy – IVy,g7 – Iy – IVy,g7 – Iy – wIIy – Vy – Iy	Dy – Gy,g7 – Dy – Gy,g7 – Dy – Ey – Ay – Dy	“Brain Damage” verse (Pink Floyd) (a different II chord than in the previous example)
Vy – yIIIy – yVIg – IVy – I	Ay – yF♯y – yBg – Gy – Dy	“Tears of a Clown” chorus (Smokey Robinson)
Ig – gVIIy – gVIy – gVIIy	Dg – gCy – gB♭y – gCy	“All Along The Watchtower” (Bob Dylan)
Ig – gIIIy – gVIIy – gIVy – gIg	Dg – gFy – gCy – gGy – gDg	“Boulevard Of Broken Dreams” verse (Green Day)

The last one is an example of a comma pump, which changes the tonic from wa D to gu D. More about that in Part IV.

Sometimes, the magnitude of the root needs to be specified with "L"or "s". Dy – F♯y – By – Ey – Ay – Dy would be written Iy – LwIIIy – wVIy – wIIy – Vy – Iy.

Here's a simple method for finding interesting chord progressions: proceed from chord to chord so that the new chord has exactly two notes in common with the old one. But if the root moves by a 4th or a 5th, there can be either one or two notes in common.

For example, from the Ih7 chord, you could go to any IV chord: IVh7, IVz7, IVs6, IVr,g7, IVg7, IVy7, etc. You could go to any V chord except Vz,y6, because that chord has three common notes with Ih7. For non-wa motion, yVIs6 or yVIg7(zg5) or gVIy7 or zIIIs6 or zIIIr,g7 or zVIIr6 would all work. However, yVIg7 or yIIIg7(zg5) would have too many common notes.

You can create a simple song by alternating between two such chords. Their notes will generally create a 6 or 7 note scale. Or you can modulate quite far by stringing together a number of such chord changes.

This method suggests some unusual progressions: Ig7 to gIIIz7, or Ih7 to zyVs6. There's also the kind of chord change in which the root doesn't change but the colors do. These work well with up to three common notes and just one shifting note, as in the classic Iy to Iy7 to Ih7. This also works with two common notes and two shifting notes.

Like chords, chord progressions can be classified by the number of colors they contain. Presumably, fewer colors creates more coherence. For example, compare Ig,r6 – Vy,z7 to Ig,r6 – Vr,g7. While the r,g7 chord is more dissonant than the y,z7 chord, it makes the overall progression tricolored rather than quinticolored, and perhaps more consonant as a whole.

Scales are loosely named after the colors of their notes. If a scale consists of only wa notes, it's called a wa scale, otherwise wa is assumed to be present and not mentioned. Because you can make different scales out of the same colors, these scale names are not unique. Hence there are several versions of yo major.

Here's some example scales; as always, 1, 4 & 5 are assumed to be wa. Minor is Aeolian or natural minor.

Table 2.5.2 – Examples of scales

wa major scale	1, w2, Lw3, 4, 5, w6, Lw7
wa minor scale	1, w2, w3, 4, 5, sw6, w7
yo major scale	1, w2 or y2, y3, 4, 5, y6, y7
gu minor scale	1, w2, g3, 4, 5, g6, g7 or w7
zo minor scale	1, w2, z3, 4, 5, z6, z7 or w7
ru major scale	1, w2 or r2, r3, 4, 5, r6, r7

Pentatonic scales: The next table is somewhat similar to the list of tetrads in table 2.4.4. In fact, the last four scales are named after the I tetrad they contain.

Table 2.5.3 – Examples of pentatonic scales

wa minor pentatonic	1, w3, 4, 5, w7
wa major pentatonic	1, w2, Lw3, 5, w6
yo pentatonic	1, w2 or y2, y3, 5, y6
gu pentatonic	1, g3, 4, 5, g7
zo pentatonic	1, z3, 4, 5, z7
ru pentatonic	1, w2, r3, 5, r6
yo zo pentatonic	1, w2, y3, 5, z7
zo yo pentatonic	1, z3, 4, 5, y6 (a mode of yo zo pentatonic)
gu ru pentatonic	1, g3, 4, 5, r6
ru gu pentatonic	1, w2, r3, 5, g7 (a mode of gu ru pentatonic)

For example, “Ash Grove” uses yo & wa notes and has a yo scale. “La Bamba” uses mostly yo and wa notes, but it has a V7 chord, and the melody uses that 7th heavily. If the V7 chord is intoned Vy,w7, the scale is yo. If it's Vy,g7, the scale is yo gu. If it's Vy,z7, it's yo zo.

To write out absolute scales, just add letters. “La Bamba” in A if using Vy,z7: wA wB yC♯ wD/zD wE yF♯ yG♯

Scales can also be classified by the number of colors they contain, for example, the yo zo scale is tricolored. The intervals between a scale's degrees will use more colors than the scale itself. For example, a yo scale contains y3 and w5, and the interval between y3 and w5 is gu. The yo zo scale will contain gu, ru, zogu and ruyo intervals between the scale degrees.

The key of a song is the note name plus the color(s) of the scale: B gu, D yo zo, etc. Like chords, keys can be classified as bicolored (A gu), tricolored (B♭ yo zo), etc.

Analogous to the relative and parallel major or minor, one can modulate to relative gu, parallel ru, etc. Modulating from a yo key to the relative gu means using gu chords on yo roots. Modulating from yo to the parallel gu means using gu chords on wa roots. Going from yo zo to the relative gu means using chords with gu and/or ru in them on yo roots. Going to the relative ru means using the same chords on zo roots. Going from yo zo to the parallel gu ru means using the same chords on wa roots. One can also modulate 4thwd or 5thwd. Modulating from a yo key to the relative gu, then from there to the parallel yo is modulating yoward. Likewise, there's guward, zoward, etc.

Table 2.5.4 – Examples of relative modulation

modulation	verse	chorus or bridge
A gu to gu-C yo	Ag – gGy – gFy – Eg – gGy	gCy – gGy – gFy – gGy ("Like a Hurricane")
A zo to zo-C ru	Az – Dz – Ez – Az	zCr – zFr – zGr – zCr
A yo zo to zo-C gu ru	Ah7 – Dh7 – Eh7 – Ah7	zCs6 – zGs6

Table 2.5.5 – Examples of parallel modulation

modulation	verse	chorus or bridge
D yo to D gu	Dy – Cy – Gy – Dy	Dg – Gy – Dg – Eg – Ay ("Norwegian Wood")
D yo zo to D gu ru	Dh7 – Gh7 – Ah7 – Dh7	Ds6 – As6

Tuning tip: If you have a yo zo scale, and you want to modulate to the relative gu, tune your tritone and semitone ruyo, so that you can use them in your gu chords (e.g. yVIs6 which uses ry4, and yIIIs6 which uses ry1). If you want to go to the relative ru, tune them zogu (for zIIIs6 using zg5, or zIIIr,g7 using zg2).

Some songs, for example "And I Love Her" (The Beatles) and "El Condor Pasa" (Simon & Garfunkel), flip between relative major/minor so fluidly that it's hard to define the tonic, and hence the key or scale. While conventional music notation can duck the issue via absolute notation, color notation unfortunately forces us to take a stand. We must pick a tonic and make it wa.

Bob Dylan's "Simple Twist of Fate" in standard notation, using slash chords to indicate the descending bass line:

C – – – Em/B – – – C/B♭ – – – F/A – – – Fm/A♭ – – – C/G – F – C – G – C – – –

Each dash is one beat. In color notation, the bass note's color is relative to the scale's tonic, not the chord's root.

Cy – – – yEg/yB – – – Cy/wB♭ – – – Fy/yA – – – Fg/gA♭ – – – C/wG – Fy – Cy – Gy – Cy – – –

There are two ways to write slash chords in relative notation. One way is with each bass note's color and degree relative to the tonic. The bass note is written as a roman numeral, since that's how scale degrees are sometimes written:

Iy – – – yIIIg/yVII – – – Iy/wVII – – – IVy/yVI – – – IVg/gVI – – – Iy/V – IVy – Iy – Vy – Iy – – –

Iy/V means Iy/wV. The color of the bass note can be omitted for wI, wIV and wV, which are assumed to be wa.

Alternatively, the color and the degree can be relative to the chord root. It's vertical, not horizontal: it tells you more about the sound of the chord, but makes it harder to read the bass melody. The bass note is written as an arabic (non-roman) number, since that's how chord components are written:

Iy – – – yIIIg/5 – – – Iy/w7 – – – IVy/3 – – – IVg/3 – – – Iy/5 – IVy – Iy – Vy – Iy – – –

The bass note's color is omitted if the note is already present in the chord. yIIIg/5 = yIIIg/w5, IVy/3 = IVy/y3 and IVg/3 = IVg/g3. As before, "/4" would imply a w4. If the number > 7, the slash means "add", e.g. Iy6/9 = Iy6add9.

Both methods are useful, but the second method is more logical, with everything up to and including the first roman numeral relative to the tonic, and everything after that roman numeral relative to the root. It also allows discussions of the sound of inversions in the abstract by omitting the root, for example y/3 vs. g/3, or y/3 vs. y/5. Omitting the chord colors as well, "/3" indicates a 1st inversion, "/5" a 2nd inversion, and "/6" or "/7" a 3rd inversion.

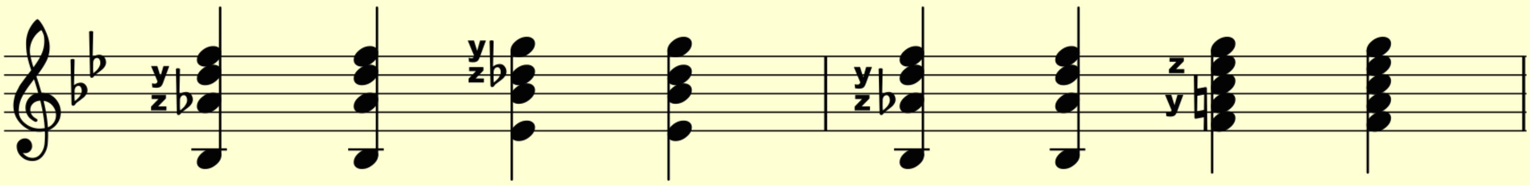
One final possibility is to combine this method with absolute notation:

Cy – – – yEg/5 – – – Cy/w7– – – Fy/3 – – – Fg/3– – – C/5 – Fy – Cy – Gy – Cy – – –

Chapter 2.6 – JI Staff Notation

All notes in staff notation are assumed to be wa. Every non-wa note is marked with a color accidental like z, g, ry, etc. Below is Ih7 – IVh7 – Ih7 – Vh9 in B♭. The music is spread out horizontally, to make room for the extra accidentals.

Figure 2.6.1 – Color notation on the staff

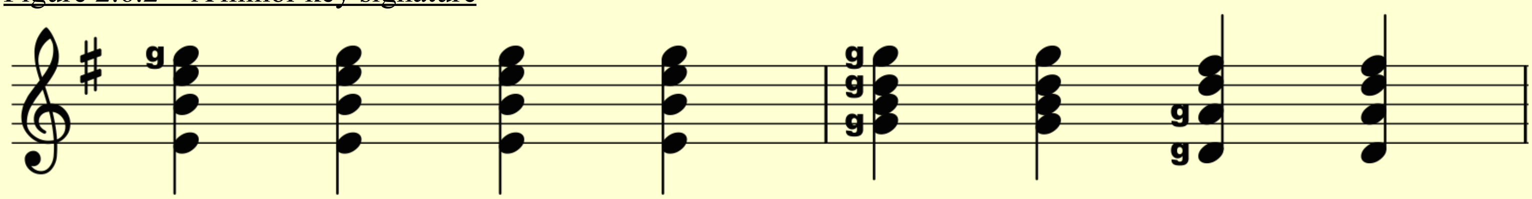


Like conventional accidentals, color accidentals carry over. For example, the the yo accidental on the 1st D note in the 1st measure applies to the 2nd D note too. Unlike conventional accidentals which apply to a note (e.g. A), color accidentals only apply to one specific "version" of that note (e.g. A flat or A natural). For example, the yo accidental in the first chord applies to all the D naturals in that measure but not to the D flats.

With handwritten scores, care must be taken that "y" doesn't look like an eighth rest, and "z" doesn't look like a quarter rest. An eighth rest should be drawn with one pen stroke, and "y" with two strokes. "z" should be drawn with the top and bottom lines slanting up slightly, like the horizontal lines in a sharp symbol do.

Minor key signatures are distinguished from the relative major key signature by the location of the wa notes. The next example is in E minor, not G major, because the E notes are wa and the the G notes aren't.

Figure 2.6.2 – A minor key signature



The two G notes in the 1st chord of the 2nd measure both require color accidentals because they are in different octaves. Like conventional accidentals, color accidentals only apply to one octave.

Large and small are implied by the color accidentals and conventional accidentals. In G, a wa B natural must be large, because the only wa major 3rd is the large wa 3rd. One wouldn't write "L" or "s" next to the notes just as one wouldn't write "major" or "minor". Magnitude is used only in relative notation, never in absolute notation.

To avoid clutter, one can use a **color signature**. It's analogous to a key signature, which defines a default accidental for each of the 7 notes. The color signature defines a default color for each version of the 7 notes.

This example uses 11 pitches: wB♭, wC, zD♭, yD, zE♭, wE♭, wF, yG, wG, zA♭ and yA. Because there are two E flats and two G naturals, the color signature only has 9 notes. When there's a color signature, color accidentals are only used for exceptions to the color signature, resulting in far fewer color accidentals:

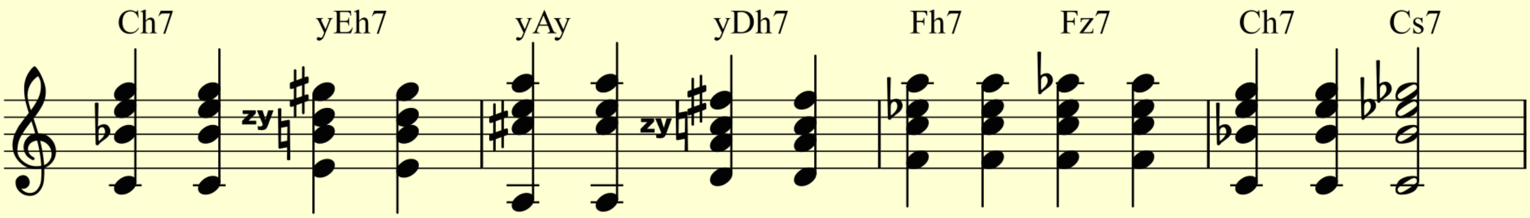
Figure 2.6.3 – A color signature example

Color signatures are most helpful for simple pieces that don't modulate much. Otherwise, a color signature only increases the amount of mental work needed to read the score. In general, only use a color signature if it would reduce the number of color accidentals by at least fourfold.

Highly chromatic music may have more than 12 notes in the color signature, as in the next example, which has both F♯ and G♭, and both G♯ and A♭.

Figure 2.6.4 – A color signature of more than 12 notes

Tuning: wC, yyC#, yD, zE♭, yE, wF, yyF#, zgG♭, wG, yyG#, zA♭, yA, zB♭, yB

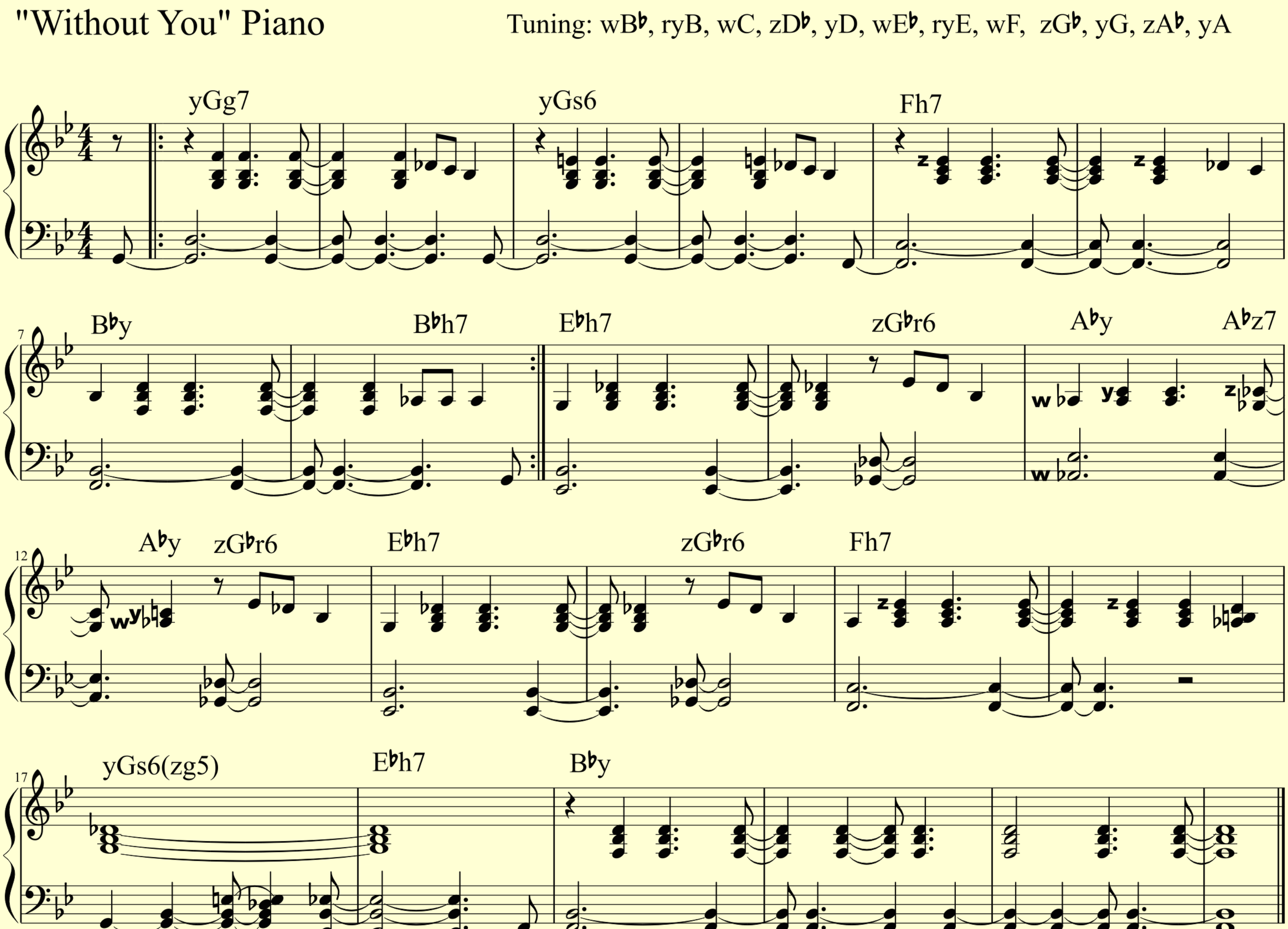


These staff notation examples were made with the free open-source MuseScore notation software. Color accidentals are made by putting fingerings on the notes, then editing the fingering text. Color accidentals can be copied from one note and pasted onto other notes. The font used is Arial Black.

Figure 2.6.5 – "Without You", a piece in B♭ that uses a 15-note scale

Below, the same piece, using a color signature. There are far fewer color accidentals. There would be none at all, if there were a second color signature for the B section.

Figure 2.6.6 – "Without You" with a color signature

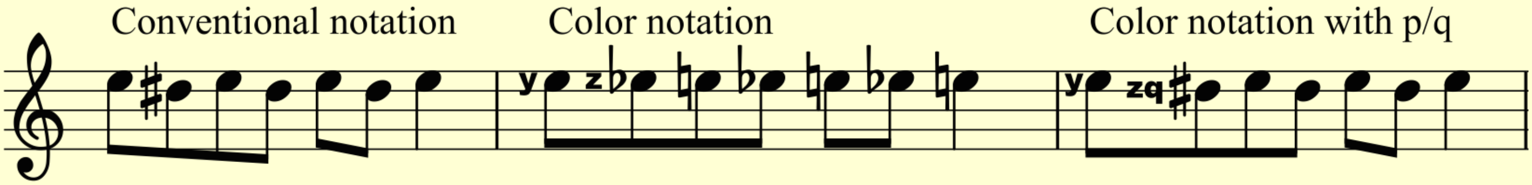


In conventional notation, a key change e.g. from C to D is clear, because there is only one D. But in color notation, there are many D's. When the key signature changes from zero to two sharps, what color is the new D? The new tonic is always assumed to be wa. Thus the new D would be the wa D from the old key. In terms of intervals, the new tonic is a w2 up from the old one. But music often modulates by a non-wa interval. For example, you might want to modulate from C major to A♭ major, with the old C becoming the yo 3rd of the new A♭. In this case, the modulation is not to the wa A♭, but to the gu A♭.

Non-wa modulations are indicated on the staff in a way that is analogous to metric modulation. When the time signature changes, there is sometimes a marking such as (♩ = ♩ .) right above the barline. The convention is "old = new", so the equals sign can be read as "becomes". The amount of time that the quarter note used to take is the amount of time that the dotted quarter note will take. Applying this convention to color notation, one would write (gA♭ = wA♭) above the new key signature. The pitch that was notated as gA♭ will henceforth be notated as wA♭.

In conventional notation, a trill between two notes a semitone apart is usually written using a diatonic semitone (i.e., a minor second), as in the first measure, to avoid the clutter of numerous accidentals:

Figure 2.6.7 – Trills in both conventional and color notation



However, if the two notes being trilled are yE and zE♭, the trill must unfortunately be written using a chromatic semitone, as in the second measure, because the zo E♭ is not the same as the zo D♯. (It's a wa comma flatter.) Likewise, a chromatic run is conventionally written to minimize accidentals: C C♯ D D♯ E, not C D♭ D♮ E♭ E♮. But depending on the exact pitches desired, color notation may require the second construction.

The 4th note in the 1st measure is sharp, because it inherits its accidental from the previous D. Likewise, the 3rd note in the 2nd measure is yo, because it inherits its color from the previous E♮. Remember, color accidentals only apply to one specific version of a note.

Although we are used to equating E♭ and D♯, historically they have been two different notes. Until the advent of well-temperaments around the time of Bach, there was no circle of fifths. Instead, there was a chain of 11 fifths. The twelfth fifth was actually a diminished 6th, and it was an unplayable wolf interval. In the 3-limit tuning of medieval times, it was about 678¢, and in the quarter-comma meantone tuning of later times, it was about 738¢.

There is an optional extension to color notation that solves the trill problem. Just as the g and y accidentals add or subtract the gu comma 81/80, the **p** and **q** accidentals add/subtract the wa comma LLw-2 = (-19, 12). This has the effect of changing the degree, since the wa comma is a negative 2nd. Mnemonics: the p stands for pythagorean (wa) comma, and the q is a backwards p. The long forms of p and q are **po** and **qu** ("ku"), because po is 3-over and qu is 3-under. The 2nd note in the last measure is a zoqu D♯. Po and qu can appear in the color signature. Popo and ququ are possible but very unlikely. Po and qu allow color notation to do everything Sagittal notation does (see appendix 3).

Po and qu can be used in relative notation too. In absolute notation, adding a gu comma to a note simply adds a "g" to its name, and vice versa: wC + g1 = gC. In relative notation, this is sometimes true: w7 + g1 = g7. But sometimes it also adds an "L": w2 + g1 = Lg2. Because Lg2 = g2 + Lw1, it follows that g2 = w2 + g1 - Lw1. Thus adding "g" to an interval's name <u>without changing the magnitude</u>, e.g. changing w2 to g2, adds g1 but sometimes also subtracts Lw1. The wa semitone is subtracted because the magnitude is calculated <u>after</u> the various commas are added or subtracted.

Adding a wa comma to a note merely adds a "p", and vice versa: wC + LLw-2 = pC = wB♯. But adding a wa comma to an interval always adds one or two "L"s to its name. Therefore adding "p" to an interval's name without changing the magnitude always means adding the wa comma and subtracting one or two wa semitones. Thus <u>adding</u> the comma paradoxically results in <u>subtracting</u> either w2 or sw2. For example, y3 + LLw-2 = LLy2 = LLyp3, thus yp3 = y2. Likewise adding qu means adding w2 or sw2: z2 - LLw-2 = ssz3 = sszq2 and zq2 = z3. Therefore the only effect of adding po or qu to an interval is to change the degree. Thus every 3rd is also a po 4th and a qu 2nd. Mnemonic: poor intervals get decreased, but cool intervals get increased. A large or small interval remains large or small after applying po or qu, e.g. Lw3 is Lp4. (Not Lwp4, the "w" is omitted because po and qu intervals are wa.)

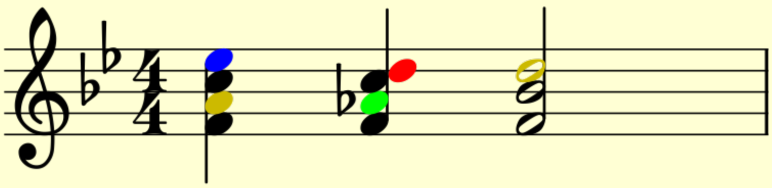
Po and qu can be used to avoid awkward chord spellings, by respelling an aug 4th or a min 6th as a 5th. For example, w1 – y3 – ry4 is a y,ry4no5 chord. But ry4 can be spelled ryp5, making a y(ryp5) chord. The name gets slightly shortened from "yo, ruyo four, no five" to '"yo, ruyopo five". Another example: Cy(gq5) is a Cy,g6no5 chord.

Po and qu can also be used to respell a scale so that each note has its own degree. For example, Erv Wilson's hexany w1 ry1 y3 ry4 w5 r6 can be spelled w1 ryp2 y3 ry4 w5 r6. Confusingly, the interval from ryp2 to ry4 is a qu 3rd which is actually a wa 4th. Another use is to rename a negative 2nd as a po unison, e.g. rry-2 becomes rryp1.

Po and qu give every ratio one or two additional names, and breaks the one-to-one correspondence between ratios and color names. Like color signatures, they can increase the amount of mental work needed to read the score, and should be used with caution.

It's possible to use actual colors in the notation, as in this example:

<u>Figure 2.6.8 – Color notation with actual colors</u>



Black is used for wa notes, and mustard-yellow for yo. While it certainly looks striking, there are several drawbacks to this method. It's hard to see the yellow, especially in whole and half notes. It's hard for people with color vision deficiency to read. Synaesthetes may find it off-putting. The biggest drawback is that there's a limit to how many different colors can be represented visually. For example, it would be hard to distinguish between yo and yoyo, or between ruyo and ruruyo and ruyoyo. The latter problem could be solved by each note head becoming a miniature pie chart, with slices of red and yellow. But creating these note heads would be very tedious.

One more possibility: color accidentals can be avoided entirely by defining several tunings that cover all the pitches used, and specifying when to switch between them. The composer does some of the planning for the performer. However, there are often several ways to retune a complex progression. For example, "I Hear Numbers" goes:

> Ch7 – zE♭s6 – Fh7 – Ch7 – zE♭s6 – Fh7 – B♭h7 – B♭h7/yD – zE♭s6 – zA♭r,g7 – Gh7

D, F and B♭ need retuning. If your retuning software lets you switch among only two tunings, set them up like so:

> Tuning #1: wC, zgD♭, wD, zE♭, yE, zF, zgG♭, wG, zA♭, yA, zB♭, yB
> **Tuning #2: wC, zgD♭, yD, zE♭, yE, wF, zgG♭, wG, zA♭, yA, wB♭, yB**

Then switch between them as you play like so (**bold** = tuning #2):

> Ch7 – zE♭s6 – **Fh7** – Ch7 – zE♭s6 – **Fh7 – B♭h7 – B♭h7/yD** – zE♭s6 – **zA♭r,g7** – Gh7

The staff notation would have the two tunings written out at the top of the page, and would indicate the places to switch in the score. These tunings reflect the basic sideways motion of the progression, with tuning #1 being fifthward and tuning #2 being fourthward. On the other hand, if you can access 3 different tunings, this method will involve less switching:

> Tuning #1: wC, zgD♭, yD, zE♭, yE, wF, zgG♭, wG, zA♭, yA, zB♭, yB
> **Tuning #2: wC, zgD♭, yD, zE♭, yE, wF, zgG♭, wG, zA♭, yA, wB♭, yB**
> <u>Tuning #3: wC, zgD♭, wD, zE♭, yE, zF, zgG♭, wG, zA♭, yA, zB♭, yB</u>

Bold = tuning #2, <u>underlined</u> = tuning #3:

> Ch7 – zE♭s6 – Fh7 – Ch7 – zE♭s6 – Fh7 – **B♭h7 – B♭h7/yD – zE♭s6 – zA♭r,g7** – <u>Gh7</u>

Here tuning #1 is central, tuning #2 is fourthward and tuning #3 is fifthward. Since the composer doesn't know what kind of retuning software the performer's using, multiple-tunings notation isn't a very practical method. Even if use of alt-tuner is assumed, the composer doesn't know how many switching pedals the performer is using. It's usually better to let the performer decide these things.

Chapter 2.7 – Chord Voicings

Chord voicings are of great importance in JI. 12-ET has a built-in out-of-tune-ness that is always present. So a well-voiced chord can only sound so good, and the difference between voicings can only have so much effect. In JI, certain chords sound so smooth when well-voiced that the contrast with a poorly-voiced one can be quite jarring. And some yaza chords can be quite dissonant, and the proper voicing is often needed to make it sound acceptable.

Although color notation makes knowing ratios less important, a working knowledge of the simpler ratios is essential for understanding chord voicings. Let's review the ratios most commonly used in chords, which mostly come from the central hexagon in the harmonic lattice, plus a few from the fifthward side.

Table 2.7.1 – Ratios commonly used in chords, octave-reduced

z3	7/6	m3 - 33¢	267¢
g3	6/5	m3 + 16¢	316¢
y3	5/4	M3 - 14¢	386¢
r3	9/7	M3 + 35¢	435¢
w4	4/3	P4 - 2¢	498¢
zg5	7/5	d5 - 17¢	583¢
w5	3/2	P5 + 2¢	702¢
y6	5/3	M6 - 16¢	884¢
r6	12/7	M6 + 33¢	933¢
z7	7/4	m7 - 31¢	969¢
g7	9/5	m7 + 18¢	1018¢
y7	15/8	M7 - 12¢	1088¢
w8	2/1	P8	1200¢
w9	9/4	M9 + 4¢	1404¢

Other ratios, like the gu minor 6th g6 = 8/5, will appear as inversions of these.

Comparing chord voicings means measuring a voicing's consonance. There is no exact measure of consonance and dissonance for a chord or even for an interval. The best we can do is discuss some of the factors involved. In chapter 3.1 I'll talk about an inexact measure I've found useful.

Certainly the timbre of the sounds has an effect on consonance. The difference between a yo 5th and a wa 5th is striking on a piano, but not as much when played with a really dirty electric guitar sound, or a buzzy electronic sound, or on an inharmonic idiophone like marimba or gamelan. The subtleties of JI are not as apparent with these timbres. I tend to like natural acoustic sounds, so I'm going to assume a harmonic timbre like voice, guitar or piano for this discussion.

Obviously the register also has an impact. Minor 3rds sound muddy in the bass but fine higher up. Let's assume the entire chord is roughly centered on middle C.

So, assuming a harmonic timbre and a middle register, how do voicings affect consonance? Let's review the math.

Widening a ratio by an octave means halving the denominator, if it's even. Otherwise, double the numerator. Inverting a ratio (moving the upper note down an octave) means flipping it, then widening it (6/5 becomes 5/6 becomes 5/3). You can confirm this by playing an interval in various voicings and watching alt-tuner's interval display.

When we discuss ratios, we tend to assume octave equivalence. 5/4 refers to both the major 3rd and the major 10th. But each ratio actually implies many others through inverting and/or widening, as this chart shows:

Table 2.7.2 – A few simple ratios, and the ratios they imply through octave equivalence

octave	2/1	4/1	8/1	16/1
wa 5th	3/2	3/1	6/1	12/1
	4/3	8/3	12/3	24/3
yo 3rd	5/4	5/2	5/1	10/1
	8/5	16/5	32/5	64/5
zo 7th	7/4	7/2	7/1	14/1
	8/7	16/7	32/7	64/7

Notice that some ratios get smaller as they grow wider (up to a point, at least) and some get bigger. As noted in chapter 1.2, the larger the numbers in a ratio, the more dissonant it is. The yo 3rd 5/4 sounds better as a yo 10th = 5/2, because the ratio's numbers are smaller. Same with the zo 3rd = 7/6 vs. the zo 10th = 7/3. But the gu 3rd 6/5 voiced wider becomes a gu 10th 12/5. Larger numbers, hence more dissonant. Only ratios with an even number on the bottom are improved with a wider voicing.

Chord voicings often span 3 or 4 octaves. Looking at wider intervals, the yo 10th = 5/2 sounds even better widened to a yo 17th = y17 = 5/1. Seventeenth?!? The conventional terminology gets really awkward, so we'll use the term **wide** for any interval that has been widened by an octave. A y10 is a wide yo 3rd, written Wy3. (Upper-case W means wide, lower-case w means wa.) 5/1 is a **double-wide** yo 3rd, written WWy3. The double octave 4/1 is written as either Ww8 or WWw1, and the triple octave 8/1 is W^3w1. A 16th like w16 = 9/2 is either a wide 9th = Ww9 or a double-wide 2nd = WWw2. Note that the ruyo double octave and the double ruyo octave are two different intervals:

15/7 = ry8 = ruyo octave = ry4 + w5
30/7 = Wry8 = ruyo double octave = ry8 + w8
225/98 = rryy8 = double ruyo octave = ry4 + ry5
225/49 = Wrryy8 = double ruyo double octave = ry4 + ry5 + w8

Another factor in consonance is the overness or underness of a ratio. Prime factors larger than 3 tend to sound better when they're on the top of the ratio. An obvious example: y3 = 5/4 is more consonant than g3 = 6/5, and the yo chord is more consonant than the gu chord.

The gu 3rd 6/5 sounds better inverted to a yo 6th 5/3. Smaller numbers, plus less under. Inverting a ratio changes not only its number size, but also its color and hence its underness. With 6/5, the two changes happen to reinforce each other, but sometimes they don't. For example, the ru 3rd 9/7 inverts to the zo 6th 14/9. Bigger numbers, but less under. It's not obvious which one sounds better; they're both fairly dissonant.

A third consideration has to do with the overtones of the harmonic series of each note. Play a wide zo 6th Wz6 = 28/9 = 1965¢. The lower note has a generally very audible overtone at 3/1, the wide wa 5th. It's only a zo 2nd = 63¢ flat of the interval's higher note and clashes with it. The Ww5 = 3/1, with very small numbers and also very over, is such a powerful interval that it usurps any nearby interval. Even voicing the z6 as a 6th, there is still a 63¢ clash between the lower note's 3/1 overtone and the higher note's 2/1 (octave) overtone.

The zo 6th inverts to the ru 3rd. If widened to Wr3, the upper note will have an audible octave overtone that clashes with the 5/1 overtone of the lower note by rg1 = 49¢. And the double-wide ru third positively howls, being usurped by WWy3 = 5/1.

The clash is less evident in a higher register, since the overtones of higher notes are quieter. Therefore the best voicing for the ru 3rd is in close position in a mid-to-high register. In this range, it sounds better than the zo 6th. That's because the overtone clash for r3 occurs 5/1= WWy3 above the lower note, but for z6 it occurs only 3/1 = W5 above.

The "clash zones" are near ratios with one or two on the bottom: 3/2, 2/1, 5/2, 3/1, 7/2, 4/1, 9/2, 5/1, 6/1, 7/1, 8/1, 9/1, etc. The effect depends on the timbre. My keyboard's piano sound has an audible 11th overtone in the lower notes. So if I play C2 C3 F5, I get a 53¢ clash. The outer interval is W^3w4 = 32/3, which clashes with the 11th overtone 11/1. The chord would actually sound better with a sharper 4th. (More on this type of 4th in chapter 3.6.) It would also sound better with a zo 4th, even though that one is usually more dissonant than the wa one. That's because the triple-wide zo 4th W^3z4 = 21/2 has smaller numbers than the triple-wide wa 4th W^3w4 = 32/3. Super-wide intervals often sound better a comma flatter or sharper than they would octave-reduced. For example, y6 = 5/3 sounds better than w6 = 27/16, but W^3y6 = 40/3 sounds worse than W^3w6 = 27/2. In other words, whether an interval is a wolf interval or not depends on the voicing.

The biggest, strongest "clash zone" of all is near 1/1, the unison, which makes 2nds generally sound better as 9ths. For example w2 = 9/8 sounds much better voiced as w9 = 9/4. It clashes with w8 = 2/1 instead. However, some 2nds are not improved with widening. The y2 = 10/9 widened becomes y9 = 20/9, even bigger numbers. It no longer clashes with 1/1, but now clashes with 9/4 by only 22¢.

More examples: g7 = 9/5 sounds worse inverted to y2 = 10/9. It becomes less under, but it has bigger numbers, and it clashes with 1/1. But widening it to the yo 9th makes 20/9, even worse. And widening g7 to Wg7 = 18/5 also makes bigger numbers. So there is no better voicing for g7.

The zg5 = 7/5 isn't improved by widening or inverting. The inversion is ry4 = 10/7, with bigger numbers. Also, although 10/7 is 5-over, 10/7 is under because the biggest prime is in the bottom of the ratio.

Considering number size, underness, and clash, here are the best voicings for each commonly used interval. If the best voicing is more than an octave wide, narrower alternatives are also listed.

Table 2.7.3 – Ratios commonly used in chords, with the most consonant voicing **bolded**

octave-reduced	widened	inverted
z3 = 7/6	**Wz3 = 7/3**	
g3 = 6/5		**y6 = 5/3**
y3 = 5/4	Wy3 = 5/2, **WWy3 = 5/1**	
r3 = 9/7		
w4 = 4/3		w5 = 3/2, **Ww5 = 3/1**
zg5 = 7/5		
w5 = 3/2	**Ww5 = 3/1**	
y6 = 5/3		
r6 = 12/7		z3 = 7/6, **Wz3 = 7/3**
z7 = 7/4	Wz7 = 7/2, **WWz7 = 7/1**	
g7 = 9/5		
y7 = 15/8	Wy7 = 15/4, WWy7 = 15/2, **W^3y7 = 15/1**	
w8 = 2/1		**w1 = 1/1**
w9 = 9/4	Ww9 = 9/2, **W^3w2 = 9/1**	

Moving on to chords: we start by looking at the intervals between each pair of notes of the chord. This is not a good measure of a chord's consonance compared to other chords. After all, both the yo triad and the gu triad in a close voicing have the same intervals (y3, g3 and w5), but the yo triad is much more consonant. But it's not a bad way to compare two voicings of the same chord.

The yo triad in close position (1st – 3rd – 5th) has three intervals, 5/4, 6/5 and 3/2. If voiced openly (1st – 5th – 10th), it has 3/2, 5/3 and 5/2. Smaller numbers and less under, thus more consonant.

These changes can be seen in the extended ratio, with 4:5:6 becoming 2:3:5. The numbers clearly become smaller. The 5 moves to the end of the extended ratio, analogous to moving from the bottom of a ratio to the top, and the chord becomes less under.

Open voicings are not always better. A 1st – 3rd – 5th gu triad contains 6/5, 5/4 and 3/2. In 1st – 5th – 10th voicing, we get 3/2, 8/5 and 12/5, which is worse. The extended ratio 10:12:15 becomes 10:15:24. The numbers become larger. Both numbers with 5-factors, 10 and 15, are at the start of the extended ratio, and the chord becomes more under. A better voicing is 3rd – 8ve – 12th, with 5/3, 3/2, and 5/2. The extended ratio is 6:10:15, with smaller numbers, and the 5-factor numbers at the end. These extended ratios are harder to read, as ratios like 3/2 are disguised as 15/10.

For a zo triad, the open voicing might seem better, as 7/6 becomes 7/3. But the 9/7 becomes a 14/9. It's hard to say definitively which voicing is better. 6:7:9 becomes 6:9:14, larger numbers but less under. To my ears, the 14/9 spoils the chord, and the best voicings for a zo triad have the 5th just above the 3rd, to avoid both the z6 and the Wr3. Finally, the resulting r3 should be in a high register, so the root must be at the bottom. 1 – Wz3 – W5 = 3:7:9 and 1 – 8 – Wz3 – W5 = 3:6:7:9 are both excellent voicings.

For the same reasons, the ru triad must have the 3rd immediately above the root, with both notes up high. This leads to a 5th – 8ve – 10th voicing, or 21:28:36, with a 4/3, a 9/7 and a 12/7, or even better, the high voicing of 8ve – 10th – 12th, or 14:18:21, with a 9/7, a 7/6 and a 3/2.

For a zo-7 tetrad, the w5 should again be just above the z3. In addition, the zo 7th should be above the root and 5th, to avoid ru intervals. That suggests 1 – Wz3 – W5 – Wz7 = 6:14:18:21 or 1 – 8 – Wz3 – W5 – Wz7 = 6:12:14:18:21.

Because z3 and w5 clash so easily, <u>a z,y6 or z7 tetrad will often sound better with the fifth omitted</u>.

Repeating notes in other octaves result in more voices, and more interval pairs. A yo triad voiced 1st – 5th – 8ve – 10th = 1/1 – 3/2 – 2/1 – 5/2, has six intervals: 3/2, 4/3, 5/4, 2/1, 5/3 and 5/2. Voicing the 5th higher, as 1st – 8ve – 10th – W5th, or 1/1 – 2/1 – 5/2 – 3/1, we get 2/1, 5/4, 6/5, 5/2, 3/2 and 3/1. We have replaced 4/3 with 3/1 (better) and 5/3 with 6/5 (worse). Hard to say which is better. The best voicing of the yo chord with the root present twice is 1:2:3:5, or root – 8ve – W5th – WW3rd, containing 2/1, 3/2, 5/3, 3/1, 5/2 and 5/1.

Is it possible to voice a chord so that each interval has the ideal voicing of table 2.7.3? Absolutely! Notice that the best interval voicings in the table are those with no even numbers in the ratio. Different chord voicings are the result of octave transpositions, which multiply or divide the ratios by 2. Since one of our goals is small numbers, it follows that a good voicing might be a **two-less** one with all the 2-factors removed. In other words, all odd numbers in all the ratios, known as **all-odd**. That implies that each note is only used once, because no even numbers means no octaves.

The few under ratios that don't sound better inverted (r3, g7, zg5) also don't sound better widened. Their best voicing is octave-reduced, and in this voicing they just happen to be all-odd ratios. So all-odd voicings also satisfy our over/under criteria.

The h7 chord's all-odd voicing is 1/1 – 3/1 – 5/1 – 7/1, or 1:3:5:7, which is indeed extremely consonant. Using octave numbers, a Ch7 chord would be C2 – G3 – yE4 – zB♭4. It's a rather impractical voicing, with a very wide gap between the lowest two notes. A more singable and playable voicing, almost as consonant, is 1/1 – 3/2 – 5/2 – 7/2, or C3 – G3 – yE4 – zB♭4, or 2:3:5:7. This voicing is more practical because it is reasonably compact.

Any two chords that are homonyms will have identical all-odd voicings. For example, Cg7(zg5) and gE♭z,y6 are homonyms. Their all-odd voicings are both gE♭3 – C4 – zgG♭4 – gB♭4. But this voicing implies a z,y6 chord, not a g7(zg5) chord. In certain musical contexts, consonance may need to be sacrificed in order to more clearly define the root. In the table below, the alternate voicings are more compact and/or define the root better.

Table 2.7.4 – All-odd chord voicings, with alternate voicings for compactness or better root definition

Chord	All-odd voicing		Alternate voicing	
Cy	1/1 – 3/1 – 5/1 1:3:5	C2 – G3 – yE4	1/1 – 3/2 – 5/2 2:3:5	C3 – G3 – yE4
Cg	3/5 – 1/1 – 3/1 3:5:15	gE♭3 – C4 – G5	3/5 – 1/1 – 3/2 6:10:15	gE♭3 – C4 – G4
Cz	1/1 – 7/3 – 3/1 3:7:9	C3 – zE♭4 – G4	same	
Cr	1/1 – 9/7 – 3/1 7:9:21	C3 – rE3 – G4	1/1 – 9/7 – 3/2 14:18:21	C4 – rE4 – G4
Cg(zg5)	3/5 – 1/1 – 7/5 3:5:7	gE♭3 – C4 – zgG♭4	1/1 – 6/5 – 7/5 5:6:7	(better root definition)
Cz(zg5)	1/1 – 7/5 – 7/3 15:21:35	C3 – zgG♭3 – zE♭4	same	
Ch7	1/1 – 3/1 – 5/1 – 7/1	C2 – G3 – yE4 – zB♭4	1/1 – 3/2 – 5/2 – 7/2	C3 – G3 – yE4 – zB♭4
Cz,y6	1/1 – 5/3 – 7/3 – 3/1	C3 – yA3 – zE♭4 – G4	same	
Cy6	1/1 – 5/3 – 3/1 – 5/1	C3 – yA3 – G4 – yE5	1/1 – 5/3 – 5/2 – 3/1	C3 – yA3 – yE4 – G4
Cz7	1/1 – 7/3 – 3/1 – 7/1	C3 – zE♭4 – G4 – zB♭5	1/1 – 7/3 – 3/1 – 7/2	C3 – zE♭4 – G4 – zB♭4
Cs6	3/7 – 3/5 – 1/1 – 3/1	rA2 – gE♭3 – C4 – G5	3/7 – 3/5 – 1/1 – 3/2	rA2 – gE♭3 – C4 – G4
Cr,g7	1/1 – 9/7 – 9/5 – 3/1	C3 – rE3 – gB♭3 – G4	same	
Cg7	3/5 – 1/1 – 9/5 – 3/1	gE♭3 – C4 – gB♭4 – G5	3/5 – 1/1 – 3/2 – 9/5	gE♭3 – C4 – G4 – gB♭4
Cr6	3/7 – 1/1 – 9/7 – 3/1	rA1 – C3 – rE3 – G4	6/7 – 1/1 – 9/7 – 3/2	rA2 – C3 – rE3 – G3
Cg7(zg5)	3/5 – 1/1 – 7/5 – 9/5	gE♭3 – C4 – zgG♭4 – gB♭4	1/1 – 6/5 – 7/5 – 9/5	(better root definition)
Cs7	1/1 – 7/5 – 7/3 – 7/1	C3 – zgG♭3 – zE♭4 – zB♭5	1/1 – 7/5 – 7/3 – 7/2	C3 – zgG♭3 – zE♭4 – zB♭4
Cy7	1/1 – 3/1 – 5/1 – 15/1	C2 – G3 – yE4 – yB5	1/1 – 3/2 – 5/2 – 15/4	C3 – G3 – yE4 – yB4
Ch9	1/1 – 3/1 – 5/1 – 7/1 – 9/1	C2 – G3 – yE4 – zB♭4 – wD5	1/1 – 3/2 – 5/2 – 7/2 – 9/2	C3 – G3 – yE4 – zB♭4 – wD5
Cs9	1/1 – 9/7 – 9/5 – 3/1 – 9/1	C3 – rE3 – rB♭3 – G4 – D6	1/1 – 9/7 – 9/5 – 3/1 – 9/2	C3 – rE3 – rB♭3 – G4 – D5
Cs6,11	1/3 – 3/7 – 3/5 – 1/1 – 3/1	F2 – rA2 – gE♭3 – C4 – G5	1/3 – 3/7 – 3/5 – 1/1 – 3/2	F2 – rA2 – gE♭3 – C4 – G4

A few caveats about this table:

There is no exact measure of consonance. It depends on timbre, register and other factors.

Vertical context matters. The highly consonant h9 chord contains a dissonant r3.

Horizontal context matters. The preceding and succeeding chords can affect the perceived consonance.

Rhythm matters. Subharmonic chords often sound better arpeggiated, since their intervals are so consonant.

The minor 3rd can be tuned not to g3 or z3, but to 19/16 = 298¢, which widens nicely to 19/4, 19/2, etc.

The last point means that a widely voiced minor chord may sound better with a 19/16 third. In general, a yaza chord in a poor voicing may sound better if tuned to a higher prime-limit. Higher primes are covered in Part III.

One more caveat: the all-odd voicing is not the best voicing if it contains a very narrow interval. For example, the all-odd voicing of a g7y6no5 chord is 9:15:25:27, with a 27/25 interval of only 133¢ between the y6 and the g7. A better voicing would be 9:15:27:50, which moves the y6 up an octave.

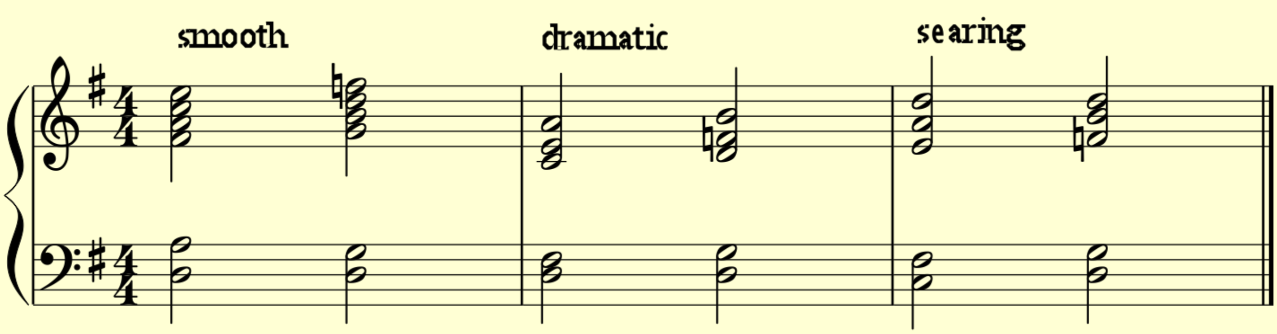
The all-odd voicings of the h7 and h9 chords are the odd harmonics in the harmonic series. The all-odd voicings of the s6, s7 and s9 chords are the odd subharmonics.

If octaves are desired in the voicing, start with the all-odd voicing, and add each new voice exactly one octave above the one it's doubling. This ensures the lowest possible numbers. For example, doubling the root of the h7 chord would give us 1/1 – 2/1 – 3/1 – 5/1 – 7/1, or 1:2:3:5:7, and doubling the fifth makes 1/1 – 3/1 – 5/1 – 6/1 – 7/1 = 1:3:5:6:7.

The same table, in staff notation:

Figure 2.7.1 – All-odd chord voicings, with alternate voicings for compactness or better root definition

A dissonant voicing is sometimes very desirable, especially with cadences. A Vh9 – Ih7 cadence can have more impact if the V chord has a dissonant voicing. Dh9 to Gh7:



What about chords not in the table? To find the all-odd voicing of any chord with alt-tuner, start with any no-octaves, each-note-only-once voicing. Play any two notes and look at alt-tuner's interval display. If the ratio is two odd numbers, you're done. If the ratio's top note is even, invert (or unwiden, if it's wide) the interval by either moving the bottom note up an octave or the top one down. If the ratio's bottom note is even, widen the interval by either moving the top note up or the bottom one down. Mnemonic: top number even, top note down; bottom number even, bottom note down. Keep going until there are no more even numbers.

For the more mathematically inclined, pick a voicing, write out the ratios, and write out the extended ratio. Then factor the twos out of any even numbers, and rearrange the numbers in order to make a new extended ratio. Divide all the numbers by the number that was first before rearranging to make ratios. If the first ratio is descending, add enough octaves to it to make it ascending, and add the same number of octaves to the rest of the ratios too. Finally, rewrite all the ratios in color notation.

Example: the diminished seventh chord s6(zg5) in close position is w1 – g3 – zg5 – r6, or 1/1 – 6/5 – 7/5 – 12/7. The two denominators are 5 and 7. Multiply by 35 to get 35:42:49:60. Reduce the even numbers to get 35:21:49:15, which rearranges to 15:21:35:49. Divide by 35 to make 15/35 – 21/35 – 35/35 – 49/35, which reduces to 3/7 – 3/5 – 1/1 – 7/5. Make 3/7 ascending by adding two octaves, to get 12/7 – 12/5 – 4/1 – 28/5, which is r6 – Wg3 – Ww8 – WWzg5.

A good voicing can help tame innate comma chords. For example, the ya 6/9 chord is y6,9, which contains a y5 = 40/27 between w2 and y6. This interval's all odd voicing is WWg4 = 27/5. The y6,9 chord's all-odd voicing is 1/1 – 5/3 – 3/1 – 5/1 – 9/1 = C3 – yA3 – G4 – yE5 – wD6, consisting of alternating y6 and g7 intervals. Some yaza chords like z7,11 or z11no3 or r6,9 will have a ru 5th. They are passable if the r5 is voiced as a wide zo 4th, e.g. as WWz4 = 21/4 or even better as W^3z4 = 21/2.

Part III – Further Out

This section goes to the very extremes of the harmonic lattice, and also looks at higher primes.

Chapter 3.1 – Remoteness Classes

I've been using the term **remote** without defining it. It's a rough measure of dissonance, or what might be called harmonic distance. It's not an exact measurement, like prime limit or odd limit. It's more a rule of thumb, categorizing intervals rather than exactly quantizing them.

Another example of a musical rule of thumb is scale degree, which describes an interval's width, or melodic distance. Unlike the exact cents value, degree puts intervals in general categories like seconds, thirds, etc. Less precise, but more musically useful. The categories overlap somewhat; from g6 to y7 in the harmonic minor is yy2 = 275¢, wider than z3 = 267¢. So sometimes aug 2nds are larger than minor 3rds. We could avoid this overlap by defining seconds as any interval between 50¢ and 250¢, thirds as 250–450¢, etc. But we don't because it would violate a very useful property of degree: intervals' degrees add up logically and consistently. Again, we sacrifice precision for musical usefulness.

Likewise, we can assign intervals into **classes** based on remoteness: Factor the ratio into primes and discard all twos. Then, let the largest prime(s) on one side of the ratio cancel out the largest prime(s) on the other side. Finally, assign each prime factor a value of (p-1)/2, and add up all these values. For example:

9/8 = 3·3 / 2·2·2 becomes 3·3 (2s are discarded) becomes 1+1 = class 2
6/5 = 2·3 / 5 becomes 1/5 (5 cancels 3) becomes class 2
15/8 = 3·5 / 2·2·2 becomes 3·5/1 becomes 1 + 2 = class 3
27/25 = 3·3·3 / 5·5 becomes 3/5·5 (each 5 cancels a 3) becomes 1+2+2 = class 5
7/5 becomes 7/1 (7 cancels 5) becomes class 3
25/21 = 5·5 / 3·7 becomes 5/7 (7 cancels one 5, the other 5 cancels 3) becomes class 5

It works for higher primes, too:
11/10 = 11 / 2·5 becomes 11/1 becomes class 5
35/33 = 5·7 / 3·11 becomes 5/11 (11 cancels 7, 5 cancels 3) becomes class 7

prime:	2	3	5	7	11	13
value:	0	1	2	3	5	6

Discarding twos makes the classification octave-equivalent, or voicing-independent. Canceling out primes is analogous to using a triangular harmonic lattice instead of a square one.

Now, this system is founded on the premise that higher prime-limits are more dissonant than lower ones. As we'll see, this only holds for primes up to 13. Ratios using higher primes will be incorrectly classified. Thus while 6/5 is class 2, the nearly as consonant 19/16 is class 9. There are more accurate measurements, for example James Tenney's harmonic distance, which for a ratio a/b is the logarithm of a·b. The logarithm of the odd limit is also used. But these are much less convenient to calculate.

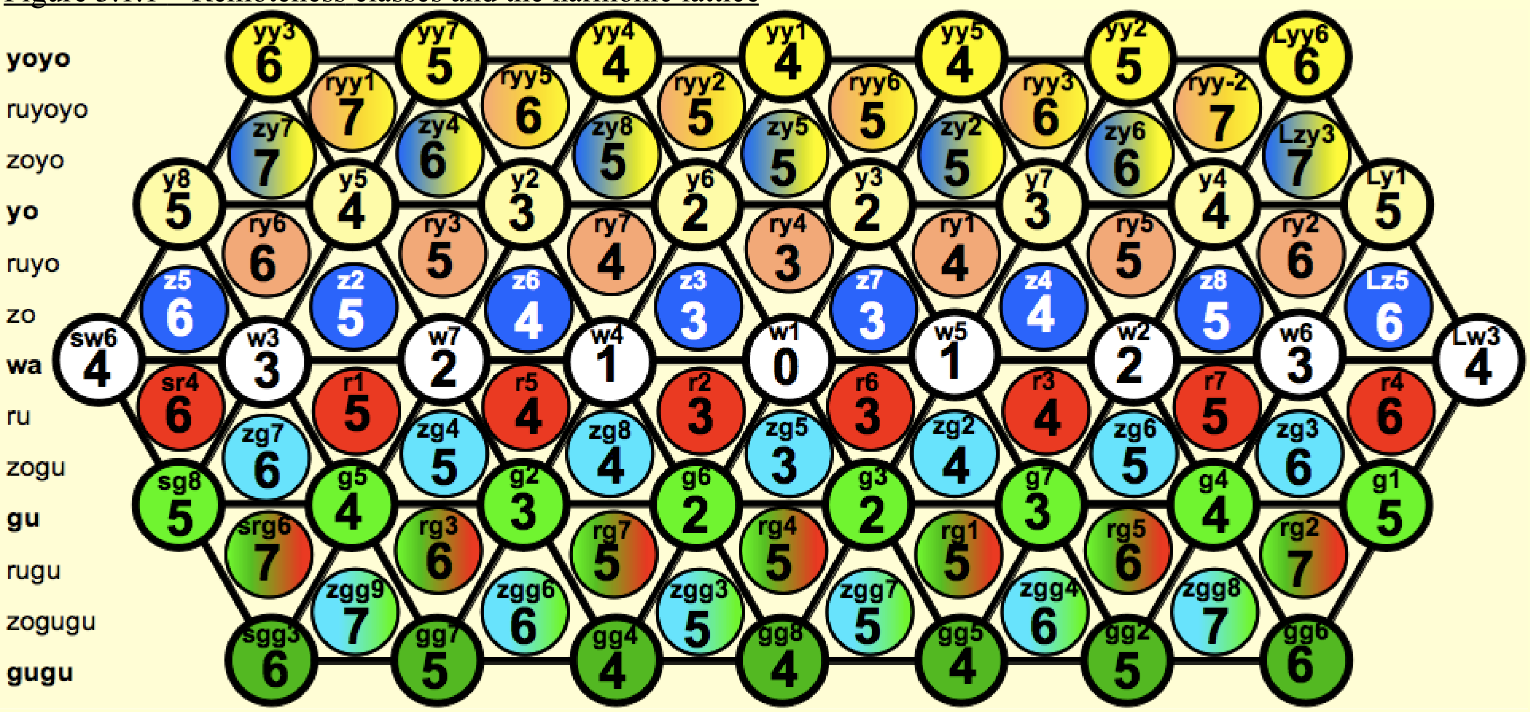
Table 3.1.1 – All yaza JI intervals of class 4 or less (octave-reduced)

class 0	w1, w8
class 1	w4, w5
class 2	w2, g3, y3, g6, y6, w7
class 3	g2, y2, r2, z3, w3, zg5, ry4, w6, r6, z7, g7, y7
class 4	yy1, zg2, ry1, Lw3, r3, z4, g4, y4, yy4, gg4, gg5, yy5, g5, y5, r5, z6, sw6, zg8, ry7, gg8

The higher the class, the more dissonant the interval, and the harder it is to tune by ear. While w2 may seem hard to tune, it's much easier voiced as w9 or Ww9. Because class is voicing-independent, class describes the consonance of an interval in its best possible voicing, generally the all-odd voicing of chapter 2.7. Note that y3 and g3 are the same class, even though the g3 is more dissonant. But g3's class is low because g3 is the very consonant y6 in a different voicing. Also, note that w7 is a lower class than z7 and g7, even though both z7 and g7 sound better in a chord than w7, in my opinion. However, w7 is the consonant Ww9 revoiced. Nevertheless, for an easy-to-calculate rule of thumb, it works well. Even comparing specific voicings, the class is never off by more than one. We sacrifice some precision for musical usefulness.

The opposite of remoteness is **nearness**, as in "the nearest 2nd is wa." With a little practice you can "see" remoteness in the harmonic lattice directly:

Figure 3.1.1 – Remoteness classes and the harmonic lattice



Classes do not add up like degrees do. However, as seen in figure 3.1.1, there's a lot of symmetry. For example octave inversions always have the same class. Also class-equivalent are what might be called vertical complements, like y3/g3 and z7/r6. The smaller the interval, the greater the class must be: Semitones are at least class 3, the main commas are class 5 or 6 (except for the wa one), and minicommas are mostly class 7 to 10.

13/7 = class 6 may be acoustically less dissonant than 15/8 = class 3, but it's much less useful musically. That's because 15/8 breaks down into two simpler intervals, 3/2 plus 5/4. 15/8 has a much lower prime limit, but only a slightly higher odd limit than 13/7. Remoteness measures both prime limit and odd limit, to give a general measure of musical usefulness. This is shown in the next table by the steady increase of remoteness from the upper left to the lower right:

Table 3.1.2 – Examples of remoteness as a function of both prime limit and odd limit

	low odd limit	medium odd limit	high odd limit
low prime limit	4/3 = class 1	27/16 = class 3	243/128 = class 5
medium prime limit	7/4 = class 3	35/32 = class 5	243/224 = class 7
high prime limit	11/6 = class 5	39/32 = class 7	256/195 = class 9

A chord is classified by the most remote interval it contains. For example, a yo chord in close position has 3 elements, w1, y3, & w5.

w1 to y3 is 5/4 is class 2
w1 to w5 is 3/2 is class 1
y3 to w5 is 6/5 is class 2

Because classes are voicing-independent, all yo chords are class 2. The class can be thought of as based on the chord's best possible voicing, the all-odd voicing of chapter 2.7. For the yo chord, that would be 1/1 – 3/1 – 5/1. Of course, the actual voicing used will affect the chord's consonance.

You don't need to write out all the intervals in a chord to find its class. You can use the lattice in figure 3.1.1. First refer to the diagrams in chapter 2.4 to find the chord's shape. For example, the yo chord is a triangle. Next, mentally place the root of the chord on the w1 in the lattice above. It covers 0, 1 and 2, for a maximum of 2. Mentally move the triangle around without rotating it so that each one of its 3 notes in turn occupies the w1, and note all the numbers covered. Their maximum is 2, the class of the chord.

Now do the same with the zo chord, a squashed leftward-leaning triangle. As you move it around, you soon find yourself covering a 4, because the zo chord is class 4.

Table 3.1.3 – Examples of chords by class

	dyads and triads	tetrads and pentads
class 1	five chord	
class 2	y chord / g chord / four chord	
class 3	z(zg5) / g(zg5)	y6 / g7 / h7 / s6 / s7 / y7 / y,9 / z7no5 / z,y6no5
class 4	z / r / (z4) / y(yy5) / g(gg5)	z7 / r6 / z,y6 / r,g7 / g7(zg5) / y,g7 / h9 / s6,11
class 5		z9 / z,y6(zg5)
class 6		g,y6(gg5) / s6(zg5)

These classes correspond closely to actual consonance of all-odd voicings. Based on my experience, I might put s6 in class 4. However, s6 voiced as r6 – Wg3 – Ww5 – Ww8 comes close to a z7 chord in consonance. Of course, then it sounds like its homonym, a s7 chord. If two chords are homonyms of each other, they have the same class.

Because classes are voicing-independent, we can classify a progression or even an entire song regardless of arrangement. A chord progression is rated by the most remote interval between all the chords' elements. For example Iy – IVy – Vy requires both w4 and y7, which make a y4 = class 4, so the whole progression is class 4.

Table 3.1.4 – Examples of chord progressions by class

	two chords	three or more chords
class 3	Iy – Vy, Iy – IVy, Iy – yVIg, Ig – gVIy	
class 4	Ig –IVy, Ig – Vy, Iy – yIIg, Ih7 – IVh7, Ih7 – Vh7	Iy – yVIg – IVy – Vy, IVy –IVg – Iy, Ig – gVIIy – gVIy – gVIIy
class 5	Is6 – Vr,g7	Iy – IVy – Vh7
class 6	Is6 – Vh7	"I Hear Numbers" riff
class 7		"Without You" verse
class 8		"Without You" entire song

All the basic two-chord rock vamps are class 3.

A scale can be similarly rated by the most distant interval it contains. If you can visualize the shape of the scale, you can use the lattice in figure 3.1.1 to find the scale's class, just like you did with chords. For scales with alternate notes, assume that only one of the alternates is used. If a chord progression fits inside a scale, the progression's class is never more than the scale's class. For example any yo-major-scale chord progression will have class 4 or less. (Unless both y2 and w2 are used, as in Iy – IVy – yIIg – yVIg – Vy = class 5.)

Table 3.1.5 – Examples of scales by class

wa major	1 w2 Lw3 4 5 w6 Lw7	class 6
wa minor	1 w2 w3 4 5 sw6 w7	class 6
yo major	1 w2/y2 y3 4 5 y6 y7	class 4
gu minor	1 w2 g3 4 5 g6 g7/w7	class 4
zo minor	1 w2 z3 4 5 z6 z7	class 6
ru major	1 w2 r3 4 5 r6 r7	class 6
wa minor pentatonic	1 w3 4 5 w7	class 4
wa major pentatonic	1 w2 Lw3 5 w6	class 4
yo pentatonic	1 w2 y3 5 y6	class 4
gu pentatonic	1 g3 4 5 g7	class 4
zo pentatonic	1 z3 4 5 z7	class 4
ru pentatonic	1 w2 r3 5 r6	class 4
yo zo pentatonic	1 w2 y3 5 z7	class 4
zo yo pentatonic	1 z3 4 5 y6	class 4
gu ru pentatonic	1 g3 4 5 r6	class 4
ru gu pentatonic	1 w2 r3 5 g7	class 4

A melody can be rated by the most remote interval between all the notes. This is equivalent to deducing a scale from the melody and rating that scale. The higher the class, the harder it should be to sing. This is unfortunately not always the case. Because of octave-equivalence, size of melodic steps and overall range aren't taken into account, even though they obviously affect the sing-ability. In other words, voicing-independence doesn't really apply to melody.

Now, one <u>could</u> just look at the intervals between any two consecutive notes. In other words, one could look at the individual melodic steps as opposed to the total ground the melody covers. But consider these two melodies: 1 – y3 – 5 (class 2) vs. 1 – y3 – yy5 (class 4). The second melody is harder to sing, even though both melodies contain only class 2 steps.

Each color or lattice row has a minimum class, easily found by looking at figure 3.1.1. This minimum class is the color's class, and we can use it to rank colors. Here we distinguish between clear (2-limit) and wa (3-limit):

class 0: clear
class 1: wa
class 2: yo, gu
class 3: zo, ru, zogu, ruyo
class 4: yoyo, gugu
class 5: zoyo, rugu, ruyoyo, zogugu
etc.

Finally, we can loosely rate entire genres by class:

Basic folk (e. g. "This Land Is Your Land"): class 2 chords, class 4 scales and progressions.
Basic rock (e.g. "Sympathy For The Devil"): class 2 & 3 chords, class 5 & 6 scales and progressions.
Basic blues (e.g. "CC Rider"): class 3 & 4 chords, class 6 scales and progressions.

Chapter 3.2 – Extremely Remote Intervals

In this chapter we will extend the harmonic lattice to truly ridiculous extremes, many more notes than can actually be distinguished by ear. The purpose is to show how to express any ratio in color notation, and to illustrate some concepts used in later chapters.

Large and small intervals are defined relative to the **midpoint** of each row. Midpoints are any ratio $2^a 3^b 5^c 7^d$ such that b + c + d = 0, as with 5/3 or 7/5. Each row has only one midpoint ratio, which is one of seven **central** (meaning neither large nor small) ratios. Central ratios are at most 3 steps along the lattice row away from the midpoint. For central ratios, b + c + d ranges from -3 to 3, and for large ratios, b + c + d ranges from 4 to 10.

Why not define the midpoint as c + d = 0, so that 5/4 and 7/4 are midpoints, so that the dotted lines in Figure 2.2.1 slant to the upper right, not the upper left? Because then a row's central ratios wouldn't have that row's smallest odd limits. For example, on the gu row, g1 = 81/80 wouldn't be central, but sg8 = 256/135 would be. Also, on the zo row, z5 = 112/81 wouldn't be central, but Lz5 = 189/128 would be. Instead, large and small are defined so that central intervals will usually have a smaller odd limit than the corresponding large or small interval. I say "usually" because unfortunately it's impossible for any simple rule like b + c + d = 0 to always produce this result. For example, yy6 = 400/243 has a slightly higher odd limit than Lyy6 = 225/128. The zyy and zzyy rows are also problematic. However, for nearby rows this definition of central works well.

To extend the rows further, use **double large** = LL and **double small** = ss. If needed, there's **triple large**, etc. To extend the number of rows, use **triple yo** for ratios involving 125, etc. Quadruple and quintuple are abbreviated **quad-** and **quint-**, as in the quadgu comma = 648/625.

Remote colors may have alternate names. Just as one can say either "Dave's not here" or "Dave isn't here", for rryyy one can say either "yo double ruyo" or "ruru triple yo". Words like "double" and "triple" apply to all subsequent colors up until another such word is used. Quad and quint affect all subsequent colors until an -a-. Double zozogu can be spoken as quadzoagugu = quadruple zo and gugu.

Six steps along the yo axis could be hextuple yo. Or is that sextuple? In my opinion, tuning theory is mentally challenging enough without having to memorize obscure Latin numeric prefixes. So for numbers above 5, I prefer to use sixfold, sevenfold, etc.

On the next page is table 3.2.3, a giant harmonic lattice that goes large, small and double, using these colors:

Table 3.2.1 – Double yaza colors

rryy = double ruyo	ryy = ruyoyo	yy = yoyo	zyy = zoyoyo	zzyy = double zoyo
rry = ruruyo	ry = ruyo	y = yo	zy = zoyo	zzy = zozoyo
rr = ruru	r = ru	w = wa	z = zo	zz = zozo
rrg = rurugu	rg = rugu	g = gu	zg = zogu	zzg = zozogu
rrgg = double rugu	rgg = rugugu	gg = gugu	zgg = zogugu	zzgg = double zogu

The column headers have three new colors purple, tho and thu, which will be explained in chapters 3.4 and 3.6. The lattice rows are written as columns. The table lists the quality, degree and cents for each ratio. Each ratio is octave-reduced so that the cents value is always between 0 and 1200. The qualities become rather extreme: dd = double diminished, AAA = triple augmented, etc. The keyspan is not listed, but can be deduced from the quality and degree using this table:

Table 3.2.2 – Degree, quality and keyspan for extreme qualities

	triple dimin	double dimin	dimin	perfect		augmt	double augmt	triple augmt
				minor	major			
unison				0		1	2	3
2nd			0	1	2	3	4	5
3rd	0	1	2	3	4	5	6	7
4th	2	3	4	5		6	7	8
5th	4	5	6	7		8	9	10
6th	5	6	7	8	9	10	11	12
7th	7	8	9	10	11	12		
octave	9	10	11	12				
9th	10	11	12					

These qualities follow a certain pattern in every color/row. For example the wa row, starting from the fourth and progressing fifthward, has 3 perfect ratios, 4 major ones, 7 augmented ones, 7 double-augmented ones, 7 triple-augmented ones, etc. Progressing fourthward, we have 3 perfect ratios, 4 minors, 7 dims, 7 double-dims, etc. The entries in bold break this pattern; this will be explained in the next chapter. Every color/row contains some part of this same **quality-chain**, which can be written:

dbl-dim – d2 d6 d3 d7 d4 d8 d5 – m2 m6 m3 m7 – P4 P1 P5 – M2 M6 M3 M7 – A4 A1 A5 A2 A6 A3 A7 – dbl-aug

It can also be written more concisely as:

...7dd – 7d – 4m – 3P – 4M – 7A – 7AA ...

The quality-chain is different from a row's **magnitude-chain**, which runs

...7ss – 7s – 7 central – 7L – 7LL...

Table 3.2.3 – Large, small and double intervals in yaza JI

	rryy = double ruyo				ryy = ruyoyo				yy = yoyo				zyy = zoyoyo				zzyy = double zoyo			
	ratio			cents	ratio			cents	ratio			cents	ratio			cents	ratio			cents
	3276800	2893401	M2	215	1638400	1240029	P4	482	819200	531441	m6	749	2867200	1594323	d8	1016	5017600	4782969	d3	83
s	1638400	964467	M6	917	819200	413343	P8	1184	204800	177147	m3	251	716800	531441	d5	518	2508800	1594323	d7	785
m	409600	321489	M3	419	204800	137781	P5	686	102400	59049	m7	953	179200	177147	m2	20	627200	531441	d4	287
a	204800	107163	M7	1121	51200	45927	M2	188	25600	19683	P4	455	89600	59049	m6	722	313600	177147	d8	989
l	51200	35721	A4	623	25600	15309	M6	890	12800	6561	P8	1157	22400	19683	m3	224	78400	59049	d5	491
l	12800	11907	A1	125	6400	5103	M3	392	3200	2187	P5	659	11200	6561	m7	926	39200	19683	m9	1193
	6400	3969	A5	827	3200	1701	M7	1094	800	729	M2	161	2800	2187	P4	428	9800	6561	m6	695
c	1600	1323	A2	329	800	567	A4	596	400	243	M6	863	1400	729	P8	1130	2450	2187	m3	197
e	800	441	A6	1031	200	189	A1	98	100	81	M3	365	350	243	P5	632	1225	729	m7	899
n	200	147	A3	533	100	63	A5	800	50	27	M7	1067	175	162	M2	134	1225	972	P4	401
t	**50**	**49**	**d-2**	**35**	25	21	A2	302	25	18	A4	569	175	108	M6	836	1225	648	P8	1102
r	75	49	AA4	737	25	14	A6	1004	25	24	A1	71	175	144	M3	338	1225	864	P5	604
a	225	196	AA1	239	75	56	A3	506	25	16	A5	773	175	96	M7	1039	1225	1152	M2	106
l	675	392	AA5	941	**225**	**224**	**d-2**	**8**	75	64	A2	275	175	128	A4	541	1225	768	M6	808
	2025	1568	AA2	443	675	448	AA4	710	225	128	A6	977	525	512	A1	43	1225	1024	M3	310
L	6075	3136	AA6	1145	2025	1792	AA1	212	675	512	A3	478	1575	1024	A5	745	3675	2048	M7	1012
a	18225	12544	AA3	647	6075	3584	AA5	914	2025	1024	A7	1180	4725	4096	A2	247	11025	8192	A4	514
r	**54675**	**50176**	**dd-2**	**149**	18225	14336	AA2	416	6075	4096	AA4	682	14175	8192	A6	949	33075	32768	A1	16
g	164025	100352	AAA4	851	54675	28672	AA6	1117	18225	16384	AA1	184	42525	32768	A3	451	99225	65536	A5	718
e	492075	401408	AAA1	353	164025	114688	AA3	619	54675	32768	AA5	886	127575	65536	A7	1153	297675	262144	A2	220
	1476225	802816	AAA5	1055	**492075**	**458752**	**dd-2**	**121**	164025	131072	AA2	388	382725	262144	AA4	655	893025	524288	A6	922

Table 3.2.3 (continued)

	rry = ruruyo (purple)				**ry = ruyo**				**y = yo**				**zy = zoyo (thu)**				**zzy = zozoyo**			
	ratio			cents	ratio			cents	ratio			cents	ratio			cents	ratio			cents
	1310720	964467	P4	531	655360	413343	m6	798	327680	177147	d8	1065	573440	531441	d3	132	2007040	1594323	dd5	399
s	327680	321489	P1	33	163840	137781	m3	300	81920	59049	d5	567	286720	177147	d7	834	1003520	531441	d9	1101
m	163840	107163	P5	735	81920	45927	m7	1002	20480	19683	m2	69	71680	59049	d4	336	250880	177147	d6	602
a	40960	35721	M2	237	20480	15309	P4	504	10240	6561	m6	771	35840	19683	d8	1038	62720	59049	d3	104
l	20480	11907	M6	939	5120	5103	P1	6	2560	2187	m3	273	8960	6561	d5	539	31360	19683	d7	806
l	5120	3969	M3	441	2560	1701	P5	708	1280	729	m7	975	2240	2187	m2	41	7840	6561	d4	308
	2560	1323	M7	1143	640	567	M2	210	320	243	P4	477	1120	729	m6	743	3920	2187	d8	1010
c	640	441	A4	645	320	189	M6	912	160	81	P8	1178	280	243	m3	245	980	729	d5	512
e	160	147	A1	147	80	63	M3	414	40	27	P5	680	140	81	m7	947	245	243	m2	14
n	80	49	A5	849	40	21	M7	1116	10	9	M2	182	35	27	P4	449	245	162	m6	716
t	60	49	A2	351	10	7	A4	617	5	3	M6	884	35	18	P8	1151	245	216	m3	218
r	90	49	A6	1053	15	14	A1	119	5	4	M3	386	35	24	P5	653	245	144	m7	920
a	135	98	A3	555	45	28	A5	821	15	8	M7	1088	35	32	M2	155	245	192	P4	422
l	**405**	**392**	**d-2**	**56**	135	112	A2	323	45	32	A4	590	105	64	M6	857	245	128	P8	1124
	1215	784	AA4	758	405	224	A6	1025	135	128	A1	92	315	256	M3	359	735	512	P5	626
L	3645	3136	AA1	260	1215	896	A3	527	405	256	A5	794	945	512	M7	1061	2205	2048	M2	128
a	10935	6272	AA5	962	**3645**	**3584**	**d-2**	**29**	1215	1024	A2	296	2835	2048	A4	563	6615	4096	M6	830
r	32805	25088	AA2	464	10935	7168	AA4	731	3645	2048	A6	998	8505	8192	A8	65	19845	16384	M3	332
g	98415	50176	AA6	1166	32805	28672	AA1	233	10935	8192	A3	500	25515	16384	A5	767	59535	32768	M7	1034
e	295245	200704	AA3	668	98415	57344	AA5	935	**32805**	**32768**	**d-2**	**2**	76545	65536	A2	269	178605	131072	A4	536
	885735	**802816**	**dd-2**	**170**	295245	229376	AA2	437	98415	65536	AA4	704	229635	131072	A6	971	535815	524288	A1	38

Table 3.2.3 (continued)

	rr = ruru				r = ru				w = wa				z = zo				zz = zozo			
	ratio			cents	ratio			cents	ratio			cents	ratio			cents	ratio			cents
	524288	321489	m6	847	262144	137781	d8	1114	65536	59049	d3	180	229376	177147	dd5	447	802816	531441	dd7	714
s	131072	107163	m3	349	65536	45927	d5	616	32768	19683	d7	882	114688	59049	d9	1149	200704	177147	dd4	216
m	65536	35721	m7	1051	16384	15309	m2	117	8192	6561	d4	384	28672	19683	d6	651	100352	59049	dd8	918
a	16384	11907	P4	553	8192	5103	m6	819	4096	2187	d8	1086	7168	6561	d3	153	25088	19683	dd5	420
l	4096	3969	P1	55	2048	1701	m3	321	1024	729	d5	588	3584	2187	d7	855	12544	6561	d9	1122
l	2048	1323	P5	756	1024	567	m7	1023	256	243	m2	90	896	729	d4	357	3136	2187	d6	624
	512	441	M2	258	256	189	P4	525	128	81	m6	792	448	243	d8	1059	784	729	d3	126
c	256	147	M6	960	64	63	P1	27	32	27	m3	294	112	81	d5	561	392	243	d7	828
e	64	49	M3	462	32	21	P5	729	16	9	m7	996	28	27	m2	63	98	81	d4	330
n	96	49	M7	1164	8	7	M2	231	4	3	P4	498	14	9	m6	765	49	27	d8	1032
t	72	49	A4	666	12	7	M6	933	1	1	P1	0	7	6	m3	267	49	36	d5	534
r	54	49	A1	168	9	7	M3	435	3	2	P5	702	7	4	m7	969	49	48	m2	36
a	81	49	A5	870	27	14	M7	1137	9	8	M2	204	21	16	P4	471	49	32	m6	738
l	243	196	A2	372	81	56	A4	639	27	16	M6	906	63	32	P8	1173	147	128	m3	240
	729	392	A6	1074	243	224	A1	141	81	64	M3	408	189	128	P5	675	441	256	m7	942
L	2187	1568	A3	576	729	448	A5	843	243	128	M7	1110	567	512	M2	177	1323	1024	P4	444
a	**6561**	**6272**	**d-2**	**78**	2187	1792	A2	345	729	512	A4	612	1701	1024	M6	879	3969	2048	P8	1145
r	19683	12544	AA4	780	6561	3584	A6	1047	2187	2048	A1	114	5103	4096	M3	381	11907	8192	P5	647
g	59049	50176	AA1	282	19683	14336	A3	549	6561	4096	A5	816	15309	8192	M7	1083	35721	32768	M2	149
e	177147	100352	AA5	984	**59049**	**57344**	**d-2**	**51**	19683	16384	A2	318	45927	32768	A4	584	107163	65536	M6	851
	531441	401408	AA2	486	177147	114688	AA4	753	59049	32768	A6	1020	137781	131072	A1	86	321489	262144	M3	353

Table 3.2.3 (continued)

	rrg = rurugu				**rg = rugu** (tho)				**g = gu**				**zg = zogu**				**zzg = zozogu** (purple)			
	ratio			cents	ratio			cents	ratio			cents	ratio			cents	ratio			cents
	1048576	535815	d8	1162	262144	229635	d3	229	131072	98415	dd5	496	458752	295245	dd7	763	1605632	885735	dd9	1030
s	262144	178605	d5	664	131072	76545	d7	931	65536	32805	d9	1198	114688	98415	dd4	265	401408	295245	dd6	532
m	65536	59535	m2	166	32768	25515	d4	433	16384	10935	d6	700	57344	32805	dd8	967	100352	98415	dd3	34
a	32768	19845	m6	868	16384	8505	d8	1135	4096	3645	d3	202	14336	10935	dd5	469	50176	32805	dd7	736
l	8192	6615	m3	370	4096	2835	d5	637	2048	1215	d7	904	7168	3645	d9	1171	12544	10935	dd4	238
l	4096	2205	m7	1072	1024	945	m2	139	512	405	d4	406	1792	1215	d6	673	6272	3645	dd8	940
	1024	735	P4	574	512	315	m6	841	256	135	d8	1108	448	405	d3	175	1568	1215	dd5	442
c	256	245	P1	76	128	105	m3	343	64	45	d5	610	224	135	d7	877	784	405	d9	1144
e	384	245	P5	778	64	35	m7	1045	16	15	m2	112	56	45	d4	379	196	135	d6	645
n	288	245	M2	280	48	35	P4	547	8	5	m6	814	28	15	d8	1081	49	45	d3	147
t	432	245	M6	982	36	35	P1	49	6	5	m3	316	7	5	d5	583	49	30	d7	849
r	324	245	M3	484	54	35	P5	751	9	5	m7	1018	21	20	m2	84	49	40	d4	351
a	486	245	M7	1186	81	70	M2	253	27	20	P4	520	63	40	m6	786	147	80	d8	1053
l	729	490	A4	688	243	140	M6	955	81	80	P1	22	189	160	m3	288	441	320	d5	555
	2187	1960	A1	190	729	560	M3	457	243	160	P5	723	567	320	m7	990	1323	1280	m2	57
L	6561	3920	A5	892	2187	1120	M7	1159	729	640	M2	225	1701	1280	P4	492	3969	2560	m6	759
a	19683	15680	A2	394	6561	4480	A4	661	2187	1280	M6	927	5103	2560	P8	1194	11907	10240	m3	261
r	59049	31360	A6	1096	19683	17920	A1	162	6561	5120	M3	429	15309	10240	P5	696	35721	20480	m7	963
g	177147	125440	A3	598	59049	35840	A5	864	19683	10240	M7	1131	45927	40960	M2	198	107163	81920	P4	465
e	**531441**	**501760**	**d-2**	**99**	177147	143360	A2	366	59049	40960	A4	633	137781	81920	M6	900	321489	163840	P8	1167
	1594323	1003520	AA4	801	531441	286720	A6	1068	177147	163840	A1	135	413343	327680	M3	402	964467	655360	P5	669

Table 3.2.3 (continued)

	rrgg = double rugu				rgg = rugugu				gg = gugu				zgg = zogugu				zzgg = double zogu			
	ratio			cents	ratio			cents	ratio			cents	ratio			cents	ratio			cents
	1048576	893025	d3	278	524288	382725	dd5	545	262144	164025	dd7	812	917504	492075	dd9	1079	1605632	1476225	ddd4	145
s	524288	297675	d7	980	131072	127575	d2	47	65536	54675	dd4	314	229376	164025	dd6	581	802816	492075	ddd8	847
m	131072	99225	d4	482	65536	42525	d6	749	32768	18225	dd8	1016	57344	54675	dd3	83	200704	164025	ddd5	349
a	65536	33075	d8	1184	16384	14175	d3	251	8192	6075	dd5	518	28672	18225	dd7	784	100352	54675	dd9	1051
l	16384	11025	d5	686	8192	4725	d7	953	2048	2025	d2	20	7168	6075	dd4	286	25088	18225	dd6	553
l	4096	3675	m2	188	2048	1575	d4	455	1024	675	d6	722	3584	2025	dd8	988	6272	6075	dd3	55
	2048	1225	m6	890	1024	525	d8	1157	256	225	d3	223	896	675	dd5	490	3136	2025	dd7	757
c	1536	1225	m3	392	256	175	d5	659	128	75	d7	925	448	225	d9	1192	784	675	dd4	259
e	2304	1225	m7	1094	192	175	m2	161	32	25	d4	427	112	75	d6	694	392	225	dd8	961
n	1728	1225	P4	596	288	175	m6	862	48	25	d8	1129	28	25	d3	196	98	75	dd5	463
t	1296	1225	P1	98	216	175	m3	364	36	25	d5	631	42	25	d7	898	49	25	d9	1165
r	1944	1225	P5	799	324	175	m7	1066	27	25	m2	133	63	50	d4	400	147	100	d6	667
a	1458	1225	M2	301	243	175	P4	568	81	50	m6	835	189	100	d8	1102	441	400	d3	169
l	2187	1225	M6	1003	729	700	P1	70	243	200	m3	337	567	400	d5	604	1323	800	d7	871
	6561	4900	M3	505	2187	1400	P5	772	729	400	m7	1039	1701	1600	m2	106	3969	3200	d4	373
L	**19683**	**19600**	**m-2**	**7**	6561	5600	M2	274	2187	1600	P4	541	5103	3200	m6	808	11907	6400	d8	1075
a	59049	39200	A4	709	19683	11200	M6	976	6561	6400	P1	43	15309	12800	m3	310	35721	25600	d5	577
r	177147	156800	A1	211	59049	44800	M3	478	19683	12800	P5	745	45927	25600	m7	1012	107163	102400	m2	79
g	531441	313600	A5	913	177147	89600	M7	1180	59049	51200	M2	247	137781	102400	P4	514	321489	204800	m6	781
e	1594323	1254400	A2	415	531441	358400	A4	682	177147	102400	M6	949	413343	409600	P1	16	964467	819200	m3	283
	4782969	2508800	A6	1117	1594323	1433600	A1	184	531441	409600	M3	451	1240029	819200	P5	718	2893401	1638400	m7	985

The next table lists some rather remote commas. The large yo minicomma Ly-2 is simply called the yo minicomma. There are other yo minicommas, but they are extremely remote. The ratios are very cumbersome, so monzos are used instead. These commas can be thought of as the difference between two more familiar commas.

Table 3.2.4 – More commas

monzo	cents	name		quality	class	derivations
(-15, 8, 1)	1.95¢	yo minicomma	Ly-2	desc dim 2nd	10	wa comma minus gu comma
(10, -6, 1, -1)	5.8¢	ruyo minicomma	sry1	perf unison	9	ru comma minus gu comma
(11, -4, -2)	19.5¢	gugu comma	sgg2	dim 2nd	8	ru comma minus the minicomma
(7, 0, -3)	41¢	triple gu comma	ggg2	dim 2nd	6	rugu comma minus the minicomma
(1, -5, 3)	49¢	triple yo comma	y^{3}1	aug unison	8	yoyo semitone minus gu comma
(-5, -1, -2, 4)	0.72¢	double zozogu **microcomma**	z^{4}gg3	double-dim 3rd	12	zozo comma minus double ruyo comma
(-1, -7, 4, 1)	0.40¢	zoquadyo microcomma	y^{4}z1	aug unison	13	triple yo comma minus gu comma minus ru comma

A microcomma is any comma too small to hear, i.e. it flunks the ear test. I define it as a comma less than 1¢, although less than 2¢ would also be a useful definition. A ratio mistuned by a microcomma will beat very slowly. For a fifth on A-220 off by a double zozogu microcomma, the beats would come about once every 4 seconds. It is impossible to tune acoustic instruments accurately enough to reflect this microcomma. Even electronic instruments can't always be tuned accurately enough. A JI tuning can't *not* temper out a microcomma, thus it creates an automatic microtempering. This microtempering is often musically useful, as we'll see in chapter 3.4. Just as minisharp/miniflat means sharpened/flattened by a minicomma, microsharp/microflat means sharpened/flattened by a microcomma.

The double zozogu microcomma (aka the deep purple microcomma) is discussed further in chapter 3.4, "Purple Intervals".

When naming commas, it's convenient to omit the magnitude (large, small, etc.) and degree, as we did with the yo minicomma. Only certain commas can have their names shortened. Each lattice row has one ratio which is the comma for that color, known as the exemplary comma. It is always under 50¢, and not a multiple of some other comma. Out of all the ratios that fit these requirements, it is the ratio with the lowest **double odd limit (DOL)**, which is found by factoring out all twos and listing the larger number first: the DOL of 50/49 is (49, 25), the DOL of 49/48 is (49, 3), and 49/48 has the lower DOL of the two.

Table 3.2.5 – The exemplary commas of 25 yaza colors

rryy-2	ryy-2	ssyy2	Lzyy1	Lzzyy1
srry1	Lry-2	Ly-2	szy2	zzy2
s^{3}rr2	r1	LLw-2	ssz3	zz2
LLrrg-3	rg1	g1	L^{3}zg-2	szzg3
Lrrgg-2	srgg2	sgg2	Lzgg1	sszzgg4

As chapter 4.5 explains, when naming a temperament that tempers out one of these commas, the magnitude can't be omitted, although the degree usually can be.

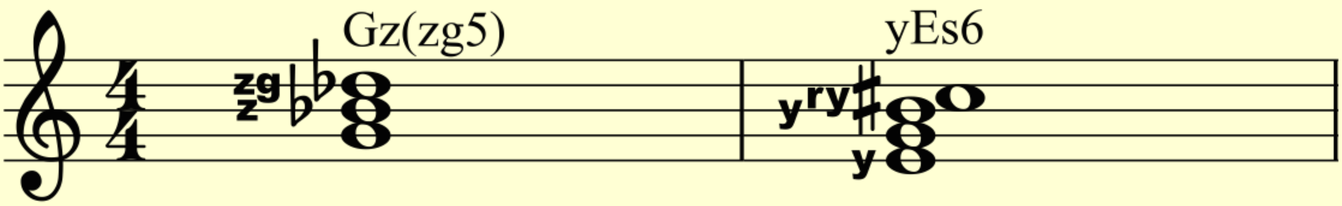
Chapter 3.3 – Paradoxical Intervals

You may have noticed some minus twos in the degree column in table 3.2.3. They're in bold, and mostly on the left-hand side. As mentioned in chapter 2.2, some intervals have a **negative** degree; they go down the scale to a higher pitch. For example, starting from the zogu 5th and going up a double ruyo comma (rryy-2 = 35¢) takes you down to the ruyo 4th. The interval from a diminished 5th to an augmented 4th is a descending diminished 2nd that actually raises the pitch! The descending aspect is why the degree is written as minus two.

More strangeness: an octave sharpened by a double ruyo comma is a double ruyo augmented 7th, rryy7 = 100/49 = 1235¢. It's a 7th, not an octave, because it's the sum of two ruyo 4ths. The sum of two zogu fifths is an octave flattened by a double ruyo comma, the double zogu diminished 9th zzgg9 = 49/25 = 1165¢. Finally there's the double zogu diminished 2nd, zzgg2 = zg5 - ry4 = 49/50 = minus 35¢. It's a diminished 2nd that's so far diminished, it's flatter than a unison. It's a descending negative 2nd that goes up the scale to a lower pitch.

The double ruyo comma is the nearest yaza negative 2nd. Other negative 2nds include the ruyoyo minicomma, the yo minicomma and the wa comma. If you venture into double large ratios and triple colors, you'll find negative 3rds, negative 4ths, etc. For example, the sum of any two negative 2nds is a negative 3rd.

Negative intervals sound ascending but look descending on paper, as in the uppermost voice here:



Remember, the ratio and the cents are the reality. The quality and degree are the theory, and the theory is based on a somewhat arbitrary choice of steps per octave and keys per octave. As we'll see in Part V, in pentatonicism, the double ruyo comma is not negative.

Most negative 2nds are diminished 2nds, with a keyspan of zero, which means that at least they make sense on a standard keyboard. You go down the scale to a higher pitch on the same key. However, there are also minor negative 2nds which go down the scale to a higher pitch on a lower key, and thus have a negative keyspan of -1. Intervals with a negative keyspan are called **upside-down** intervals. The nearest yaza one is the triple-ru gu comma, a minor negative 2nd, r^3g-2 = 1728/1715 = (6, 3, -1, -3) = 13¢ = g1 + r1 - zz2 = class 11.

What's the musical significance of an upside-down interval? Consider the large double rugu negative 2nd Lrrgg-m2 = 19683/19600 = (-4, 9, -2, -2) = 7¢. It takes you from a zoyo minor 3rd 280/243 = 245¢ to a rugu major 2nd 81/70 = 253¢. Thus in A, zyC is slightly flatter than rgB! Also, in the melody A – zyC – D, the A–C step is 7¢ narrower than the C–D step. Same with the melody gD – zF – gG.

Now, all these examples are quite contrived; one would have to modulate like crazy to use both zy3 and rg2 in the same song (class 15 scale!), the melodies seem unlikely (class 9), and a 7¢ difference in melodic step sizes is barely audible. Because of their small size and extreme remoteness, upside-down intervals are not a practical problem in 7-limit JI. But as we'll see in chapter 3.6, 11-limit and 13-limit JI produce much less remote upside-down intervals.

Is there a way to avoid negative 2nds entirely? What if we defined 7/4 as an aug 6th? Then 7/5 would be an aug 4th, 10/7 a dim 5th, and the double ruyo comma would be a diminished 2nd. 15/14 would be a minor 2nd, just like 16/15, and the ruyoyo minicomma would be a unison. However, 12/7 would be a dim 7th, and the zozo comma from 12/7 to 7/4 would unfortunately be a negative 2nd. It would be a double-diminished negative 2nd with a keyspan of 1!

Likewise, we could avoid the yo minicomma being negative by defining 5/4 as a dim 4th. But not only would that be very counterintuitive, it would make 81/80 a negative 2nd. Furthermore, the wa comma would still be negative. As we'll see in the "Various Mathematical Formulas and Proofs" section following part V, negative intervals and upside-down intervals are unavoidable.

There are also **diminished unisons**, which diminish the quality but raise the pitch. (Diminished unisons which lower the pitch, such as gg1 = 24/25, are actually descending augmented unisons.) Diminished unisons take you to the next lower key, thus they have a keyspan of -1, and are upside-down. For example, in the key of A, the zoyoyo 3rd C♯ is zyyM3 = 175/144 = 338¢, but the rugu 3rd C is rgm3 = 128/105 = 343¢. The minor is sharper than the major by the small ruru triple-gu diminished unison srrg^3d1 = 6144/6125 = (11, 1, -5, -2) = 5.4¢. This makes C♯ slightly flatter than C, thus adding a sharp flattens the note. In the chord Azyy,y5 = A – zyyC♯ – yE, A – C♯ is slightly narrower than C♯ – E. Again, these examples are quite contrived (class 12 melody, class 7 chord). Yaza diminished unisons are too small and remote to be a practical problem. The ruru triple-gu minicomma is the nearest yaza dim unison, at class 12.

An octave sharpened by a diminished unison is a diminished octave that is sharper than 1200¢. An octave minus a diminished unison is an augmented octave flatter than 1200¢. The inverse of a diminished unison, e.g. Lzzy^3A1 = (-11, -1, 3, 2) = -5¢, is a descending diminished unison which augments the quality but lowers the pitch.

The sum of any two diminished unisons is a doubly-diminished unison. Needless to say, these are extremely remote!

Every interval has a degree and a quality, which define its keyspan. Here's what conventional music theory has to say about these three concepts:

Table 3.3.1 – Degree, quality and keyspan

7 steps, 12 keys	double dimin	dimin	perfect minor	perfect major	augmt	double augmt
unison			0		1	2
2nd		0	1	2	3	4
3rd	1	2	3	4	5	6
4th	3	4	5		6	7
5th	5	6	7		8	9
6th	6	7	8	9	10	11
7th	8	9	10	11	12	13
octave	10	11	12		13	14
9th			etc.			

Here's the same chart, expanded to include negative and upside-down intervals. Every interval now has an additional property, its **sign**, which is **positive** or negative. A negative interval has the same keyspan as the corresponding positive interval, but with the opposite sign. For example, a dim 3rd is 2 semitones, and a dim negative third is -2 semitones.

Table 3.3.2 – Degree, quality and keyspan

7 steps, 12 keys	double dimin	dimin	perfect minor	perfect major	augmt	double augmt
neg. 4th			etc.			
neg. 3rd	-1	-2	-3	-4	-5	-6
neg. 2nd	1	0	-1	-2	-3	-4
unison	-2	-1	0		1	2
2nd	-1	0	1	2	3	4
3rd	1	2	3	4	5	6
4th	3	4	5		6	7
5th	5	6	7		8	9
6th	6	7	8	9	10	11
7th	8	9	10	11	12	13
octave	10	11	12		13	14
9th			etc.			

The naming paradoxes that negative seconds create are a mere annoyance; they make your note names run out of order, so that F♯ is sharper than G♭. The real headache is upside-down intervals. They make your keyboard run backwards! Unfortunately, as we'll see in part V, they are inevitable no matter how you define your ratios, if you modulate far enough. Notice from the chart that not all negative intervals are upside-down and not all upside-down ones are negative. Negative refers to degree and upside-down refers to keyspan.

For prime limits lower than 7, the negative intervals are more remote. Only in yaza are they a problem. The nearest ya ones are the yo minicomma and the fivefold-yo comma, both class 10. The nearest wa one is the LLw-2 comma. As the next table shows, the higher the prime limit, the less remote the negative intervals are. In chapter 3.6, we'll see that 11-limit and 13-limit break this pattern.

Table 3.3.3 – The nearest negative intervals for various prime limits

prime limit	ratio or monzo	cents	name		quality & degree	keyspan	class
3	(-19, 12)	24¢	wa comma	LLw-2	desc dim 2nd	0	12
5	(-15, 8, 1)	1.95¢	yo minicomma	Ly-2	desc dim 2nd	0	10
5	(-10, -1, 5)	30¢	fivefold-yo comma	Ly^5-2	desc double-dim 2nd	1	10
7	50/49	35¢	double ruyo comma	rryy-2	desc dim 2nd	0	6

The next table explores the nearest upside-down intervals. Again, ya JI has two. Note that the sevenfold-gu comma isn't negative, because it takes you up the scale from C to D♭♭♭. The wa minicomma, 53 steps fifthward, is a descending sevenfold-diminished sixth! Again, higher prime limits have less remote paradoxes.

Table 3.3.4 – The nearest upside-down intervals for various prime limits

prime limit	monzo	cents	name		quality & degree	keyspan	class
3	(-84, 53)	3.6¢	wa minicomma	L^8w-6	desc dim^7 6th	-1	53
5	(2, 9, -7)	13¢	sevenfold-gu comma	$g^7$2	double-dim 2nd	-1	16
5	(17, 1, -8)	11¢	eightfold-gu comma	$sg^8$3	dim^4 3rd	-1	16
7	(6, 3, -1, -3)	13¢	gu triple-ru comma	gr^3-2	desc min 2nd	-1	11

To sum up: every JI ratio has eight properties: a color, a quality, a degree, a size (measured in cents), a keyspan, a sign, a magnitude (large, small, central, etc.) and a class. Thus the yo minicomma Ly-d2 = 2¢ = 0 semitones = class 10. We've seen some pretty crazy intervals, and there's more to come, so let's review how to add and subtract intervals:

Cents add up directly. 702¢ + 498¢ = 1200¢, etc.

Degrees add up as usual if the sign is positive. An Xth and a Yth make a (X + Y - 1)th, and an Xth minus a Yth is a (X - Y + 1)th. If X - Y + 1 is zero or negative, the new interval is a descending (Y - X + 1)th, *unless* the width is positive, in which case it's a negative (Y - X + 1)th. When adding or subtracting negative intervals, treat them like descending ones. Thus adding/subtracting a negative 2nd is like subtracting/adding a regular 2nd.

Sign: add up the cents and the degrees, and if the cents are positive and the degree is descending, the sign is negative. Otherwise it's positive.

Keyspans are added/subtracted directly, X semitones and Y semitones make X + Y semitones.

Qualities don't add together, instead you add/subtract the keyspans and degrees and that tells you the quality. Thus d3 plus A2 = a 2-semitone 3rd plus a 3-semitone 2nd = a 5-semitone 4th = a perfect fourth.

Colors add together directly, with opposites gu/yo and zo/ru canceling each other out. To subtract a color, change it to its opposite and add it. Wa is its own opposite.

Ratios multiply/divide directly. A/B plus C/D = AC/BD, and A/B minus C/D = AD/BC.

Magnitudes don't add together, instead you multiply/divide the ratios and the magnitude is determined by the distance from that color's midpoint.

Classes don't add together either, instead you multiply/divide the ratios and the class is determined by the prime factors.

So far I've stressed consistency and logic in tuning. Interval X is always N semitones wide, no matter where it's played, and if interval Y is M semitones wide, the interval of X plus Y is always N + M semitones wide. I'd like to acknowledge other approaches, such as Michael Harrison's. On his CD "Revelations", he tunes his piano with several upside-down intervals. He plays it very well, and it sounds beautiful, which is of course the whole point of music. The audience doesn't care that the notes run out of order. On the other hand, only Michael Harrison can play that piano, almost every other pianist would be lost. There are certain advantages to consistency and logic!

Chapter 3.4 – Purple Intervals

Something interesting happens when you mix ruyo and zo intervals in a scale. Consider the melody z4 – ry4 – z6. The two melodic steps are 160/147 = rry1 = 147¢ and 49/45 = zzg3 = 147¢. That's a ruruyo unison and a zozogu third. How can two such different intervals be the same size? (The same thing happens when combining zogu and ru: zg2 – r2 – zg4 has the exact same intervals.) The difference between the two is a microcomma:

(-5, -1, -2, 4)	2401/2400	0.72¢	double zozogu microcomma	z^4g^23

This microcomma creates an automatic micro-tempering that equates ruruyo and zozogu to a **pseudocolor** called **purple**. A pseudocolor is simply a handy name for two real colors that are separated by only a microcomma. Pseudocolors are never used in staff notation or chord names. They are for description, not notation. Hence there is no abbreviation, especially since p already stands for po. However, p is used loosely for purple in a few diagrams in this chapter and the next one, for reasons of space.

zozogu = ruruyo = purple

The double zozogu microcomma has another, cooler name, the deep purple microcomma. It's a double diminished 3rd, spanning one semitone. Therefore two intervals differing by a deep purple microcomma are different notes of the scale and also different keys on the keyboard. For example 49/40 and 60/49 are both about 351¢, but one is a 4th and one is a 2nd. The purple 3rd is defined as the interval that lies exactly in between them. Since they add up to a fifth, a purple 3rd equals exactly half a fifth, and its irrational "ratio" is the square root of 3/2.

49/40	351.34¢	zzg4	dim 4th = 4 semitones
60/49	350.62¢	rry2	aug 2nd = 3 semitones
√(3/2)	350.98¢	purple 3rd	??? 3rd = ??? semitones

For the first time, logic has broken down and we don't know how many semitones a purple 3rd spans. For example, consider a keyboard tuning in D that includes a zo F, a ruyo G♯ and a zo C. These three notes make a purple triad with two purple 3rds, one 3 semitones and the other one 4, but they sound almost exactly the same.

Table 3.4.1 – Purple intervals

interval	ratio	cents	keyspan	theoretical quality/degree	actual quality/degree
purple 2nd	√ w3 = √(32/27)	147¢	1 or 2	A1 or d3	m2 or M2
purple 3rd	√ w5 = √(3/2)	351¢	3 or 4	A2 or d4	m3 or M3
purple 4th	√ Lw7 = √(243/128)	555¢	5 or 6	A3 or d5	P4 or A4
purple 5th	√ sw9 = √(512/243)	645¢	6 or 7	A4 or d6	d5 or P5
purple 6th	√ w11 = √(8/3)	849¢	8 or 9	A5 or d7	m6 or M6
purple 7th	√ w13 = √(27/8)	1053¢	10 or 11	A6 or d8	m7 or M7

The keyspan, quality and degree depend on whether it's considered ruruyo (aug) or zozogu (dim). Purple intervals fall exactly halfway between gu and yo. The purple 3rd is exactly one-half of a wa 5th. Having purple in the middle of the rainbow might not seem to follow our rainbow analogy. But purple intervals actually come from neighboring rainbows: the purple 3rd is both an infrared 2nd and an ultraviolet 4th.

The purple 3rd has a quality known in conventional music theory as **neutral**, abbreviated n, halfway between major and minor. However, not all purple intervals are neutral. The purple 4th is halfway between perfect and augmented, and is **half-augmented**, written hA. Likewise the purple 5th is **half-diminished**, written hd (not to be confused with conventional music theory's half-diminished tetrad 1 – m3 – d5 – m7).

The 6 purple intervals (along with r1, z2, ry3 & their inverses) neatly fill the gaps in the list of intervals in table 2.1.4, making rainbows with seven bands, and reducing the steps between intervals to 36¢ or less. This gap-filling makes purple intervals very useful. Read down the columns to follow the scale:

Table 3.4.2 – Seven-banded rainbows (*p stands for purple, not po)

zo		z2	z3	z4	z5	z6	z7	z8
zogu or wa		zg2	w3	w4	zg5	zg6	w7	w8
gu		g2	g3	g4	g5	g6	g7	
purple		p2*	p3*	p4*	p5*	p6*	p7*	
yo		y2	y3	y4	y5	y6	y7	
ruyo or wa	w1	w2	ry3	ry4	w5	w6	ry7	
ru	r1	r2	r3	r4	r5	r6	r7	

This "scale of commas" is somewhat similar to Harry Partch's 43-tone scale. It implies, and is fairly well approximated by, 41-tone equal temperament. Why 41-edo and not 45-edo, when there's 45 ratios in the table? Because the 4th and 5th rainbows overlap, with the ratios in the gray areas only a minicomma apart:

purple 4 + sry1 = z5
y4 - ryy-2 = zg5
ry4 - ryy-2 = g5
r4 + sry1= purple 5

Purple intervals may seem abstract and theoretical, but they can be very useful musically. I used them when writing "Without You", which is in a yo-zo key and uses the relative gu (g7 and s6 chords on yo roots). The basic melody in the verse goes w5 – ry4 – z4 – y3, one note per measure. The descending melodic steps are zg2 – purple 2 – zg2. The extra large 2nd step creates a feeling of unexpected falling on the third line. Since the third line of each verse is generally sadder than the first two, the use of purple reinforces this sadness and adds extra poignancy.

The purple 2nd step in this song is a neutral 2nd which is actually a ruruyo aug unison, notated as a chromatic semitone. "Without You" is written out in chapter 2.6 in B♭. The purple 2nd is the interval from zE♭ to ryE. Purple intervals are always notated one degree either larger or smaller than they sound. Unless po and qu are used, in which case, purple 3rd = zozoguku 3rd = ruruyopo 3rd.

The purple row has a quality-chain of 7 half-dim intervals, 4 neutral ones, and 7 half-aug ones. Then comes the large purple 4th = 669¢, halfway between aug and double-aug, which is **extra-augmented**, xA. (**Extra-diminished** is xd.) The purple quality-chain is 7xd – 7hd – 4n – 7hA – 7xA. All these qualities are **ambiguous** qualities, with an ambiguous keyspan. In full, it's:

extra-dim – hd2 hd6 hd3 hd7 hd4 hd8 hd5 – n2 n6 n3 n7 – hA4 hA1 hA5 hA2 hA6 hA3 hA7 – extra-aug

Purple can be combined with other colors. Purple-yo notes can also be thought of as zozo or double ruyo. They are halfway between wa and yoyo. (w5=702¢ < purple-y5=737¢ < yy5=773¢.) Likewise purple-gu notes are double zogu or ruru. Then there's purple-yoyo, etc. All these colors have the same quality-chain as purple, so the purple-yo 5th is a hA 5th. Purple-zo notes are better thought of as ruyo, likewise purple-ru ones are really zogu. Deep purple notes are by definition wa notes.

The zozo comma zz2 = 49/48 = 35.7¢ and the double ruyo comma rryy-2 = 50/49 = 35.0¢ differ by only a deep purple microcomma. They both can be approximated by the purple-yo comma = √yy1 = √25/24 = 35.3¢ = half-augmented unison = hA1.

The trouble with defining large and small purple intervals is that both the purple 6th = 49/30 and the purple 3rd = 60/49 are midpoint ratios. If we arbitrarily make the purple 3rd the midpoint, we would have the purple unison = 57¢ and the small purple octave = 1143¢. But they are octave inverses, and inverses *always* have complimentary magnitudes. The inverse of a small interval is always large and the inverse of a central one is always central. But if they're both central, we'd have two central purple octaves of 1143¢ and 1257¢. There is only one logical solution. There are only 6 central purple ratios and there is no central purple unison or octave. Instead there is the large purple unison = √LwA1 = 57¢ = hA1 and its inverse the small purple octave = √swd15 = 1143¢ = hd8. Thus the magnitude-chain for purple is different from any other color: 7ss – 7s – 6 central – 7L – 7LL. Every other color, even those containing purple, has 7 central intervals. The midpoint on the purple-yo row is the purple-yo comma = 35¢, and the purple-gu midpoint is the purple-gu half-dim 8ve = 1165¢. Other midpoints are the purple-yoyo half-aug 6th = 920¢ and the purple-gug half-dim 3rd = 280¢.

There is no small purple unison, just as there is no yo unison or zo unison, because it would be a descending interval. An alternate name for the large purple unison is the **purple quartertone**. (Quartertone is a conventional music theory term meaning half of a semitone.) Every purple interval is a purple quartertone away from a wa interval. For example, the purple 3rd is a quartertone larger than w3 and a quartertone smaller than Lw3. This holds with other colors too: every purple-yo interval is a purple quartertone away from a yo interval.

If you mix zo and gu intervals in a scale, or yo and ru ones, you get two other types of neutral-sounding interval, zoyo and rugu:

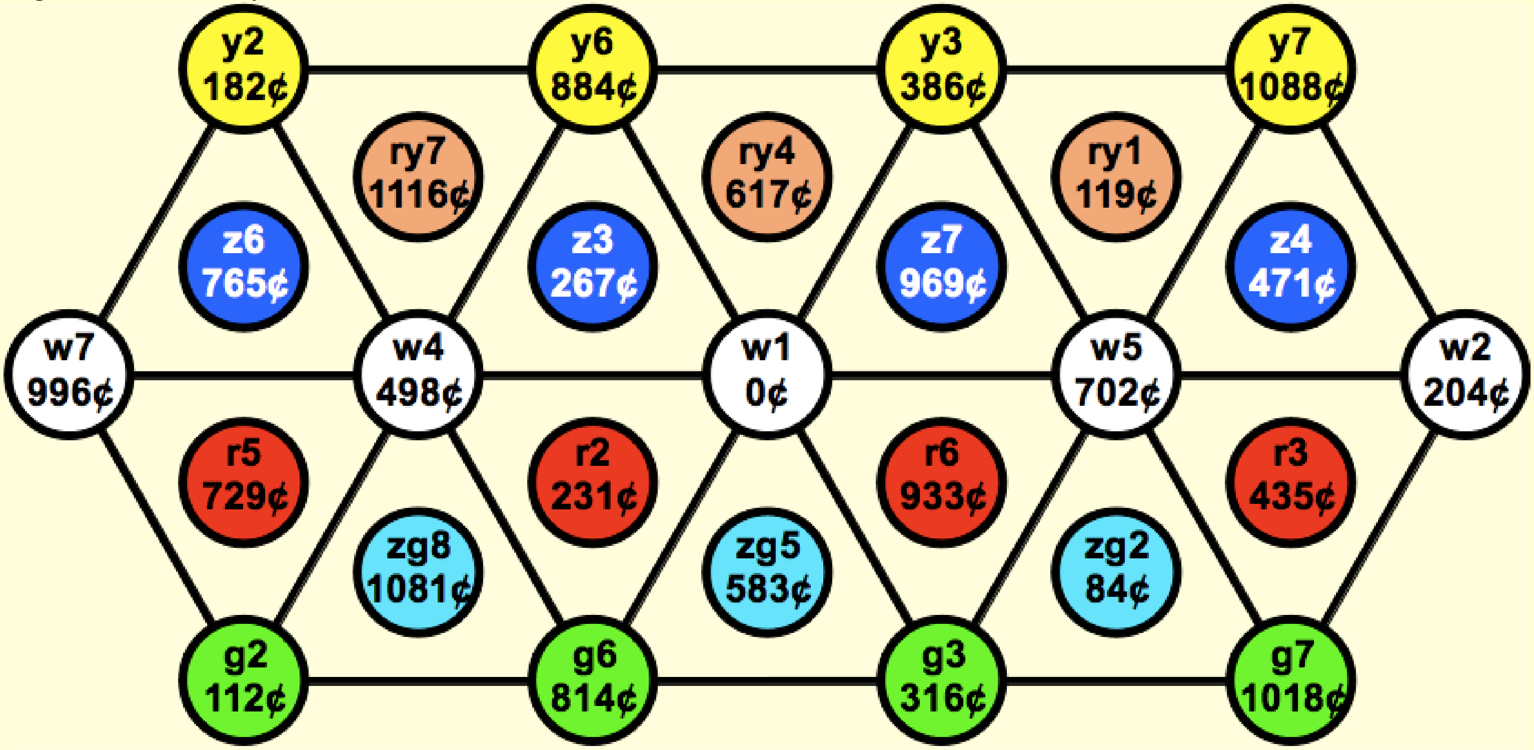
> large zoyo 3rd = LzyM3 = 315/256 = 359¢
> Possible derivations: LzyM3 = z4 - g2, LzyM3 = y7 - r5
> Difference from other 3rds: LzyM3 = y3 - r1, LzyM3 = purple 3rd + ryy-2
>
> rugu 3rd = rgm3 = 128/105 = 343¢
> Derivations: rgm3 = g6 - z4 = r9 - y7 = g3 + r1 = purple 3rd - ryy-2

Unlike purple intervals, these intervals have a well-defined quality and keyspan. The Beatles song "You Never Give Me Your Money" tuned yaza might use a zoyo 2nd for the z7 in the III chord of the verse: Iy – yIIIh7 – yVIg – Ih7 – IVy – Vy – Iy.

Chapter 3.5 – The Expanded Harmonic Lattice

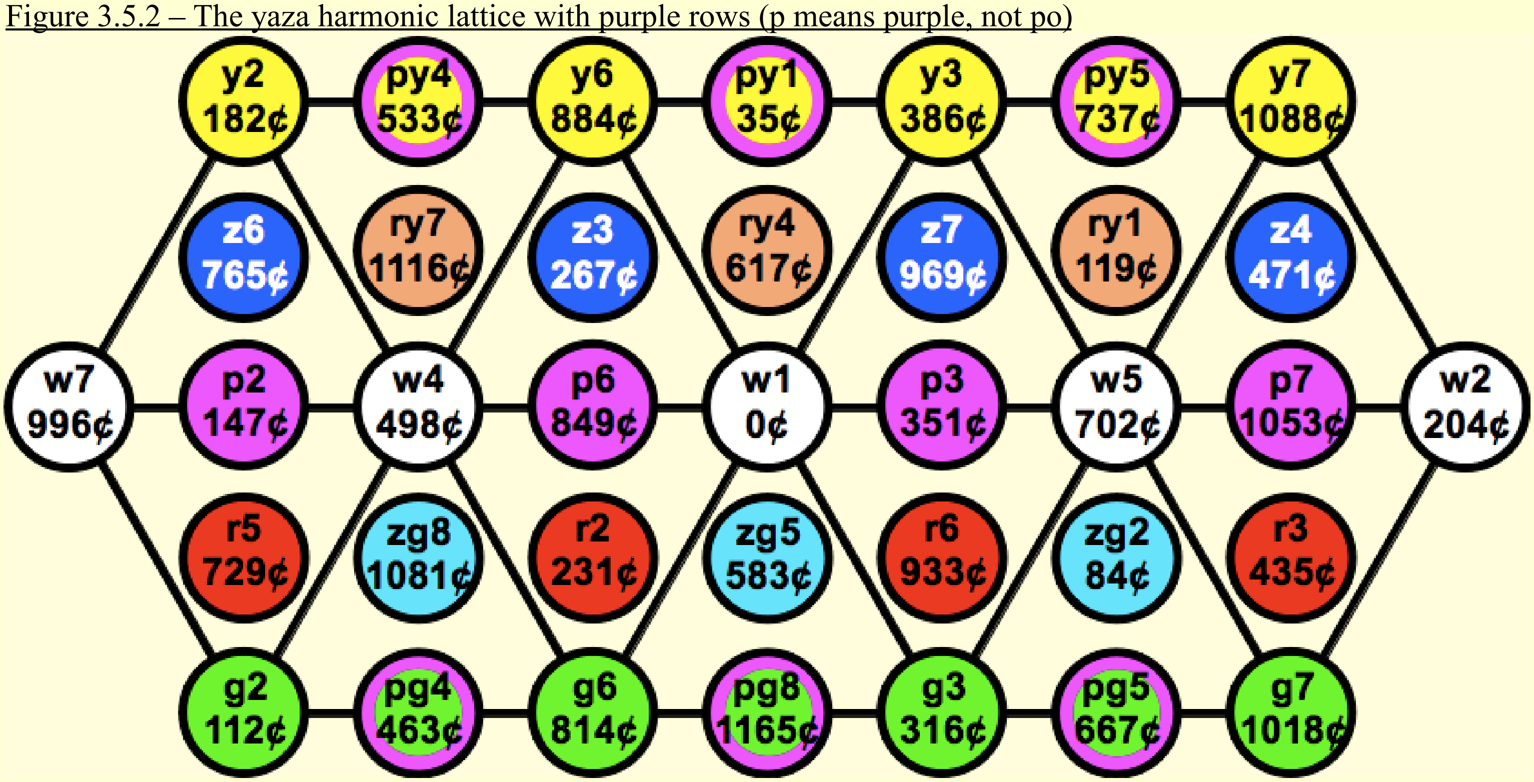
The usual tetrahedral harmonic lattice is a 2-D diagram of a 3-D object.

Figure 3.5.1 – The yaza harmonic lattice



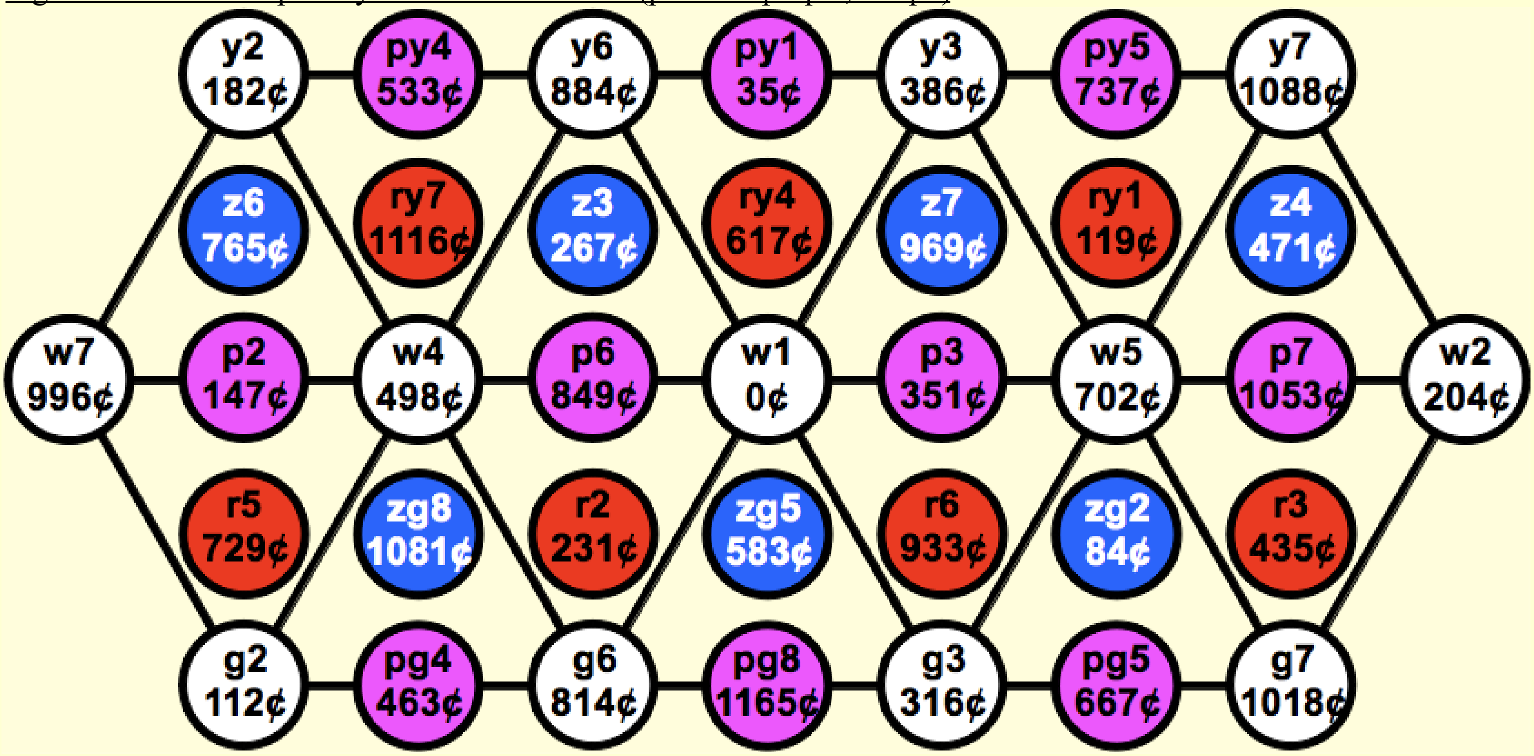
While it can be expanded to the right and left (large & small) and up and down (yoyo & gugu), there's no room to expand it septimally (zozo & ruru). This makes it hard to visualize certain chord progressions. Fortunately, tempering out an inaudible microcomma reduces the lattice to two dimensions, yet it still accurately represents JI. The best one is the deep purple microcomma. It modifies the lattice so that four z7 steps covers the same ground in two dimensions as one w5 step and two y3 steps (g6 → zg5 → p3 → ry1 → y7 equals g6 → g3 → w5 → y7).

Figure 3.5.2 – The yaza harmonic lattice with purple rows (p means purple, not po)



There's ten colors in the lattice. It might be easier to read if we use a different color scheme:

Figure 3.5.3 – The 4-plane yaza harmonic lattice (p means purple, not po)



The key to understanding this chart is to view it simultaneously in both two dimensions and three. Two-dimensionally, a sideways step = a purple 3rd, and an upwards step = ry4 = 10/7. Over-two-steps = a fifth, over-and-up = z7 = 7/4, and over-and-up-two = y3 = 5/4. These relationships hold no matter which color you start on. As noted previously, in all lattices, if three notes line up, the center one is exactly halfway between the other two melodically as well (allowing for octave transpositions).

Three-dimensionally, there are four planes. The ya plane, i.e the 5-limit plane, contains all the ya ratios (wa, yo and gu). Above it is the zo plane (zo and zogu). Below is the ru plane (ru and ruyo). The purple plane contains purple, purple-yo and purple-gu. It's both the zozo plane and the ruru plane, and it's both above the zo plane and below the ru one! The h7 chord on the purple 6th has a note on the ru plane (ry4), and the s6 on the purple 6th chord has a note on the zo plane (zg5).

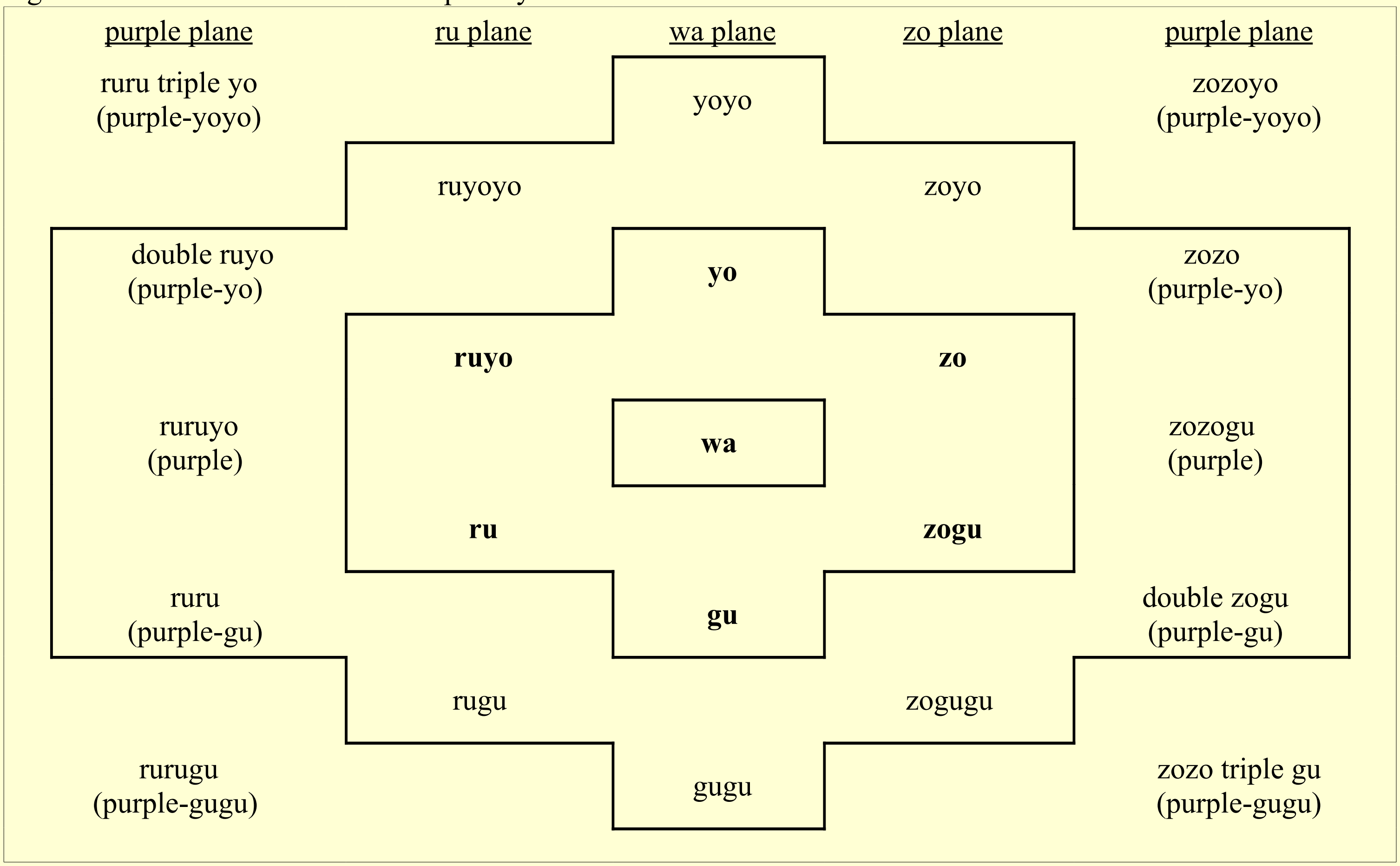
Just as octaves are invisible in the lattice, here the deep purple microcomma is also invisible. Just as C3 and C4 occupy the same spot in the lattice, so does the zozo plane and the ruru plane. Unfortunately, if viewed two-dimensionally, the simplest ratios are no longer the closest ones. ry4 = 10/7 and purple 3rd = 60/49 or 49/40 are closer to w1 than w5 = 3/2 and y3 = 5/4.

We've reduced the rank-4 lattice to rank-3 by tempering out a microcomma. Rank and temperaments are covered in Part IV. An example tuning: the octaves and fifths are pure, 5/4 is sharpened by one-sixth of a deep purple microcomma = 0.12¢ and 7/4 is flattened by the same amount. Thus a sideways step equals √(3/2), and an upwards step equals 10/7 plus a third of a microcomma.

The harmonic lattice can also be sliced up into vertical planes. The za plane contains ruru (purple-gu), ru, wa, zo & zozo (purple-yo). The yo plane is parallel to the za plane and contains the purple, ruyo, yo and zoyo rows. The parallel gu plane has rugu, gu, zogu and purple. Note that purple is on both the yo and the gu planes.

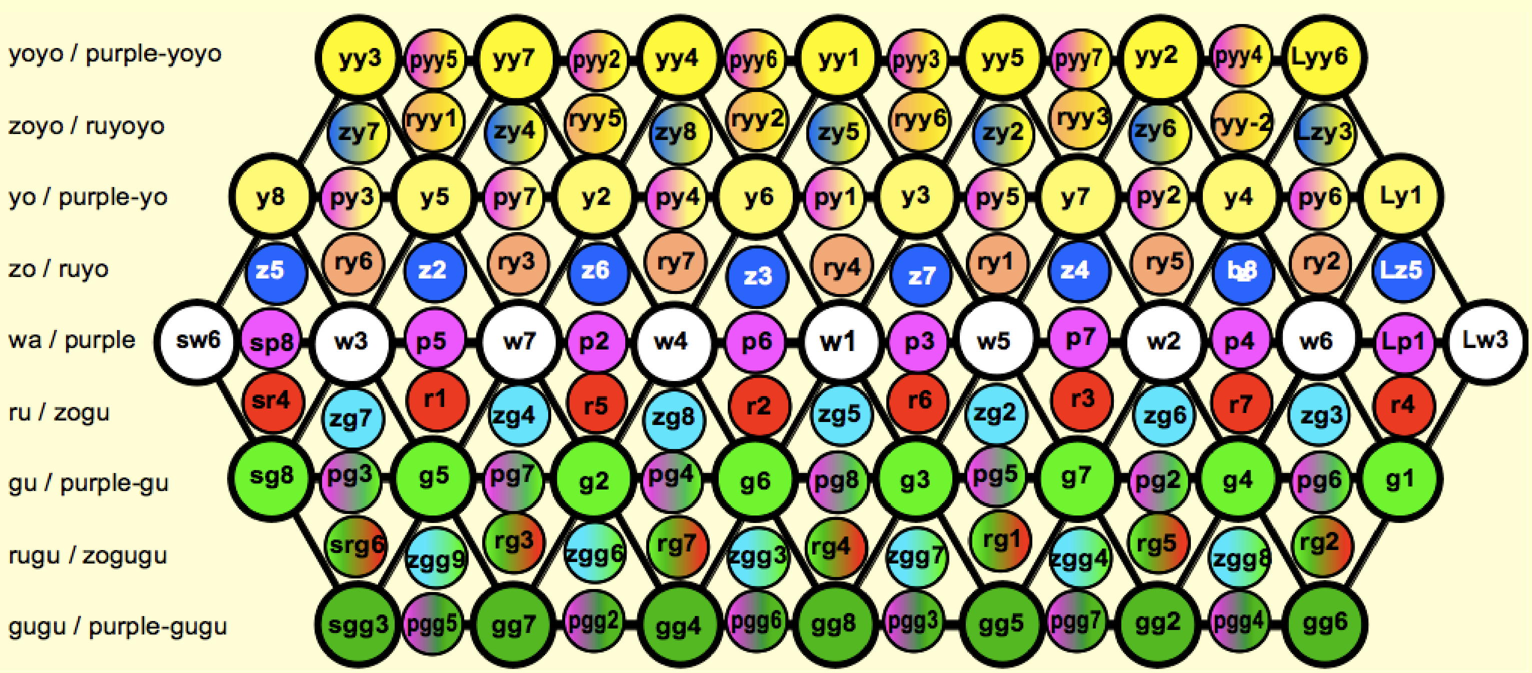
In the next figure, each entry is a row and each column is a horizontal plane, with the purple plane shown twice. The lattice is extended to include yoyo and gugu.

Figure 3.5.4 – Cross section of the 4-plane yaza harmonic lattice



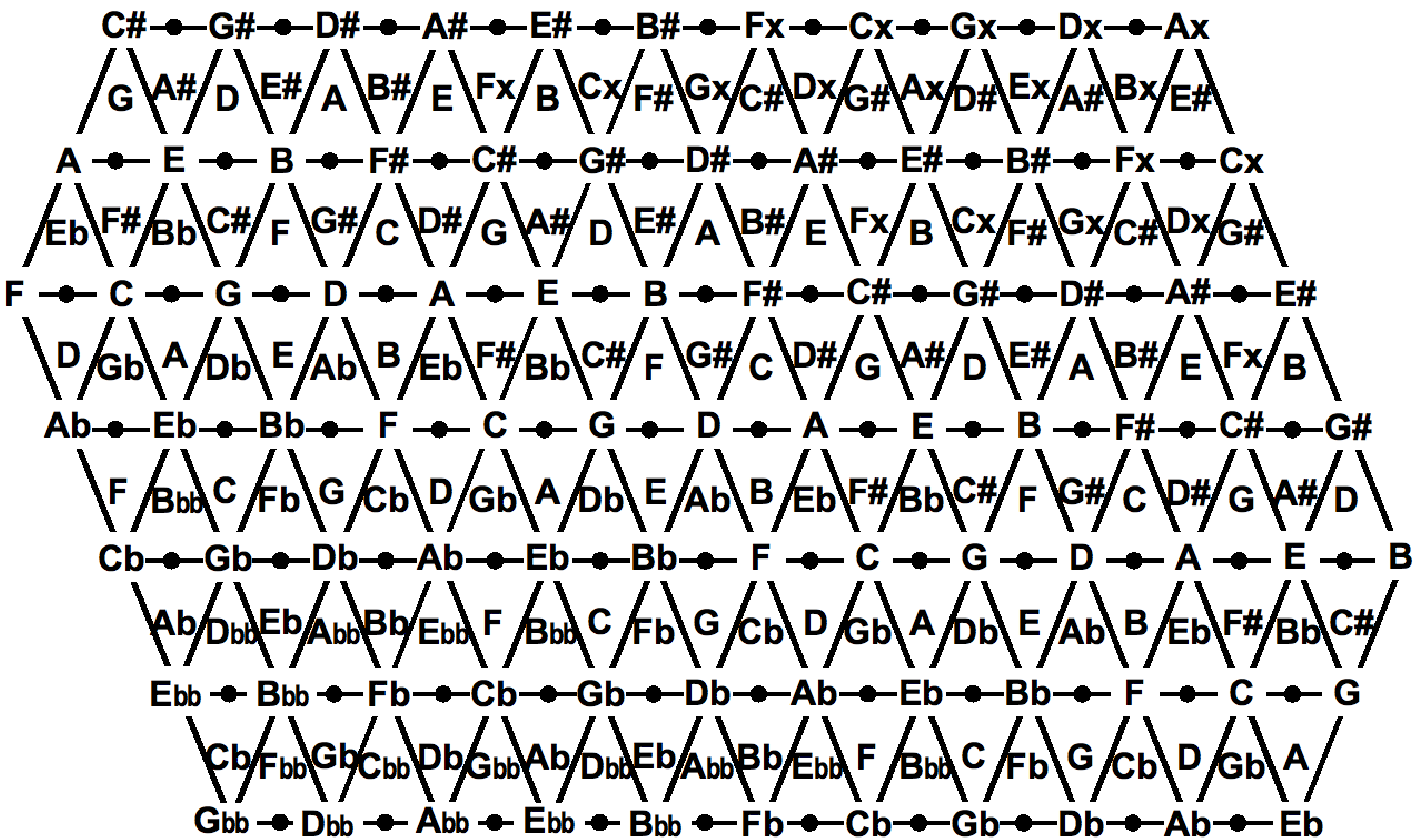
The same lattice, extended sideways to include large and small:

Figure 3.5.5 – The harmonic lattice with double colors and purple rows (p means purple, not po)



Because purple is both zozogu and ruruyo, purple notes have two names. To avoid this problem, the following printable lattice diagram shows the purple plane as nameless dots. The two possible names are directly above and below the dot, on the wa plane. For example, the dot between G and D can be either a B♭ or a B. As we'll see in the next chapter, this lattice can also be used for higher-limit primes.

Figure 3.5.6 – The expanded harmonic lattice with neutral intervals



Chapter 3.6 – 11-limit and 13-limit Intervals

For certain intervals and chords, slightly adjusting the intonation will increase the prime limit up to the next prime and greatly decrease the odd limit. I think of these chords as "deal-breakers"; IF they are perceived as consonant (or "interesting", see the limits discussion in chapter 1.2), they will force us to use a higher prime limit. In the Renaissance, when the major third became accepted as a consonance, and the major triad became used more, it "broke" 3-limit, as 81/64 was effectively overshadowed by the nearby 5/4. Likewise, as the dom7 chord and the dim triad come to be accepted as consonant, they break ya. Diminished 5ths like 36/25 and 64/45 are overshadowed by 7/5. Among yaza's deal-breakers are the neutral 3rd (60/49 or 49/40 overshadowed by 11/9) and the half-augmented 4th (135/98 overshadowed by 11/8). If these intervals are considered consonant, 11-limit JI becomes desirable.

Color notation is easily expanded to higher primes. The rainbow analogy breaks down, because 11-over and 11-under ratios are clumped tightly together midway between yo and gu. In fact, the 11-over 2nd, 3rd, 6th and 7th are only 7.1¢ flat of the 11-under 2nd, 3rd, 6th and 7th. Rather than two distinct colors, the impression is of two slightly different shades of the same color. This color is the pseudocolor that results from tempering out 243/242 and merging 11-over and 11-under. This pseudocolor is named **lavender** (mnemonic: e-leven-der). The 11-over shade is **lovender**, and the 11-under shade is **luvender**. Lovender is the lower (flatter) of the two (mnemonic: low-vender). 11-over and 11-under aka lovender/luvender are abbreviated as **lo** and **lu**. In order to distinguish "lo C" from "low C", lo has an alternate form, **ilo** ("ee-LOW"). The disambiguation prefix **i-** is only used if lo appears alone; 11/10 is logu not ilogu. 11-all or 2.3.11 is **la**, or **ila** to distinguish it from the solfege syllable La, and 11-limit is yazala. La or ila is not an abbreviation for lavender. La is a prime subgroup and lavender is a pseudocolor. Lavender sounds identical to purple, but is derived differently.

Unfortunately, lo and lu can't be used in interval names and chord names, because the letter l looks too much like the roman numeral I. The I chord with an ilo 3rd would be Ilo. 11-over and 11-under can't be abbreviated as 11o and 11u, because 11o and 11u in a chord name would imply an 11th chord. The C chord with an ilo 3rd would be C11o, which appears to be C11 diminished. Instead, 11o and 11u are shortened to **1o** and **1u**. The I lo chord is I1o, which in a sans-serif font like arial is I1o. Lolo is written 1oo, triple lu is $1u^3$, etc. On the staff, 1o or 1u is written before the notehead.

13-over and 13-under ratios also clump together midway between yo and gu, but they are 17¢ apart, a little too far apart to feel like shades of the same color. Thus there is no pseudocolor for 13. The color names for 13 are derived directly from the word thirteen, and don't indicate any actual hue or tint. 13-over is **tho**, 13-under is **thu**, and 13-all is **tha**. Tho is lower than thu. In chord names and interval names, tho and thu are written **3o** and **3u**, because 13o or 13u would imply a 13th chord. Thotho is written 3oo. On the staff, 3o or 3u is written before the notehead.

The ilo, lu, tho and thu midpoints are 11/6, 12/11, 13/12 and 24/13, which creates these central intervals:

Table 3.6.1 – Ilo, lu, tho and thu central intervals

ilo				lu				tho				thu			
1o1	hA1	33/32	53¢									3u1	hA1	27/26	65¢
1o2	n2	88/81	143¢	1u2	n2	12/11	151¢	3o2	n2	13/12	139¢	3u2	n2	128/117	155¢
1o3	n3	11/9	347¢	1u3	n3	27/22	355¢	3o3	n3	39/32	343¢	3u3	n3	16/13	359¢
1o4	hA4	11/8	551¢	1u4	hd4	128/99	445¢	3o4	hd4	104/81	433¢	3u4	hA4	18/13	563¢
1o5	hA5	99/64	755¢	1u5	hd5	16/11	649¢	3o5	hd5	13/9	637¢	3u5	hA5	81/52	767¢
1o6	n6	44/27	845¢	1u6	n6	18/11	853¢	3o6	n6	13/8	841¢	3u6	n6	64/39	857¢
1o7	n7	11/6	1049¢	1u7	n7	81/44	1057¢	3o7	n7	117/64	1045¢	3u7	n7	24/13	1061¢
1o8	hA8	33/16	1253¢	1u8	hd8	64/33	1147¢	3o8	hd8	52/27	1135¢	3u8	hA8	27/13	1265¢

Central ilo and thu intervals are neutral or half-augmented. Central lu and tho intervals are neutral or half-diminished.

All yaza JI concepts can be applied to higher primes: large lu, triple tho, yala and zatha, ilo triads, etc. There are compound colors, such as 14/13 = 3uz2 = thuzo 2nd, and 243/242 = 1uu1 = the lulu minicomma = 7.1¢.

When using 11 and/or 13, some people like to omit 5 and/or 7 from the prime subgroup, for simplicity. Subgroups which omit 5 or 7 are **noya** or **noza**, and those which omit both are **noyaza**. Unlike nowa, these terms aren't used in subgroup names. They are general descriptive terms, e.g. zala, latha and zalatha are all noya.

The augmented triad (which could be considered one of yaza's deal-breakers) finds a low odd-limit representation as the ru loru-5 chord w1 – r3 – 1or5, or 7:9:11. The 4:5:6:7:9:11:13 chord (which could be considered one of yazala's deal-breakers) is written y,z7,9,1o11,3o13. Remoteness is calculated as before with the (p-1)/2 formula. The quality-chain for all these colors is the same as for purple, ambiguous (i.e. neutral, half-aug, etc).

How to place la and tha ratios onto a standard keyboard? That depends entirely on the keyspan of the ilo rung and the tho rung, which is not obvious. Unfortunately, whatever keyspans we choose, there will be non-remote diminished unisons that will cause minor to be sharper than major.

The ilo rung's keyspan is either 5 or 6, and 1o4 is either perfect or augmented. In C, the 11/8 is either F or F♯. Supposing it's F♯, consider the chord D1o = D – 1oF♯ – A. The lower 3rd is narrower than the upper one, but spans more semitones. The difference between the minor lu 3rd and the smaller major ilo 3rd is the lulu diminished unison = 243/242 = 7¢.

If instead 11/8 = F, in the melody D – yE – 1oF – G – A, the yE – 1oF step is wider, but spans fewer semitones, than 1oF – G. The difference between the minor logu 2nd and the smaller major lu 2nd is the lologu diminished unison = 121/120 = 14¢.

The correlation between an interval's size and its keyspan, already weakened by purple intervals, is further eroded.

The same problem arises with the tho rung, which has a keyspan of either 8 or 9, and a quality of either minor or major. In C, 13/8 is either A♭ or A. If it's A, in the chord F3o = F – 3oA – C, F – A is narrower than A – C. The difference between the minor thu 3rd and the smaller major tho 3rd is the thuthu dim1 = 512/507 = 16¢.

If instead 13/8 is A♭, in the melody F – G – 3oA♭ – zB♭– C, G – A♭ is wider than A♭– B♭. The difference between the minor tho 2nd and the smaller major thuzo 2nd is the thothoru dim1 = 169/168 = 11¢.

These diminished unisons are small, but they are not as remote as the ones in chapter 3.3, and the examples are not at all contrived.

There is no obvious way to notate these intervals. If la and tha ratios are used only occasionally, their notation and keyspans may be determined on a case-by-case basis, depending on the piece. There will usually be contradictions, so take the lesser-of-the-two-evils approach. But if la or tha ratios are used regularly, it's best to choose one notation and stick with it. There is no consensus among microtonalists for 1o4 = P4/A4 or 3o6 = m6/M6.

For example, take the harmonics 8-16 scale w1, w2, y3, 1o4, w5, 3o6, z7, y7, w8. We look at the intervals between all the notes: 1or5 = 11/7 = 782¢ is a fifth that's too big to be a perfect fifth. It should be augmented, so 1o4 = 1orA5 - rM2 must be augmented too. That takes care of ilo, what about tho? The tholu 3rd 13/11 = 289¢ should be a minor 3rd. Likewise 3uz2 = 14/13 = 128¢ should also be minor, not major. This leads to the tho 6th = 13/8 being major. That gives us wP1, wM2, yM3, 1oA4, wP5, 3oM6, zm7, yM7, wP8. In C, that would be wC, wD, yE, 1oF♯, wG, 3oA, zB♭, yB, wC.

There are still contradictions. For example 3oM2 = 13/12 = 139¢ and 1um2 = 12/11 = 151¢, thus the major tho 2nd is flatter than the minor lu 2nd by the thulu dim unison = 144/143 = 12¢. However, as 3o2 and 1u2 both sound fairly neutral, I find 3o2 being major and 1u2 being minor less problematic than 1or5 being perfect or 3o1u3 being diminished or 3uz2 being major. The 3u1u dim1 also shows up as 1oM3 < 3um3 and 1oA4 < 3uP4.

By the same logic, the subharmonic-series scale becomes wP1, gm2, rM2, 3um3, wP4, 1ud5, gm6, wm7, wP8. In C, that's wC, gD♭, rD, 3uE♭, wF, 1uG♭, gA♭, wB♭, wC.

Another example is the arabic maqam Rast, which runs P1, M2, n3, P4, P5, M6, n7, P8. It might be tuned w1, w2, 1o3, w4, w5, w6, 1o7, w8. In this case 1o3 and 1o7 could be either major or minor intervals, as long as they both have the same quality, so that 1o3 to 1o7 makes a perfect 5th. However, the descending scale uses a wa minor 7th, forcing both

1o3 and 1o7 to be major: wP1, wM2, 1oM3, wP4, wP5, wM6, wm7, 1oM7, wP8. In C, that would be wC, wD, 1oE, wF, wG, wA, wB♭, 1oB, wC. Note that 1oM3 to wP5 is 1um3, and 1um3 > 1oM3.

Maqam Sikah runs P1, n2, n3, hA4, P5, n6, n7, P8 ascending, with a hd5 in the descending form. The neutral intervals form a chain of 5ths: hd5 – n2 – n6 – n3 – n7 – hA4. Tuning the hA4 as 1o4, and tuning the chain by w5's creates w1, 1o2, 1o3, 1o4, s1o5, w5, 1o6, 1o7, w8. The w5 is of course perfect, so the s1o5 must be diminished, which means the 1o4 can't be augmented and must be perfect. Mapping all the 5ths in the chain as perfect 5ths results in wP1, 1om2, 1om3, 1oP4, s1od5, wP5, 1om6, 1om7, wP8. In C: wC, 1oD♭, 1oE♭, 1oF, s1oG♭, wG, 1oA♭, 1oB♭, wC.

This tuning actually has no upside-down intervals, although some minor 2nds & 3rds are only 1uu1 = 7¢ narrower than some major ones. It can be extended with wa intervals to 12 chromatic notes:

> wP1, 1om2, wM2, 1om3, LwM3, 1oP4, s1od5, wP5, 1om6, wM6, 1om7, LwM7, wP8
> In C: wC, 1oD♭, wD, 1oE♭, wE, 1oF, s1oG♭, wG, 1oA♭, wA, 1oB♭, wB, wC

A full 12-note la tuning without contradictions! But if we change Lw3 to y3, contradictions result. It seems that to avoid upside-down intervals, the ilo 4th must be perfect, and yo must not be present. Gu can be used; we could reduce the odd limit by replacing s1o5 = 352/243 and 1o2 = 88/81 with g5 and g2.

Let's explore this tuning further. Here are some of its modes:

> Starting on G: wG, 1oA♭, wA, 1oB♭, wB, wC, s1oD♭, wD, 1oE♭, wE, 1oF, s1oG♭, wG
> wP1, 1om2, wM2, 1om3, LwM3, wP4, s1od5, wP5, 1om6, wM6, 1om7, s1od8, wP8
>
> Starting on D: wD, 1oE♭, wE, 1oF, s1oG♭, wG, s1oA♭, wA, 1oB♭, wB, wC, s1oD♭, wD
> wP1, 1om2, wM2, 1om3, s1od4, wP4, s1od5, wP5, 1om6, wM6, wm7, s1od8, wP8

When using staff notation, the keyspans of the ilo and tho rungs should be clearly specified. One can write "1o4 = P4" or "1o4 = A4" at the top of the page, or possibly above the key signature. A color accidental always represents a comma with zero keyspan and stepspan, so that 1oE is on the same key as wE. Thus "1o" either raises wa by 1o1 = 33/32 = 53¢ (if 1o4 = P4), or lowers it by L1u1 = 729/704 = 60¢ (if 1o4 = A4). And "3o" either raises wa by L3o1 = 1053/1024 = 48¢ (if 3o6 = m6) or lowers it by 3u1 = 27/26 = 65¢ (if 3o6 = M6).

When you invite 11 and 13 to the party, it gets rather crowded. Every ambiguous interval contains a miniature rainbow, with bands only 4-6¢ wide. We are like astronomers, finding a large gap between Mars and Jupiter, looking for a planet there and instead finding the asteroid belt. To my ears, this lack of one clearly preferable color creates a "so out of tune it's in tune" quality to neutral intervals. Shown here high to low, omitting the more remote intervals:

Table 3.6.2 – Select neutral, half-augmented and half-diminished intervals in yazalatha JI (*p means purple, not po)

Colors	half-aug unisons		neutral seconds		neutral thirds		half-aug fourths		half-dimin fifths		neutral sixths		neutral sevenths		half-dimin octaves	
1ug, zg	21/20	84¢			56/45	379¢	7/5	582¢	81/55	670¢			28/15	1081¢		
1or, zzgg, rr	22/21	81¢	54/49	168¢			88/63	579¢	72/49	666¢	81/49	870¢			49/25	1165¢
1og, 3or			11/10	165¢	26/21	370¢			22/15	663¢	33/20	867¢	13/7	1072¢	88/45	1161¢
yy	25/24	71¢			100/81	365¢	25/18	569¢					50/27	1067¢		
3u, zy	27/26	65¢	35/32	155¢	16/13	359¢	18/13	563¢	35/24	653¢	64/39	857¢	24/13	1061¢	35/18	1151¢
1u			12/11	151¢	27/22	355¢			16/11	649¢	18/11	853¢			64/33	1147¢
purple	**Lp1***	**57¢**	**p2***	**147¢**	**p3***	**351¢**	**p4***	**555¢**	**p5***	**645¢**	**p6***	**849¢**	**p7***	**1053¢**	**sp8***	**1143¢**
1o	33/32	53¢			11/9	347¢	11/8	551¢			44/27	845¢	11/6	1049¢		
3o, rg	36/35	49¢	13/12	139¢	39/32	343¢	48/35	547¢	13/9	637¢	13/8	841¢	64/35	1045¢	52/27	1135¢
gg			27/25	133¢					36/25	631¢	81/50	835¢			48/25	1129¢
1uy, 3uz	45/44	39¢	14/13	128¢	40/33	333¢	15/11	537¢			21/13	830¢	20/11	1035¢		
1uz, rryy, zz	49/48	36¢			98/81	330¢	49/36	534¢	63/44	621¢			49/27	1032¢	21/11	1119¢
1oy, ry			15/14	119¢			110/81	530¢	10/7	618¢	45/28	821¢			40/21	1116¢

The chart is symmetrical around the purple row. The purple, ilo, lu, tho and thu neutral 3rds all have roughly the same remoteness, class 6 or 7. A purple triad is class 6, ilo and tho triads are class 7.

The next few tables summarize the paradoxical intervals of yazala and yazalatha limit. Surprisingly, the negative intervals are more remote than those of yaza, necessitating excluding yaza intervals like 50/49. The reasons for this, as we'll see in Part V, is that 7-edo approximates 11/8 and 13/8 better than it does 7/4 and 5/4.

The nearest upside-down interval depends on which keyspan is chosen for the la and tha rungs. Yazala is worse than yaza unless the ilo 4th is rounded up to A4. Notice how much yazalatha varies. Rounding la up and tha down, or vice versa, results in dangerously near paradoxes. To avoid upside-down intervals, it's better to round them off in the same direction. Since rounding la up is better, this suggests 1o4 = A4 and 3o6 = M6 as the keyspans that best avoid upside-down intervals.

Table 3.6.3 – The nearest negative intervals for various prime limits

prime limit	ratio or monzo	cents	name		quality	keyspan	class
wa	(-19,12)	23.5¢	wa comma	LLw-2	desc dim 2nd	0	12
ya	(-15,8,1)	1.95¢	yo minicomma	Ly-2	desc dim 2nd	0	10
yaza z7 = m7	50/49	35¢	double ruyo comma	rryy-2	desc dim 2nd	0	6
yaza z7 = A6	49/48	36¢	zozo comma	zz2	desc dbl-dim 2nd	1	6
yazala	99/98	18¢	loruru comma	1orr-2	desc 2nd (dim or minor)	0 or -1	9
yazalatha	275/273 = (0,-1,2,-1,1,-1)	13¢	thulo-ruyoyo comma	3u1oryy-2	desc 2nd (dbl-dim or dim or minor)	1, 0 or -1	11

Table 3.6.4 – The nearest upside-down intervals for various prime limits

prime limit	ratio or monzo	cents	name		quality	keyspan	class
wa	(-84,53)	3.6¢	wa minicomma	L^8w-6	desc dim^7 6th	-1	53
ya	(2,9,-7)	13¢	sevenfold-gu comma	$g^7$2	double-dim 2nd	-1	16
yaza	(6,3,-1,-3)	13¢	gu triple ru comma	r^3g-2	desc min 2nd	-1	11
yazala ilo 4th = P4	99/98	18¢	loruru comma	1orr-2	desc min 2nd	-1	9
yazala ilo 4th = A4	540/539 = (2,3,1,-2,-1)	3.2¢	lururuyo minicomma	1urry-2	desc min 2nd	-1	12
yazatha tho 6th = m6	351/350 = (-1,3,-2,-1,0,1)	4.9¢	thorugugu minicomma	3orgg1	dim unison	-1	11
yazatha tho 6th = M6	729/728 = (-3,6,0,-1,0,-1)	2¢	thuru comma	L3ur-2	desc min 2nd	-1	13
yazalatha 1o4=P4, 3o6=m6	143/140 = (-2,0,-1,-1,1,1)	37¢	tholorugu comma	3o1org1	dim unison	-1	11
yazalatha 1o4=A4, 3o6=m6	78/77	22¢	tholuru comma	3o1ur1	dim unison	-1	9
yazalatha 1o4=P4, 3o6=M6	66/65	26¢	thulogu comma	3u1og1	dim unison	-1	8
yazalatha 1o4=A4, 3o6=M6	144/143 = (4,2,0,0,-1,-1)	13¢	thulu comma	3u1u1	dim unison	-1	11

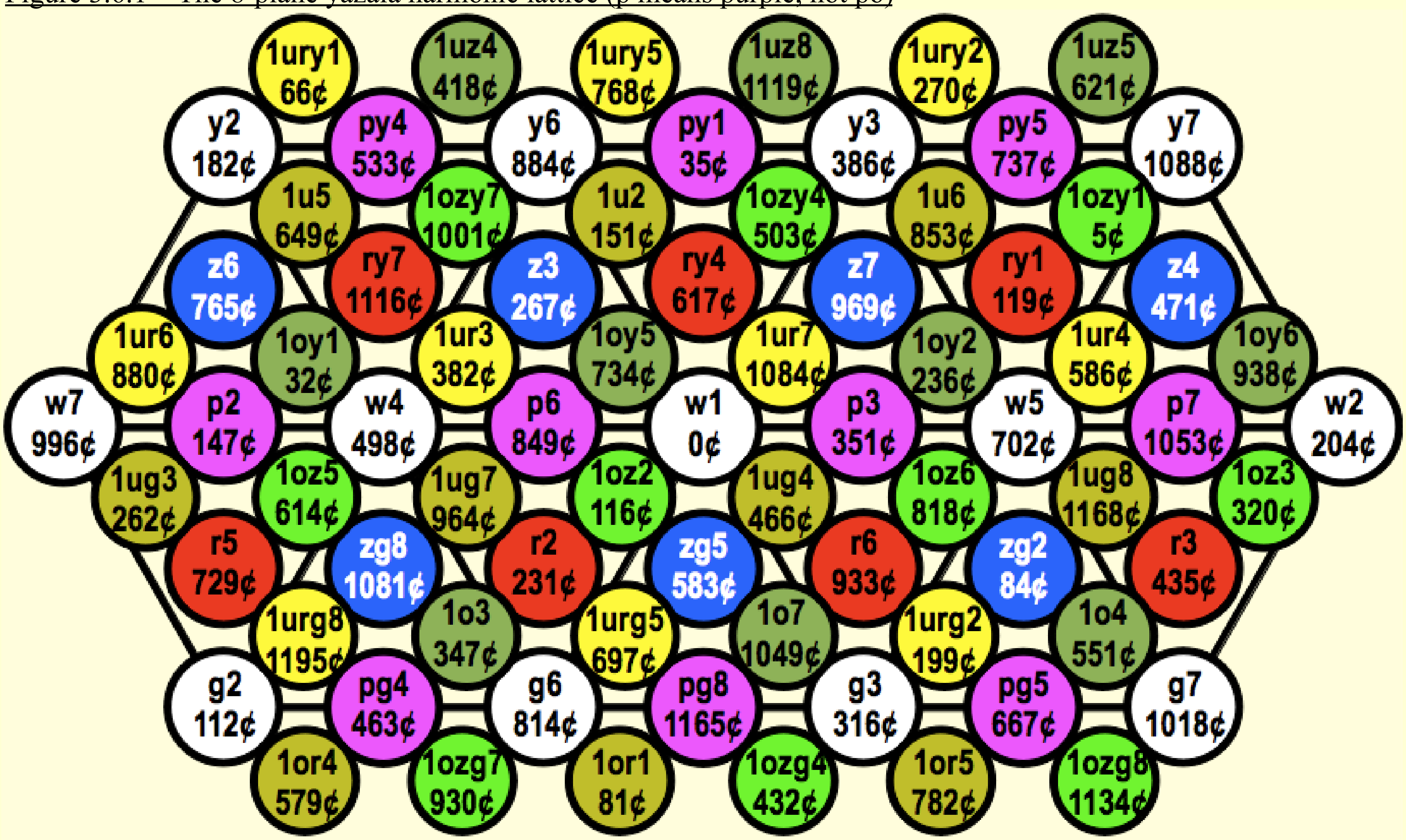
Each prime adds a dimension to the lattice. Wa, ya and za create a 3-D lattice of tetrahedrons. Where to put these new ratios in the harmonic lattice? If you can visualize 4 or 5 dimensions, you can put them anywhere. If not, using the same approach as the previous chapter, the layout of higher primes' rungs should reflect small commas with lolo or lulu. This collapses the lattice down to 3-D. For example, using the lulu minicomma, the ilo 4th is placed between w2 & w6. Alt-tuner defaults to this lavender placement of la ratios. With this placement, and the deep purple placement of za from the previous chapter, the 2-D lattice in figure 3.5.6 can be used for yazala JI, with the nameless dots representing both lavender and purple.

Two tho 6ths are about a 4th plus an octave, so the tho 6th might be placed between w1 and w4 (alt-tuner's default). The comma is the thuthu comma = 512/507 = 17¢, clearly audible. If la is placed via the lulu comma, the ilo and tho rows will coincide if lengthened, because the thulo comma = 3u1o1 = 352/351 = 4.9¢ becomes invisible. With these placements, the nameless dots in figure 3.5.6 represent purple, ilo, lu, tho and thu. Or, each nameless dot could be replaced with a short column of 5 dots, for these 5 colors. Double colors like lolo and thotho wouldn't appear.

Since two 1o4's are also about equal to y7 (missing by 1oog1 = 14¢), the ilo 4th could instead be set to half a y7, with horiz = 150 and vert = 86. See the "Full 11-limit" example in chapter 6.9. The 1o4 would be placed halfway between y3 and w5. Because 14¢ is clearly audible, 1o4 and 1uy4 might be placed side by side on the y3–w5 lattice segment.

For a more accurate placement of la, we need an inaudible microcomma containing double (or triple, etc.) ilo or lu. One possibility is the loloruyoyo microcomma = 1ooryy-2 = 3025/3024 = (-4, -3, 2, -1, 2) = 0.57¢. This makes two 1o4's equal to zgg8 = 189/100, and places 1o4 midway between zg2 = 21/20 and g7 = 9/5. Combining this placement with the deep purple placement of zo makes this lattice:

Figure 3.6.1 – The 8-plane yazala harmonic lattice (p means purple, not po)



This lattice is 4-D, but can be viewed as 3-D, with 8 planes. They can be traced by going from the purple-gu 4th in the lower left to the purple-yo 5th in the upper right: purple-gu, ilo, ru, lozo, wa, luru, zo, lu, and purple-yo. The ilo plane contains the ilo, loyo and luzo ratios, shown here in dull green. Two steps above it is the lozo plane, containing lozo, lozogu and lozoyo, shown here in bright green. Above that is the luru plane, in yellow. Above that is the lu plane, in brown, which includes loru.

An alternate interpretation traces the planes from the purple-gu 5th up to the purple-yo 4th: purple-gu, luru, ru, lugu, wa, loyo, zo, lozoyo, and purple-yo. Either way, two loyo intervals add up to a zo interval. Adding purple to any color moves you one step sideways to a new color. For example, purple-zo is ruyo, and purple-ruyo is zo.

This lattice can also be viewed as 2-D. The w5 is two steps in the northeast direction and two steps in the southeast direction. y3 is 3 NE steps minus 1 SE step. z7 is 2 NE steps. 1o4 is 1 NE plus 4 SE steps. To locate any ratio, add up the steps: 385/384 = 1ozy1 = y3 + z7 - w5 + 1o4 = (3, -1) + (2, 0) - (2, 2) + (1, 4) = (4, 1).

An exampe tuning: the wa 5th and the yo 3rd are sharpened by 1/7 of a deep purple microcomma = 0.1¢, and the zo 7th is flattened by the same amount. The ilo 4th is flattened by half a 1ooryy-2 = 0.3¢. A NE step = T1ur7 = descending 115.64¢, and a SE step = T1ug4 = 466.67¢. However, the cents shown in Figure 3.6.1 are untempered.

Chapter 3.7 – Higher Primes: 17, 19, and Beyond

Primes higher than 13 are also useful. For example, 19/16 has a high prime limit but not too large an odd limit, and widens well, so it gives 6/5 and 32/27 some competition in wider voicings as a consonant minor third. 17/8 makes a good minor 9th. See the next chapter for more on such chords.

Every prime higher than 13 uses the prime number itself, plus -o, -u and -a for over/under/all. This is the short form, the long form is the first letter of the number followed by the usual vowel.

seventeen-over: short form **17o**, long form **so** (or perhaps **iso** when by itself)
seventeen-under: short form **17u**, long form **su** ("sue")
nineteen-over: short form **19o**, long form **no** (or **ino** when by itself)
nineteen-under: short form **19u**, long form **nu** (or **inu** when by itself)

17a = sa = the 2.3.17 prime subgroup
yasa = 2.3.5.17
yasana = 2.3.5.17.19

The disambiguation prefix i- is sorely needed: "so" can be confused with the solfege syllable So. "No 3rd" can mean either use the 19o 3rd, or omit the 3rd. "The nu key" sounds like "the new key". The i- prefix is only used if needed: "large omit 3rd" makes no sense, so "large no 3rd" is clear, as is nosu 4th, triple no 5th, and nono 6th.

17/16 = 17o2 = iso 2nd
18/17 = 17u1 = su semitone
19/16 = 19o3 = ino 3rd
20/19 = 19uy1 = nuyo semitone

Table 3.7.1 – 17o, 17u, 19o and 19u central intervals

iso				su				ino				inu			
				17u1	A1	18/17	99¢					19u1	A1	81/76	110¢
17o2	m2	17/16	105¢	17u2	A2	81/68	303¢	19o2	m2	19/18	94¢	19u2	M2	64/57	201¢
17o3	m3	153/128	309¢	17u3	M3	64/51	393¢	19o3	m3	19/16	298¢	19u3	M3	24/19	404¢
17o4	d4	34/27	399¢	17u4	A4	24/17	597¢	19o4	P4	171/128	501¢	19u4	A4	27/19	608¢
17o5	d5	17/12	603¢	17u5	A5	27/17	801¢	19o5	d5	38/27	592¢	19u5	P5	256/171	699¢
17o6	m6	51/32	807¢	17u6	M6	256/153	891¢	19o6	m6	19/12	796¢	19u6	M6	32/19	902¢
17o7	d7	136/81	897¢	17u7	M7	32/17	1095¢	19o7	m7	57/32	999¢	19u7	M7	36/19	1106¢
17o8	d8	17/9	1101¢	17u8	A8	36/17	1299¢	19o8	d8	152/81	1090¢	19u8	A8	81/38	1310¢

Central iso and ino intervals are all minor or diminished, except for ino's perfect 4th. Central su and inu intervals are all major or augmented, except for inu's perfect 5th.

Sa and na ratios sound rather ordinary compared to zo's subminor ratios, or ru's supermajor ratios, or la and tha's neutral ratios. A more positive view is that 12-edo does a very good job approximating sa and na ratios.

323/320 = 16¢ = $2^{-6} \cdot 5^{-1} \cdot 17 \cdot 19$ = 19o17og2 = nosogu comma
324/323 = 5.4¢ = $2^{2} \cdot 3^{4} \cdot 17^{-1} \cdot 19^{-1}$ = 19u17u-2 = nusu minicomma

The nusu minicomma is all that separates iso and inu, or su and ino, in Table 3.7.1.

All JI concepts apply to higher primes, thus there is soso, large su, the na rung, etc. Chords are named as usual: 1/1 – 19/16 – 3/2 = w1 – 19o3 – w5 is an ino triad. If the root is C, it's a C19o chord, "C ino". On the staff, 17o or 19u is written right before the notehead. Double colors are signified by repeating the -o or -u:

289/288 = 6.0¢ = 17oo2 = soso comma
6912/6859 = 13.3¢ = $19u^3$-2 = triple-nu comma

How to place these new rungs on the lattice? Proceeding as in the two previous chapters, find a microcomma that uses soso or triple so, and make it invisible. For example, the triple so 4th = $(17/16)^3$ = 314.9¢ is very close to g3 = 6/5, narrower only by the gu triple-su microcomma $17u^3$g-2 = 0.8¢. Thus the iso 2nd can be placed 1/3 of the way between the w1 and the g3.

The 19th harmonic falls nearly midway between the 18th and 20th harmonics. This puts 19/16 midway between 9/8 and 5/4. The ino 3rd can be placed between the w2 and the y3 on the lattice, or equivalently between w5 and y7. This makes the nonogu minicomma 19oog2 = 361/360 = 4.8¢ invisible.

Primes above 19 are treated similarly. Words like twenty-three-over are shortened by replacing the last digit with w-, th-, s- and n- for 1, 3, 7 and 9. Iso, ino and inu are never used: twenty-no not twenty-ino.

23o and 23u and 23a = twenty-tho and twenty-thu and twenty-tha
29o and 29u and 29a = twenty-no and twenty-nu and twenty-na
31o and 31u and 31a = thirty-wo and thirty-wu and thirty-wa
37o and 37u and 37a = thirty-so and thirty-su and thirty-sa

yasana23a = 2.3.5.17.19.23 subgroup
sana23a29a = 2.3.17.19.23.29 subgroup

23/16 = 23o5 = twenty-tho fifth
23/20 = 23og3 = twenty-thogu third
29/23 = 29o23u3 = twenty-no-twenty-thu third

529/512 = 23oo2 = double-twenty-tho second
841/512 = 29oo6 = double-twenty-no sixth

To extend color notation to cover every possible JI ratio, we need only determine the quality and degree of each prime's rung. This done by mapping each prime rung to a wa ratio. From these rungs, we can deduce the quality, degree and keyspan of any ratio. For most of the primes above 7, the keyspan and/or the degree are somewhat arbitrary. The table below defines the quality and degree relative to 12-edo the usual way (50-150¢ is a minor 2nd, 150-250¢ is a major 2nd, etc.) See Table 3.7.3. If the cents fall within 10¢ of the boundary, an alternate quality and degree is added in parentheses.

Table 3.7.2 – Prime rungs from 2 to 61

rung ratio	cents	shorthand	quality and degree	keyspan	accidentals	comma
2/1	1200¢	w8	perfect octave	12 semitones	—	—–
3/2	702¢	w5	perfect 5th	7 semitones	♯, ♭	Lw1 = (-11, 7) = 114¢
5/4	386¢	y3	major 3rd	4 semitones	y, g	g1 = 81/80 = 22¢
7/4	969¢	z7	minor 7th	10 semitones	z, r	r1 = 64/63 = 27¢
11/8	551¢	1o4	augmented 4th (or perfect 4th)	6 (or 5) semitones	1o, 1u	L1u1 = $2^{-6}\cdot3^{6}\cdot11^{-1}$ = 60¢ (if A4) 1o1 = 33/32 = 53¢ (if P4)
13/8	841¢	3o6	minor 6th (or major 6th)	8 (or 9) semitones	3o, 3u	L3o1 = $2^{-10}\cdot3^{4}\cdot13$ = 48¢ (if m6) 3u1 = 27/26 = 65¢ (if M6)
17/16	105¢	17o2	minor 2nd	1 semitone	17o, 17u	L17o1 = $2^{-12}\cdot3^{5}\cdot17$ = 15¢
19/16	298¢	19o3	minor 3rd	3 semitones	19o, 19u	L19o1 = 513/512 = $2^{-9}\cdot3^{3}\cdot19$ = 3¢
23/16	628¢	23o5	diminished 5th	6 semitones	23o, 23u	L23o1 = $2^{-14}\cdot3^{6}\cdot23$= 40¢
29/16	1030¢	29o7	minor 7th	10 semitones	29o, 29u	29o1 = 261/256 = 38¢
31/16	1145¢	31o7 (or 31o8)	major 7th (or perfect octave)	11 (or 12) semitones	31o, 31u	s31o1 = 248/243 = 35¢ (if M7) 31u1 = 32/31 = 55¢ (if P8)
37/32	251¢	37o3 (or 37o2)	minor 3rd (or major 2nd)	3 (or 2) semitones	37o, 37u	s37u1 = $2^{10}\cdot3^{-3}\cdot37^{-1}$ = 43¢ (if m3) 37o1 = 37/36 = 47¢ (if M2)
41/32	429¢	41o3	major 3rd	4 semitones	41o, 41u	41o1 = 82/81 = 21¢
43/32	512¢	43o4	perfect 4th	5 semitones	43o, 43u	43o1 = 129/128 = 13¢
47/32	666¢	47o5	perfect 5th	7 semitones	47o, 47u	47u1 = 48/47 = 36¢
53/32	874¢	53o6	major 6th	9 semitones	53o, 53u	53u1 = 54/53 = 32¢
59/32	1059¢	59o7	major 7th (or minor 7th)	11 (or 10) semitones	59o, 59u	L59u1 = 243/236 = 51¢ (if M7) 59o1 = $2^{-9}\cdot3^{2}\cdot59$ = 63¢ (if m7)
61/32	1117¢	61o7	major 7th	11 semitones	61o, 61u	s61o1 = 244/243 = 7¢

The comma in the table above is the interval that the color accidental raises or lowers wa by. This comma is always a perfect unison, with zero keyspan and degree of one. The quality and degree of the prime rung determines the comma, and vice versa. One approach to choosing the quality and degree is to minimize the comma's odd limit. For example, the ilo 4th would be a P4 because 33/32 is a much smaller ratio than 729/704. But in practice, the comma ratio doesn't really matter. To get 11/8, one just flattens the wa A4 instead of sharpening the wa P4. Mathematically, flattening the A4 means 729/512 (Lw4) divided by 729/704 (L1u1) equals 11/8 (1o4). Those are big numbers! But the whole point of color notation is to hide all the ratios, and present the musician with only conventional notes and small inflections of them. From that point of view, 11/8 = A4 is just as easy to read as 11/8 = P4. Using the A4 makes 11/8 in the key of B become E♯ not E, which is a little awkward. But on the other hand, 11/9 in the key of A♭ would become C, not an awkward C♭.

The ambiguity of 11/8 being either P4 or A4 disappears in relative notation, it's simply a 4th. But 31/16 and 37/32 have ambiguous degrees, which cause even the relative notation to be ambiguous.

In the following table, one can look up the rung ratio's cents and find the default quality and degree. This table applies only to rung ratios, those ratios with a prime number above a power of two. The quality and degree of other ratios must be derived from their component rungs.

Table 3.7.3 – Lookup Table For Default Quality and Degree of Rung Ratios

cents range	quality and degree	keyspan	degree
0-50¢	perfect unison	0 semitones	unison
50-150¢	minor 2nd	1 semitone	2nd
150-250¢	major 2nd	2 semitones	
250-350¢	minor 3rd	3 semitones	3rd
350-450¢	major 3rd	4 semitones	
450-550¢	perfect 4th	5 semitones	4th
550-600¢	augmented 4th	6 semitones	
600-650¢	diminished 5th		5th
650-750¢	perfect 5th	7 semitones	
750-850¢	minor 6th	8 semitones	6th
850-950¢	major 6th	9 semitones	
950-1050¢	minor 7th	10 semitones	7th
1050-1150¢	major 7th	11 semitones	
1150-1200¢	perfect octave	12 semitones	octave

No rung ratio will ever land exactly on a boundary, because of the unique factorization theorem. However, some come very close, such as 11/8 = 551.3¢.

The default quality and degree for primes 31, 37, 41 or 61 is not the same as alt-tuner's defaults. Alt-tuner's default keyspans are the same, but the degree is derived from the nearest 7-edo note. The quality comes from the keyspan and the degree. This method is not recommended here, it's better suited for creating non-heptatonic and non-12-tone notations. If using this method, six new intervals would be added to the cents lookup table:

> from 50¢ to 1\14 (85.71¢) would be an aug unison, not a min 2nd
> from 250¢ to 3\14 (257.14¢) would be an aug 2nd, not a min 3rd
> from 5\14 (428.57¢) to 450¢ would be a dim 4th, not a maj 3rd
> from 750¢ to 9\14 (771.43¢) would be an aug 5th, not a min 6th
> from 11\14 (942.86¢) to 950¢ would be a dim 7th, not a maj 6th
> from 13\14 (1114.29¢) to 1150¢ would be a dim octave, not a maj 7th

The quality or degree chosen for any prime rung will usually be clear from context, because interpreting the accidental the wrong way would create ratios with extremely large numbers, in the thousands or even higher. For example, upon seeing a wD – zF – wA – 1oC chord, one can deduce that the ilo 4th must be perfect not augmented. Otherwise, the wD – 1oC interval would be not 11/6 but 11264/6561! However, it's safer to write at the top of the page "1o4 = P4".

What do higher primes sound like? 17o ones and 19o ones sound quite ordinary, being so close to 12-ET intervals. The other ones seem rather obscure to me personally. I doubt I could tune even the simplest intervals like 23/1 by ear in isolation. The easiest way to hear them is in the harmonic series scale:

> Harmonics 16-32: w1, 17o2, w2, 19o3, y3, z4, 1o4, 23o5, w5, yy5, 3o6, w6, z7, 29o7, y7, 31o7, w8

Tuning a 23o5 is easier when it's part of a scale like this, as you can tune melodically, not harmonically, and aim for the midpoint between 1o4 and w5.

The next table shows a harmonic series and a subharmonic series scale with 32 notes per octave. In the previous chapter, I said that certain scales may be better notated with alternate mappings such as 1o4 = P4. But for harmonic series scales of more than 16 notes, it's better to standardize the notation and use the default qualities and degrees.

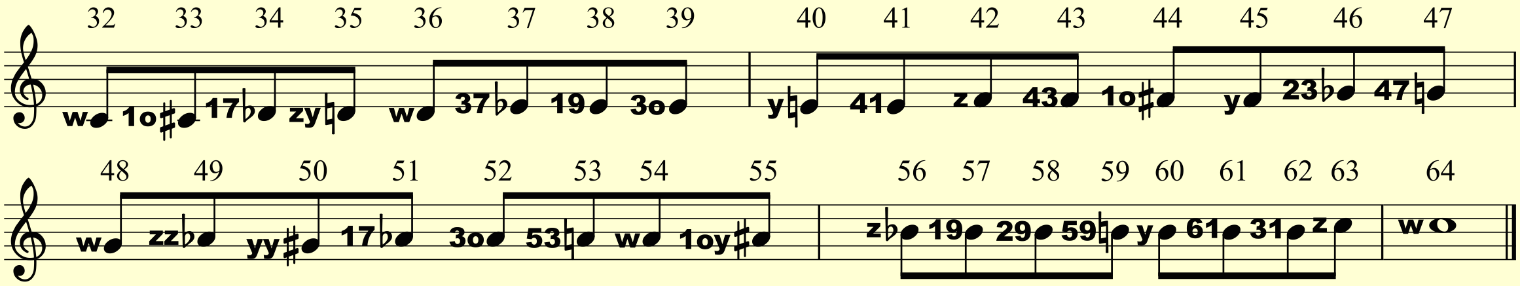
Table 3.7.4 – Harmonics and subharmonics 1-64

	Harmonic series in C				Subharmonic series in C			
(sub)harmonic #	cents	interval	quality & degree	note	cents	interval	quality & degree	note
1, 2, 4, 8, 16, 32	0¢	w1	perfect unison	wC	1200¢	w8	perfect 8ve	wC
33	53¢	1o1	aug unison	1oC♯	1167¢	1u8	dim 8ve	1uC♭
17, 34	105¢	17o2	minor 2nd	17oD♭	1095¢	17u7	major 7th	17uB
35	155¢	zy2	major 2nd	zyD	1045¢	rg7	minor 7th	rgB♭
9, 18, 36	204¢	w2	major 2nd	wD	996¢	w7	minor 7th	wB♭
37	251¢	37o3	minor 3rd	37oE♭	949¢	37u6	major 6th	37uA
19, 38	298¢	19o3	minor 3rd	19oE♭	902¢	19u6	major 6th	19uA
39	343¢	3o3	minor 3rd	3oE♭	857¢	3u6	major 6th	3uA
5, 10, 20, 40	386¢	y3	major 3rd	yE	814¢	g6	minor 6th	gA♭
41	429¢	41o3	major 3rd	41oE	771¢	41u6	minor 6th	41uA♭
21, 42	471¢	z4	perfect 4th	zF	729¢	r5	perfect 5th	rG
43	512¢	43o4	perfect 4th	43oF	688¢	43u5	perfect 5th	43uG
11, 22, 44	551¢	1o4	aug 4th	1oF♯	649¢	1u5	dim 5th	1uG♭
45	590¢	y4	aug 4th	yF♯	610¢	g5	dim 5th	gG♭
23, 46	628¢	23o5	dim 5th	23oG♭	572¢	23u4	aug 4th	23uF♯
47	666¢	47o5	perfect 5th	47oG	534¢	47u4	perfect 4th	47uF
3, 6, 12, 24, 48	702¢	w5	perfect 5th	wG	498¢	w4	perfect 4th	wF
49	738¢	zz6	minor 6th	zzA♭	462¢	rr3	major 3rd	rrE
25, 50	773¢	yy5	aug 5th	yyG♯	427¢	gg4	dim 4th	ggF♭
51	807¢	17o6	minor 6th	17oA♭	393¢	17u3	major 3rd	17uE
13, 26, 52	841¢	3o6	minor 6th	3oA♭	359¢	3u3	major 3rd	3uE
53	874¢	53o6	major 6th	53oA	326¢	53u3	minor 3rd	53uE♭
27, 54	906¢	w6	major 6th	wA	284¢	w3	minor 3rd	wE♭
55	938¢	1oy6	aug 6th	1oyA♯	262¢	1ug3	dim 3rd	1ugE♭♭
7, 14, 28, 56	969¢	z7	minor 7th	zB♭	231¢	r2	major 2nd	rD
57	999¢	19o7	minor 7th	19oB♭	201¢	19u2	major 2nd	19uD
29, 58	1030¢	29o7	minor 7th	29oB♭	170¢	29u2	major 2nd	29uD
59	1059¢	59o7	major 7th	59oB	141¢	59u2	major 2nd	59uD
15, 30, 60	1088¢	y7	major 7th	yB	112¢	g2	minor 2nd	gD♭
61	1117¢	61o7	major 7th	61oB	83¢	61u2	minor 2nd	61uD♭
31, 62	1145¢	31o7	major 7th	31oB	55¢	31u2	minor 2nd	31uD♭
63	1173¢	z8	perfect 8ve	zC	27¢	r1	perfect unison	rC
2, 4, 8, 16, 32, 64	1200¢	w8	perfect 8ve	wC	0¢	w1	perfect unison	wC

Below is the harmonic series from 32 to 64 in staff notation. Harmonic 49, zzA♭, appears higher on the staff than harmonic 50, yyG♯. This is unavoidable, because 50/49 is a negative interval, as discussed in chapter 3.3. Table 3.6.3 indicates that a similar problem would occur with harmonics 98 (minor 6th) and 99 (aug 5th). Table 3.6.4 indicates that a keyspan problem would occur with 77 (major 3rd) and 78 (minor 3rd).

If it's made clear at the top of the page that the piece only uses the harmonic series, the notation can be simplified somewhat. The "o" after prime numbers above 13 can be omitted.

Figure 3.7.1 – Harmonics 32-64, in C, with an implied "-o" suffix for numbers 17 and greater

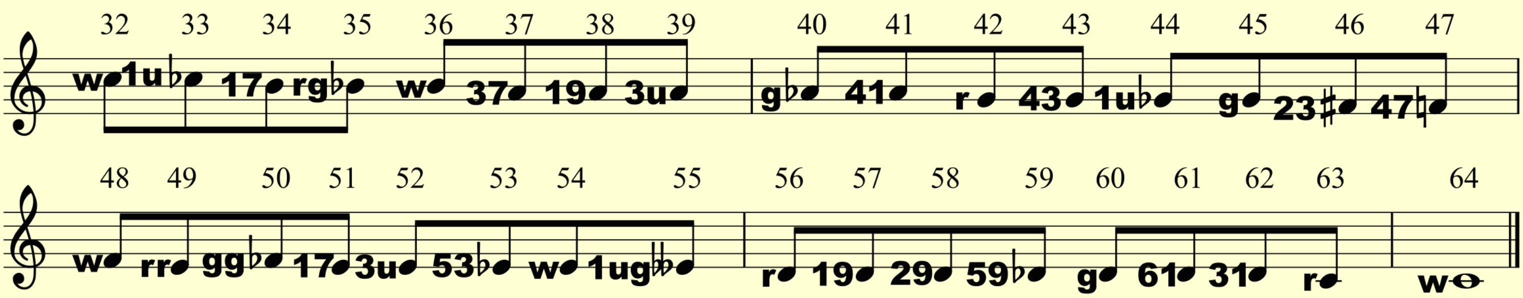


Only the over colors yo, zo, lo, etc. are used, but the under ones gu, ru, lu, etc. are needed for relative notation. Even though there are no gu notes, there are still gu intervals such as yE – G and gu chords such as yEg7.

One could simply write the actual harmonic number after every note. But it can be argued that when notating harmonic series scales, as any JI scale, the notation should indicate the sound of the intervals. Two intervals that sound the same should look the same, and two intervals that sound different should look different. For example, harmonics 32 and 40 make a y3 = 5/4. So do harmonics 36 and 45. Both intervals appear on the staff as a yo major 3rd. If harmonic numbers were used instead of colors, one would appear as a 32–40 major 3rd, and the other as a 36–45 major 3rd. One shouldn't have to mentally reduce the ratios 40/32 and 45/36 to ascertain that they are the same interval. Likewise, one shouldn't have to reduce 42/32 and 48/36 (both perfect 4ths) to ascertain that they are different intervals.

The subharmonic series is notated similarly to the harmonic series, using the under colors. The numbers have an "-u" suffix, which can be omitted.

Figure 3.7.2 – Subharmonics 32-64, in C, with an implied "-u" suffix for numbers 17 and greater



Chapter 3.8 – JI Chord Names Part II

Color chord names are based on jazz chord names (see "A Player's Guide to Chords & Harmony" by Jim Aiken), but are meant to apply to all genres, not just jazz. In jazz, triads are very rare, so it's safe to refer to a dim7 chord as a dim chord. But in many genres, triads are common, Cdim is a triad, and a dim tetrad must be called Cdim7. In jazz, chords have 11ths and 13ths, not added 4ths and 6ths. But in color notation, add-four chords and add-six chords are allowed.

In jazz, an 11th chord sometimes doesn't contain a 3rd, and a 13th chord sometimes doesn't contain an 11th, depending on whether the 3rd is major or minor, whether the 11th is perfect or aug, etc. But the ilo 3rd can be considered as either major or minor, and the ilo 11th can be either perfect or aug, so such rules break down. For simplicity's sake, a new rule is adopted:

A higher degree always implies all lower degrees. Unless otherwise specified,

a 7th chord contains a 3rd and a 5th,
a 9th chord contains a 3rd, 5th and 7th,
an 11th chord contains a 3rd, 5th, 7th and 9th, and
a 13th chord contains a 3rd, 5th, 7th, 9th and 11th.

An absent 3rd is indicated by "no3" or "5" (e.g. C5g7) or by rewriting the chord as a sus chord (e.g. C11no3 might become C9sus4). An absent 5th is indicated by "no5". An absent 7th/9th/11th is indicated by "no7", etc., or by rewriting the chord as an add-something chord. For example, C11no9 might become C7add11. An added 11th or 13th can also be notated as an added 4th or 6th.

The most basic chord names are formed from stacked 3rds. 6th chords are also a stack of 3rds, if you think of the 6th as being below the root. These chords are named similar to CM7, Cm9, etc., but with a color replacing "M" or "m". The chord is formed by two chains of wa 5ths. One chain has the root, the 5th, perhaps the 9th, and perhaps the 13th too, all wa. The other chain has the 3rd, the 6th or 7th, and perhaps the 11th, all the same color.

Cy	C yo	w1 y3 w5	the triad is named after the color of the 3rd
Cy6	C yo six	w1 y3 w5 y6	the 6th's color matches the 3rd
Cy7	C yo seven	w1 y3 w5 y7	the 7th's color matches the 3rd
Cy9	C yo nine	w1 y3 w5 y7 w9	the 9th is assumed to be wa
Cy11	C yo eleven	w1 y3 w5 y7 w9 y11	the 11th's color matches the 7th
Cy13	C yo thirteen	w1 y3 w5 y7 w9 y11 w13	the 13th is assumed to be wa

Added notes are listed after the stacked-3rds chord, using commas as needed:

Cy,9	C yo, add nine	w1 y3 w5 w9	needs a comma to distinguish it from Cy9
Cy6,9	C yo six, nine	w1 y3 w5 y6 w9	innate comma chord, y6–w9 is a wolf 4th
Cy6,11	C yo six, eleven	w1 y3 w5 y6 w11	an added 11th is assumed to be wa...
Cy7,11	C yo seven, eleven	w1 y3 w5 y7 w11	...even when there's a non-wa 7th
Cy7y11	C yo seven, yo eleven	w1 y3 w5 y7 y11	could instead be written Cy11no9

13th chords and 6,9 chords tend to have an innate comma, unless something is omitted.

A comma is used to separate two colors (e.g. Cy,z7) or two numbers (Cy6,9). It may also separate a color and a number sometimes (Cy,9). Commas can optionally be used before every added note, for readability (Cy7,y11).

Alterations are always enclosed in parentheses, and additions never are. Thus "sus" and "add" can be avoided, greatly shortening chord names. However, "add" must sometimes be spoken, to avoid implying an alteration.

Cg7(zg5)	C gu seven, zogu five	w1 g3 zg5 g7	"zogu five" is assumed to be an alteration
Cg7zg5	C gu seven, add zogu five	w1 g3 zg5 w5 g7	must say "add" to specify an addition

"Add" also usually needs to be spoken when adding to a triad, to avoid implying a larger stacked-3rds chord:

Cy,9	C yo, add nine	w1 y3 w5 w9	"C yo, nine" sounds too much like Cy9
Cz,11	C zo, add eleven	w1 z3 w5 w11	"C zo, eleven" sounds too much like Cz11
C4,9	C four, nine	w1 w4 w5 w9	no "add" needed, because C4 ends with a number

In conventional practice, we write Cm7♭9 not Cm9♭9, because additions are preferred over alterations. Likewise, in color notation, we write Cg7zg9, not Cg9(zg9). Therefore if the highest degree is spoken twice, the 2nd time must be an addition, and "add" isn't needed:

Cg9zg9	C gu nine, zogu nine	w1 g3 w5 g7 zg9 w9	no "add", saying "nine" twice implies two 9ths

A chord's full name is the stacked-3rds chord, followed by any added or altered notes, followed by any omitted notes:

Cy11(z7)3o13no5	C yo eleven, zo seven, tho thirteen, no five	w1 y3 z7 w9 y11 3o13

Before any added, altered or omitted notes, the speaker should insert a slight pause, for clarity. Especially with thirds with a compound color, which are fortunately quite rare, due to their high odd limit:

C3o,1u6	C tho, lu six	w1 3o3 w5 1u6	1/1 – 39/32 – 3/2 – 18/11
C3o1u6	C tholu six	w1 3o1u3 w5 3o1u6	1/1 – 13/11 – 3/2 – 52/33

Added and altered notes are usually listed in order of degree, but can also follow conventional practice:

Cz,y6(zg5)	C zo, yo six, zogu five	w1 z3 zg5 y6	analogous to C6♭5

If the 3rd and the 7th are different colors, often stacking 3rds beyond a triad doesn't work, and every component except the root and 5th must be explicitly listed. This is called the explicit format.

Cz,y6	C zo, yo six	w1 z3 w5 y6
Cz,y6,9	C zo, yo six, nine	w1 z3 w5 y6 w9

However, larger chords might be named as an altered stacked-3rds chord:

Cy11(z7)	C yo eleven, zo seven	w1 y3 w5 z7 w9 y11
Cz11(y3)	C zo eleven, yo three	w1 y3 w5 z7 w9 z11

Theoretically, Cr,g7 could be named Cg7(r3). But since the altered format isn't any shorter, the explicit format is preferred, for clarity.

If the 3rd and the 6th are different colors, the explicit format is almost always preferred:

Cz,y6	C zo, yo six	w1 z3 w5 y6
Cz,y6,9	C zo, yo six, nine	w1 z3 w5 y6 w9
Cz,y6,11	C zo, yo six, eleven	w1 z3 w5 y6 w11

Enharmonic substitutions are not allowed in color notation (unless po and qu are used). Every ratio has a specific degree, which can't be changed to aid chord spelling. For example, 7/3 is not a ♯9 and must be a ♭10, which is notated as an added 3rd or 10th. Two tunings of the Hendrix chord:

Ch7,z10no5	C har-seven, zo ten, no five	w1 y3 z7 z10
Ch7,19o10no5	C har-seven, ino ten, no five	w1 y3 z7 19o10

A ♭9 must sometimes be notated as a ♯8, an augmented octave, or perhaps as a po 9th. Recall from chapter 2.6 that p and q merely change the degree, and every octave is also a po 9th and a qu 7th.

Cy7ry8	C yo seven, ruyo eight	w1 y3 w5 y7 ry8	Cy7ryp9	C yo seven, ruyopo nine	w1 y3 w5 y7 ryp9

A $\sharp$11 may need to be a $\flat$5. A $\flat$13 may need to be a $\sharp$5. A major 7th may even need to be a $\flat$8, a diminished octave:

Cz(zg5)zg8	C zo, zogu five and eight	w1 z3 zg5 zg8	a mM7$\flat$5 chord

A $\flat$9$\sharp$11 chord is usually tuned as either $\sharp$8$\sharp$11 or $\flat$9$\flat$5, with both notes the same color, to avoid a wolf 4th. "And" can be used with alterations/additions with the same color:

Ch7,zg5zg9	C har-seven, add zogu five and nine	w1 y3 w5 z7 zg9 Wzg5

A **sus4 chord** has "(4)", and an add4 chord has "4". Either way, the 4th is assumed to be wa. C(4) is written C4, to match conventional practice.

C4 or C(4)	C four	w1 w4 w5	the 4th is assumed to be wa
C(z4)	C zo-four	w1 z4 w5	
Cz,4	C zo, add four	w1 z3 w4 w5	6:7:8:9, a good example of an add-4 chord
Cy(4)	*invalid, write C4 instead, don't mention yo if it isn't present*		
Cy(z4)	*invalid, write C(z4) instead, don't mention yo if it isn't present*		
Cz4	*invalid, because it's not clear whether the 3rd or the 4th is zo*		
Cz7(4)	C zo seven, four	w1 w4 w5 z7	could be written C4z7
Cz7(z4)	C zo seven, zo four	w1 z4 w5 z7	could be written C(z4)z7

The 4th in a sus4 or add4 chord needn't always be perfect. In these examples, it's augmented:

C(ry4)	C ruyo-four	w1 ry4 w5	a sus$\sharp$4 chord, could be written C5ry11
Cy,ry4	C yo, ruyo four	w1 y3 ry4 w5	an add$\sharp$4 chord, could be written Cy,ry11
Cy,ry4no5	C yo, ruyo four, no five	w1 y3 ry4	more consonant than Cy,ry4
Cy6(ry4)no5	C yo six, ruyo four, no five	w1 ry4 y6	a homonym of ryF$\sharp$z(zg5)

Sus2 and add2 chords are similar:

C2 or C(2)	C two	w1 w2 w5	the 2nd is assumed to be wa
C(y2)	C yo-two	w1 y2 w5	a wolf chord
Cy,2	C yo, add two	w1 w2 y3 w5	
Cz7(2)	C zo seven, two	w1 w2 w5 z7	a homonym of Gz,4
Cy6(y2)	C yo six, yo two	w1 y2 w5 y6	a wolf chord
C(zg2)	C zogu-two	w1 zg2 w5	a sus-flat-2 chord, could be written C5zg9
Cg,zg2	C gu, zogu-two	w1 zg2 g3 w5	an add-flat-2 chord, could be written Cg,zg9

Except for sus2 and sus4 chords, alterations are invalid if the degree is absent: Cz7(y6) is invalid.

A **thirdless** chord can be either a "5" chord or a "no3" chord. Smaller chords tend to be 5 chords:

C5	C five	w1 w5	no third-color in the chord name implies no 3rd
C(zg5)	C zogu-five	w1 zg5	"
C5zg5	C five, zogu five	w1 zg5 w5	no "add" because saying "five" twice implies two 5ths
C5z7	C five, zo seven	w1 w5 z7	could also be written Cz7no3, "C zo seven, no three"

Larger chords tend to use "no3" instead:

Cz9no3	C zo nine, no three	w1 w5 z7 w9
Cz9(zg5)no3	C zo nine, zogu five, no three	w1 zg5 z7 w9
Cz9no3,5	C zo nine, no three, five	w1 z7 z9

Occasionally, the magnitude (large, small, etc.) is part of the chord name.

CLw	C large wa	w1 Lw3 w5	Lw3 = 81/64
Cw9	C wa nine	w1 w3 w5 w7 w9	
CLw9	C large wa nine	w1 Lw3 w5 Lw7 w9	the 7th is a w5 above the 3rd, thus large
CLw9(w7)	C large wa nine, wa seven	w1 Lw3 w5 w7 w9	could also be written Cw9(Lw3)
Cw6	C wa six	w1 w3 w5 sw6	the 6th is a w4 above the 3rd, thus small
Cw,w6	C wa, wa six	w1 w3 w5 w6	the 6th is central

<u>Harmonic-series chords</u>, if named explicitly, would have cumbersome names. So there is a special format for them: "h" followed by a number means harmonic. Not to be confused with "hA" (half-aug) or "hd" (half-dim).

Ch7	4:5:6:7	w1 y3 w5 z7	Cy,z7	"C har seven"
Ch8	*invalid, no even numbers allowed*			
Ch9	4:5:6:7:9	w1 y3 w5 z7 w9	Cy,z7,9	"C har nine"
Ch11	4:5:6:7:9:11	w1 y3 w5 z7 w9 1o11	Cy,z7,9,1o11	
Ch9,11	4:5:6:7:9 + 8/3	w1 y3 w5 z7 w9 w11	Cy,z7,9,11	add the 11th degree, not the 11th harmonic
Ch11no3	4:6:7:9:11	w1 w5 z7 w9 1o11	Cz9,1o11no3	omit the 3rd degree, not the 3rd harmonic
Ch13	4:5:6:7:9:11:13	w1 y3 w5 z7 w9 1o11 3o13	Cy,z7,9,1o11,3o13	
Ch9(zg5)	4:5:7:9 + 7/5	w1 y3 zg5 z7 w9	Cy,z7,9(zg5)	alter the 5th degree, not the 5th harmonic

13 is the highest degree used in conventional chord names. By a happy coincidence, most of the first 13 odd harmonics when octave-reduced have matching degrees: h1 = w1, h7 = z7, h9 = w9, h11 = 1o11 and h13 = 3o13. Thus Ch11 is an actual 11th chord, and "no9" means both omit the major 9th and omit the 9th harmonic. But starting with harmonic 15, the correspondence ends. All numbers in harmonic-series chords that are 15 or higher refer to harmonics, not degrees:

Ch15	4:5:6:7:9:11:13:15	w1 y3 w5 z7 w9 1o11 3o13 Wy7	Cy9,z7,1o11,3o13
Ch9,15	4:5:6:7:9:15	w1 y3 w5 z7 w9 Wy7	Cy9,z7
Ch17	4:5:6:7:9:11:13:15:17	w1 y3 w5 z7 w9 1o11 3o13 Wy7 W17o9	Cy9,z7,1o11,3o13,17o9
Ch17no15	4:5:6:7:9:11:13:17	w1 y3 w5 z7 w9 1o11 3o13 W17o9	Cy,z7,9,1o11,3o13,17o9
Ch27	4:5:6:7:9:11:13:15:17: 19:21:23:25:27	w1 y3 w5 z7 w9 1o11 3o13 y7 17o9 19o3 z11 23o5 yy5 w6	

A "no1" harmonic chord implies a root other than the 1st harmonic:

Ch9no1	5:6:7:9	y3 w5 z7 w9	implies	yEg7(zg5)	w1 g3 zg5 g7	5:6:7:9
				Gz,y6	w1 z3 w5 y6	6:7:9:10

Subharmonic-series chords: "s" followed by a color means small, but "s" followed by a number means subharmonic. The chords are pronounced "C subharmonic seven" or "C sub seven". The root of a sub-N chord is the Nth subharmonic, to ensure the presence of a 3rd, 5th, 7th etc. With the exception of the s6 chord, a sub-N chord is N/(1:3:5:...N), where N must be odd. All numbers in subharmonic-series chords that are 15 or higher refer to harmonics, not degrees. Beware, the s9 chord is not a s7 chord plus a 9th, it's a completely different chord.

Cs6	6/(7:6:5:4)	w1 g3 w5 r6	Cg,r6	"C sub six"
Cs6,11	6/(7:6:5:4) + 8/3	w1 g3 w5 r6 w11	Cg,r6,11	add the 11th degree, not the 11th subharmonic
Cs6(zg5)	6/(7:6:4) + 7/5	w1 g3 zg5 r6	Cg,r6(zg5)	alter the 5th degree, not the 5th subharmonic
Cs7	7/(7:6:5:4)	w1 z3 zg5 z7	Cz7(zg5)	
Cs9	9/(9:7:6:5:4)	w1 r3 w5 g7 w9	Cr,g7,9	
Cs9no5	9/(9:7:5:4)	w1 r3 g7 w9	Cr,g7,9no5	omit the 5th degree, not the 5th subharmonic
Cs11	11/(11:9:7:6:5:4)	w1 1o3 1or5 1o7 1og9 1o4	C1o11(1or5,1og9)	
Cs13	13/(13:11:9:7:6:5:4)	w1 3o1u3 3o5 3or7 3o9 3og11 3o13		
Cs15	15/(15:13:11:9:7:6:5:4)	w1 3uy2 1uy4 y6 ry8 y10 w12 Wy7		the upper chord is yEs6
Cs9,15	9/(9:7:6:5:4:15)	w1 r3 w5 g7 w9 g10	Cg9,r3	add the 15th subharmonic
Cs17no15	17/(17:13:11:9:7:6:5:4)			omit the 15th subharmonic

Alternate roots for subharmonic chords:

Cs6	6/(7:6:5:4)	w1 g3 w5 r6	→	rAs7	7/(7:6:5:4)	w1 z3 zg5 z7
Cs7	7/(7:6:5:4)	w1 z3 zg5 z7	→	zE♭s6	6/(7:6:5:4)	w1 g3 w5 r6
Cs9	9/(9:7:6:5:4)	w1 r3 w5 g7 w9	→	Gs6,11	6/(9:7:6:5:4)	w1 g3 w5 r6 w11
Cs6,11	6/(9:7:6:5:4)	w1 g3 w5 r6 w11	→	Fs9	9/(9:7:6:5:4)	w1 r3 w5 g7 w9

Polychords have two roots, an upper one and a lower one. The two roots may not be the same color. If so, when the chord is written in isolation, the lower root is colorless, and the upper root is colored relative to the lower root.

Dy/Cy	D yo over C yo	w1 y3 w5 w9 y11 w13	Cy13no7
wIIy/Iy	wa two yo over yo	"	y13no7 chord
zgG♭y/Cy	zogu G-flat yo over C yo	w1 y3 w5 z7 zg9 Wzg5	Ch7zg9zg5
zgVy/Iy	zogu-five yo over yo	"	h7zg9zg5 chord

In the context of a song, both root colors are always relative to the tonic, which is always wa.

Gy9 – zFy/yBy	G-yo-nine to zo-F-yo over yo-B-yo	wG yB wD yF# wA – yB yyD# yF# zyA zC zF

In relative notation, again everything is relative to the tonic:

Iy9 – zVIIy/yIIIy	one-yo-nine to zo-seven-yo over yo-three-yo	w1 y3 w5 y7 w9 – y3 yy5 y7 zy9 z11 Wz7

The Cs15 chord voiced as 15/(15:13:11:9:7:6:5:4) can be written yEs6/3uyD3o1u,3og7(3o5). Polychords are not to be confused with slash notes, which were covered in chapter 2.5.

To sum up: besides polychords, there are three basic formats, the stacked-3rds format, the harmonic format and the subharmonic format. Each one can have additions, alterations and omissions. The explicit format is a type of stacked-3rds format: a triad with many additions, and perhaps an altered 5th. Every note is explicitly spelled out except the root, and the fifth if it's wa.

The stacked-3rds format is best for bicolored chords. The harmonic and subharmonic formats ane best for chords from the harmonic or subharmonic series. The explicit format is the least concise, and the last resort. The stacked-3rds name is preferable to the harmonic/subharmonic name unless it's much longer; Cz11(y3) is preferable to Ch9,z11. Avoid high harmonics if possible; Ch9,13 is preferable to Ch9,27. The latter requires familiarity with the ratio 27/16, and the whole point of color notation is to avoid memorizing large ratios. Even Ch9,15 might be better written Ch9,y7.

Chapter 3.9 – Using Higher Primes in Chords

The minor triad in certain voicings and registers will sound smoother with an ino 3rd instead of a gu one. The ino triad's extended ratio in a close voicing is 16:19:24, whereas the gu triad's is 10:12:15. Recall from chapter 2.7 that the all-odd voicing is theoretically the most consonant. The all-odd voicing is 1:3:19, a rather far-flung w1 – Ww5 – $W^4$19o3. A more compact voicing would be 4:6:8:19 = w1 – w5 – w8 – WW19o3. The gu triad in the same voicing is 10:15:20:48, much larger numbers. The gu triad's all-odd voicing is 3:5:15 = g3 – w8 – WWw5. This compacts to 6:10:15 = g3 – w8 – Ww5. The ino triad in the same voicing is 19:32:48, again much larger numbers.

Register also matters. Higher registers favor otonal chords and lower registers favor utonal chords. In a higher register, an otonal chord's difference tones will create an audible bass tone, reinforcing the root. In a lower register, the common harmonic of a utonal chord's notes will be more audible.

Voicing a minor triad with the minor 3rd in an upper voice and/or with the whole chord in a high register will favor using the ino 3rd. Conversely, voicing it with the minor 3rd in a lower voice and/or with the whole chord in a lower register will favor the gu 3rd.

A composer might use this information to find a good voicing and register for a chord with a specific JI tuning. A choir director working with an arranged piece might work the other way, and use the existing voicing and register to find a good JI tuning. They may want to choose the tuning that has the lowest integer limit in the actual voicing of the chord.

Armed with higher primes, let's look at some chords which don't have an obvious yaza tuning. (See Table 2.4.11 for those that do.)

The **augmented chord** can be tuned in ya as a stack of two yo 3rds, as discussed in chapter 2.4. The interval from the aug 5th up to the octave is a gugu 4th = gg4 = 32/25 = 427¢.

yo yoyo-5 chord	y(yy5)	w1 – y3 – yy5	1/1 – 5/4 – 25/16	16:20:25

The all-odd voicing, presumably the most consonant voicing, is 1:5:25 = w1 – WWy3 – W^4yy5, extremely far-flung. A more compact voicing is 10:16:25 = y3 – w8 – Wyy5.

The 12-edo augmented chord is symmetrical. It's theoretically possible to consider any of the three notes as the root, and thus write the chord out in three different ways. In practice, the chord names become quite cumbersome:

yo gu-6 no-5 chord	y,g6no5	w1 – y3 – g6	1/1 – 5/4 – 8/5	20:25:32
gugu-4 gu-6 no-5 chord	(gg4)g6no5	w1 – gg4 – g6	1/1 – 32/25 – 8/5	25:32:40

The gugu 4th is very close to the much simpler ratio r3 = 9/7 = 435¢. This suggests a yaza tuning which stacks two yo 3rds and a ru 3rd. Because this falls short of an octave by ryy-2 = 8¢, a slight tempering of some or all of the intervals is usually necessary. Tempered chords are discussed in chapter 4.9.

A zala tuning combines a ru 3rd = 435¢ and an ilo 3rd = 347¢ to make a loru 5th = 782¢. The component intervals are r3, 1o3 and 1uz4 = 418¢, plus their inverses z6, 1u6 and 1or5. This tuning has the smallest odd limit possible for an augmented chord, and can be virtually beatless. But it's debatable if this really qualifies as an augmented chord, as the upper 3rd (1o3) hardly sounds major.

ru loru-5 chord	r(1or5)	w1 – r3 – 1or5	1/1 – 9/7 – 11/7	7:9:11 (all-odd)

A yatha tuning, also nearly beatless, combines a yo 3rd and a thogu 4th = 13/10 = 454¢ to make a tho 6th = 3o6 = 13/8 = 841¢. This chord is also debatable, because the upper interval is so wide. The other 3rd is a thu 3rd = 16/13 = 359¢. A good nearly all-odd voicing is 5:8:13 = y3 – w8 – W3o6.

yo, tho-six, no-5 chord	y,3o6no5	w1 – y3 – 3o6	1/1 – 5/4 – 13/8	8:10:13

A zasa tuning combines a suzo 3rd = 17uz3 = 21/17 = 366¢ and a ru 3rd to make a su 5th = 17u5 = 27/17 = 801¢. The other 3rd is actually a 4th, the iso 4th = 17o4 = 34/27 = 399¢.

suzo su-5 chord	17uz(17u5)	w1 – 17uz3 – u5	1/1 – 21/17 – 27/17	17:21:27 (all-odd)

A yana tuning combines a yo third and a nogu 4th = 19og4 = 19/15 to make an ino 6th = 19/12. The component intervals are y3 = 386¢, 19og4 = 409¢ and the inu 3rd = 19u3 = 404¢, very close to the 12-edo augmented chord. A good nearly-all-odd voicing is 6:15:19 = w1 – Wy3 – W19o6.

yo ino-6 no-5 chord	y,19o6no5	w1 – y3 – 19o6	1/1 – 5/4 – 19/12	12:15:19

Other possibilities with a higher odd limit include a zalatha one 21:26:33 = 3or(1or5), two yazala ones 28:35:44 = y(1or5) and 35:44:55 = 1org(1or5), and three yaza ones: 28:35:45 = y(ry5), 28:36:45 = r(ry5) and 40:50:63 = y,zg6no5. The first two yaza chords imply the y3 + y3 + r3 + rry-2 chord discussed above.

For a close 1–3–5 voicing, the tuning with the lowest odd limit is 7:9:11 = r(1or5). For an open voicing of 1–5–10, the lowest odd limit tuning is 5:8:13 = y,3o6,no5.

The **augmented 7th chord** contains an aug triad, so we can find tunings by adding a 7th to the previous examples and their homonyms.

A yaza tuning adds a zo 7th to the y(yy5) chord. A good voicing is 10:14:16:25 = y3 – z7 – w8 – Wyy5.

h7(yy5) chord	w1 – y3 – yy5 – z7	1/1 – 5/4 – 25/16 – 7/4	16:20:25:28

Two zala tunings build on homonyms of the r(1or5) chord:

1o,z6,w7,no5 chord	w1 – 1o3 – z6 – w7	1/1 – 11/9 – 14/9 – 16/9	9:11:14:16
(1uz4)lu6,1uy7,no5 chord	w1 – 1uz4 – 1u6 – 1uy7	1/1 – 14/11 – 18/11 – 20/11	11:14:18:20

Voicing them as all-odd or nearly so implies a different root. Both chords become 11th chords. The 1st chord is w1 – z7 – w9 – 1o11 = 4:7:9:11, an h11no3,5 chord. The 2nd chord is y3 – z7 – w9 – 1o11 = 5:7:9:11, an h11no1,5 chord.

A yatha tuning adds a g7 to a homonym of the y,3o6,no5 chord:

(3og4)g6,g7,no5 chord	w1 – 3og4 – g6 – g7	1/1 – 13/10 – 8/5 – 9/5	10:13:16:18

A good voicing is 4:5:9:13, which implies that the g6 is the root. It becomes a y,9,3o13,no5 chord.

Three 19-limit tunings (yazana, yana and yasana) build on the y,19o6,no5 chord and its homonyms:

y,19o6,z7,no5 chord	w1 – y3 – 19o6 – z7	1/1 – 5/4 – 19/12 – 7/4	12:15:19:21
(19og4)g7,g6,no5 chord	w1 – 19og4 – g6 – g7	1/1 – 19/15 – 8/5 – 9/5	15:19:24:27
19u,19u17o7(19uy5) chord	w1 – 19u3 – 19uy5 – 19u17o7	1/1 – 24/19 – 30/19 – 34/19	19:24:30:34

A nearly all-odd close voicing for the first chord is the one given above. The second chord has 12:15:19:27, which implies that the aug 5th becomes the root, making an aug,add9 chord: w1 – y3 – 19o6 – w9. The third chord has 12:15:17:19, implying that the 3rd becomes the root, making an aug,add♭5 chord: w1 – y3 – 17o5 – 19o6.

The **minor-major chord** also contains an augmented triad. The new note must be a w5 below one of the other notes.

The obvious ya tuning, with an odd limit of 25.

g,y7 chord	w1 – g3 – w5 – y7	1/1 – 6/5 – 3/2 – 15/8	40:48:60:75

A zalatha tuning uses the tholu 3rd = 3o1u3 = 289¢ and the luzo 8th = 1uz8 = 1119¢.

3o1u,1uz8 chord	w1 – 3o1u3 – w5 – 1uz8	1/1 – 13/11 – 3/2 – 21/11	22:26:33:42

Two yana tunings build on the y,19o6,no5 chord, using the nogu 8ve = 19og8 = 1111¢, and the ino 3rd = 19o3 = 295¢.

g,19og8 chord	w1 – g3 – w5 – 19og8	1/1 – 6/5 – 3/2 – 19/10	10:12:15:19
19o,y7 chord	w1 – 19o3 – w5 – y7	1/1 – 19/16 – 3/2 – 15/8	16:19:24:30

The **dim seven chord**, like the augmented triad, is a symmetrical chord in 12-edo. It can be notated four ways, by taking any one of the four notes as the root. A good yazasa tuning with a low odd limit which closely approximates the familiar 12-edo chord is a rootless pentad with an iso 9th. The root can be thought of as suspended up a minor 2nd, similar to the suspended 3rd in a sus4 chord. There are three good homonyms for this chord:

g,17og7(zg5) chord	w1 – g3 – zg5 – 17og7	1/1 – 6/5 – 7/5 – 17/10	10:12:14:17
z,y6(17o5) chord	w1 – z3 – 17o5 – y6	1/1 – 7/6 – 17/12 – 5/3	12:14:17:20
h7,17no1 chord	17o2 – y3 – w5 – z7	17/16 – 5/4 – 3/2 – 7/4	17:20:24:28

The component intervals are g3, z3, soru 3rd = 17or3 = 17/14 = 336¢ and suyo 2nd = 17uy2 = 20/17 = 281¢. The tritones are zg5 and iso 5th = 17o5 = 17/12 = 603¢, and their inverses. The all-odd voicing is w1 – y6 – z10 – WW17o5 = 1/1 – 5/3 – 7/3 – 17/3 = 3:5:7:17. A more compact voicing would be w1 – y6 – z10 – W17o5 = 1/1 – 5/3 – 7/3 – 17/6 = 6:10:14:17.

The tuning with the lowest odd limit is a yalatha tuning. Some of the intervals are almost too wide or narrow to be called minor 3rds. The component intervals from widest to narrowest are 1o3 = 347¢, the gu 3rd, the tholu 3rd = 3o1u3 = 13/11 = 289¢, and thuyo 2nd = 3uy2 = 15/13 = 248¢. The tritones are the tho 5th = 3o5 = 13/9 = 637¢ and the logu 5th = 1og5 = 22/15 = 663¢.

1o,y6(3o5) chord	w1 – 1o3 – 3o5 – y6	1/1 – 11/9 – 13/9 – 5/3	9:11:13:15 (all-odd)

A yaza tuning stacks a gu 3rd, a zo 3rd, a gu 3rd, and a ruyoyo 2nd = ryy2 = 25/21 = 302¢. The ryy2 is only 4.3¢ sharper than the ino 3rd 19/16. Both tritones are zogu 5ths. The all-odd voicing is w1 – zg5 – y3 – z10 = 15:21:25:35.

z,y6(zg5) chord	w1 – z3 – zg5 – y6	1/1 – 7/6 – 7/5 – 5/3	30:35:42:50
g,zgg7(zg5) chord	w1 – g3 – zg5 – zgg7 ≈ 19u6	1/1 – 6/5 – 7/5 – 42/25 ≈ 32/19	25:30:35:42

Yet another yaza tuning stacks the same four intervals, but in a different order. Again, 19o3 can be substituted for ryy2. The all-odd voicing is w1 – y6 – z10 – Wyy4 = 9:15:21:25. This chord has smaller numbers in its extended ratio than the previous one, but the yy4/gg5 tritone is more dissonant. gg5 = 36/25 is only 3.01¢ sharper than 23o5 = 23/16.

g,y6(zg5) chord	w1 – g3 – zg5 – y6	1/1 – 6/5 – 7/5 – 5/3	15:18:21:25
g,zgg7(gg5) chord	w1 – g3 – gg5 – zgg7 ≈ 19u6	1/1 – 6/5 – 36/25 – 42/25 ≈ 32/19	25:30:36:42

This yaza tuning stacks a zo 3rd, a gu 3rd, a zo 3rd, and a purple 3rd = 351¢ = 49/40 or 60/49. I consider zo, gu and purple to be the coolest JI 3rds! The all-odd voicing is w1 – zg5 – z10 – Wzzg7 = 15:21:35:49. Again, both tritones are zogu 5ths. There are two good homonyms.

s7(zzg7) chord	w1 – z3 – zg5 – zzg7	1/1 – 7/6 – 7/5 – 49/30	30:35:42:49
s6(zg5) chord	w1 – g3 – zg5 – r6	1/1 – 6/5 – 7/5 – 12/7	35:42:49:60

The "double purple" yaza chord stacks two purple 3rds with a zo 3rd and a ru 2nd. It uses notes from all 4 planes of the lattice in figure 3.5.7. In this lattice, it forms a square of adjacent notes. It consists of two zg5 tritones a purple 3rd apart. Unlike the previous chord, every voicing contains a purple interval from the root. The extended ratio changes depending on whether that purple interval is ruruyo or zozogu. The numbers are deceptively high, the chord is less dissonant than one would expect. The deep purple equivalences make it impossible to specify an all-odd voicing.

(rry2,zg5)r6 chord	w1 – rry2 – zg5 – r6	1/1 – 60/49 – 7/5 – 12/7	245:300:343:420 = 280:343:392:480
(r2,zg5)zzg7 chord	w1 – r2 – zg5 – zzg7	1/1 – 8/7 – 7/5 – 49/30	210:240:294:343 = 245:280:343:400
(rry2)ry4z7no5	w1 – rry2 – ry4 – z7	1/1 – 60/49 – 10/7 – 7/4	196:240:280:343 = 280:343:400:490
z,ry4,zzg7no5	w1 – z3 – ry4 – zzg7	1/1 – 7/6 – 10/7 – 49/30	210:245:300:343 = 294:343:420:480

For a close 1–3–5–7 voicing, the dim7 tuning with the lowest odd limit is 9:11:13:15 = 1o,y6(3o5), followed closely by 10:12:14:17 = g,17og7(zg5). For an open 1–5–W3–W7 voicing, the best tuning is 5:7:12:17 = g,17og7(zg5).

The **dom7 sus4 chord** has an obvious wa tuning:

w7(4) chord	w1 – w4 – w5 – w7	1/1 – 4/3 – 3/2 – 16/9	18:24:27:32

A za tuning uses a zo 4th, a z7(z4) chord. It's best to voice the 4th as an 11th, to make it more consonant.

z11no3,9 chord	w1 – w5 – z7 – z11	1/1 – 3/2 – 7/4 – 21/8	8:12:14:21

A zala tuning is possible, if the ilo 4th can be accepted as a sus 4th. It too is better voiced as an 11th. The chord contains a loru 5th = 1or5 = 11/7 = 782¢.

h7,11no3 chord	w1 – w5 – z7 – 1o11	1/1 – 3/2 – 7/4 – 11/4	4:6:7:11

The **dom7 flat-5 chord** is another chord that's symmetrical in 12-edo. A yazasa tuning uses the iso 5th = 17/12 = 603¢.

h7(17o5) chord	w1 – y3 – 17o5 – z7	1/1 – 5/4 – 17/12 – 7/4	12:15:17:21 (nearly all-odd)

A yazala tuning is possible, if the ilo 4th = 11/8 can be accepted as a flat 5th. A compact near-all-odd voicing is w1 – y3 – z7 – 1o11 = 4:5:7:11.

h7,1o4no5 chord	w1 – y3 – 1o4 – z7	1/1 – 5/4 – 11/8 – 7/4	8:10:11:14

Many **flat-nine chords** can be tuned yasa or yazasa with the iso 9th = 17o9 = 17/8 = 1305¢. A good voicing is 1 – 5 – W3 – W7 – W9.

maj,♭9	y,17o9 chord	w1 – y3 – w5 – 17o9	1/1 – 5/4 – 3/2 – 17/8	8:10:12:17
dom7,♭9	h7,17o9	w1 – y3 – w5 – z7 – 17o9	1/1 – 5/4 – 3/2 – 7/4 – 17/8	8:10:12:14:17
maj7,♭9	y7,17o9	w1 – y3 – w5 – y7 – 17o9	1/1 – 5/4 – 3/2 – 15/8 – 17/8	8:10:12:15:17

Some flat-9 chords can be tuned yaza with the zogu 9th = zg9 = 21/10 = 1285¢.

min7,♭9	g7,zg9 chord	w1 – g3 – w5 – g7 – zg9	1/1 – 6/5 – 3/2 – 9/5 – 21/10	10:12:15:18:21
dom7,♭9	h7,zg9	w1 – y3 – w5 – z7 – zg9	1/1 – 5/4 – 3/2 – 7/4 – 21/10	20:25:30:35:42

Another yaza possibility for the flat 9th is not technically a 9th: the ruyo aug 8ve = ry8 = 15/7 = 1319¢. We can't call it a 9th, because ry9 = 135/56. But calling it an octave is misleading, so it's called an 8th. It works well in the y7,ry8 chord. The all-odd voicing is w1 – ry8 – Ww5 – WWy3 – WWy7. This chord is a homonym both of the s7,zg8 chord = 7/(7:6:5:4:15) and of the s6 chord with a g6 in the bass = s6/g6 = 12/(7:6:5:4:15).

maj7,♭9	y7,ry8 chord	w1 – y3 – w5 – y7 – ry8	1/1 – 5/4 – 3/2 – 15/8 – 15/7	56:70:84:105:120

A yazana tuning of the min6(♭9) chord uses the ino 9th = 19o9 = 19/9 = 1294¢. The all-odd voicing is w1 – y6 – 19o9 – z10 – Ww5 = 9:15:19:21:27. Because of the narrow 173¢ interval between the 9th and the 10th, a better voicing might be w1 – y6 – z10 – Ww5 – W19o9 = 9:15:21:27:38.

min6,♭9	z,y6,19o9	w1 – z3 – w5 – y6 – 19o9	1/1 – 7/6 – 3/2 – 5/3 – 19/9	18:21:27:30:38

The **Hendrix chord** (dom7 sharp-9 no-5 chord) can be tuned yaza with a zo 10th = 7/3. Color notation requires the sharp 9th to be notated as an added minor 10th. The all-odd voicing is w1 – z10 – Wy10 – WWz7, which is far from the usual voicing of 1–3–7–9. In this voicing, the ♯9 makes a high-integer-limit ratio of 28/15 with the 3rd.

h7,z10no5 chord	w1 – y3 – z7 – z10	1/1 – 5/4 – 7/4 – 7/3	12:15:21:28

A yazana tuning uses the ino 10th = 19/8 = 1200¢ + 298¢. It's a harmonic series chord, h7,19no5. A good near all-odd voicing is w1 – y10 – Wz7 – W19o10 = 4:10:14:19. The 19o10 makes a wolf 4th with the z7, making a noru 4th = 19/14 = 529¢, wider than w4 by a noru comma 19or1 = 31¢. However, in the usual voicing, the interval between the 3rd and the ♯9 is 19/10, better than 28/15 and perhaps making up for the wolf.

h7,19o10no5	w1 – y3 – z7 – 19o10	1/1 – 5/4 – 7/4 – 19/8	8:10:14:19

Finally, let's look at alternate tunings of the basic chords covered in chapter 2.4. First, any ya chord can be tuned za by substituting zo for gu and ru for yo. And any yaza chord has an alternate yaza tuning, found by swapping yo with ru and gu with zo.

For example, the **minor triad** can be tuned not gu but zo. Chapter 2.7 discussed the ideal voicing for any JI chord. If a minor triad is in the ideal voicing for a zo triad, it suggests a zo tuning. A voicing with a minor 10th suggests this substitution, because Wz3 = 7/3 has a lower integer limit than Wg3 = 12/5. However, in any voicing, the ru 3rd contained in the zo triad will always have a higher integer limit than the yo 3rd contained in the gu triad.

The **major triad** can be tuned not yo but ru. This is less likely because the dissonant ru 3rd is more prominent in a ru triad than in a zo one.

The **dom7 chord** can be tuned yaza as either h7 or r,g7. Two ya tunings are y,w7 or y,g7. The y,g7 chord's tritone is more dissonant, but the other intervals are better, especially if there is a wa 9th. The y,g7 tuning is better for Oye Como Va (Im7 – IV7) or for I – VIm – IIm – V7 – I tuned to adaptive JI. The w7 tuning is better for the I – IV – V7 – I progression.

See also chapter 4.9, Tuning Innate Comma Chords.

Part IV – Temperaments

In Parts II and III we looked at the notes in the harmonic lattice. In Part IV we'll look at the spaces between the notes – the rungs of the lattice. Several chapters assume the use of alt-tuner, so that we can discuss some methods of tempering that only alt-tuner can do. (Part IV is unfinished)

Chapter 4.1 – Basic Tempering

To **temper** means to slightly alter the size of the primary rungs that make up the harmonic lattice, and doing so creates a **temperament**. Alt-tuner's tempering sliders let you adjust the interval size of each primary rung, thus warping the harmonic lattice and retuning all the intervals.

Slide the wa tempering slider half way to the right and the yo and zo ones halfway to the left. Play a few chords, and hear how dissonant everything has become! Why would you want to do this? Three reasons:

The first reason is to extend the range of the lattice in alt-tuner. This is a technique specific to alt-tuner of retuning a JI interval to another JI interval, in effect transforming one color into another. It can also be thought of as moving an entire row from one part of the lattice to another. These tunings are not really temperaments, just JI tunings under a different name.

The second reason is that there's nothing wrong with dissonance per se. In fact, music needs some sort of dissonance to be interesting. Otherwise music would be a repetitive, plodding, droning bore. Various cultures and genres favor certain kinds of dissonance over others. African music favors the rhythmic dissonance of polyrhythms and syncopation. Balkan music favors the rhythmic dissonance of irregular time signatures. Arabic music favors the melodic dissonance of quarter-tones. Classical music favors the dissonance of frequent modulations, extreme dynamics, etc. Jazz favors the harmonic dissonance of 12-ET tetrads and pentads. Rock favors the timbral dissonance of loud raunchy sounds. Many microtonalists deliberately choose tunings that are nowhere near JI, just for the interesting dissonances they produce. In my opinion, this type of dissonance works better with inharmonic timbres. These tunings are not strictly speaking temperaments, because they aren't conceptually based on adjusting JI rungs. I call this non-JI-centric tempering.

The third reason is to allow greater freedom in modulating. Every JI keyboard tuning has wolf intervals like yo 5ths and ru 5ths. So certain chords will sound bad, and certain chord progressions won't work in JI. Alt-tuner can get around this by retuning the keyboard on the fly, but that isn't always desirable. Tempering allows us to spread around the out-of-tune-ness among all the intervals so that they are only slightly mistuned. This is the usual approach to tempering in Western music. I call this JI-centric tempering.

Let's look at the first method. Suppose you're in C and you want this chord progression: Fy – Cy – yEy – yAg – Gy. The E chord needs a yoyo interval, which is not on the lattice. You could expand the lattice and add a yoyo row. Or you could set your center note to E and use the gu C as your tonic. There's a third way: you can turn zo into yoyo by setting the zo slider to yy6 = 976.5¢. Set your center note to C and set up this scale: wC zD♭ wD gE♭ yE wF zgG♭ wG zA♭ yA gB♭ yB. The Ey chord will look funny on the lattice, but will sound fine. (This can also be done with a slight tempering; see below.)

In effect we moved the entire row of zo notes up to the yoyo row. Moving the zo slider affects all septimal intervals. We also moved the ru row down to the gugu row, and the zogu row over to the yo row on the fifthward side, and the ruyo row over to the fourthward gu row.

Suppose you want your wa keys to be this scale: w1 1o2 1o3 w4 w5 1o6 1o7 w8. Since alt-tuner defaults to treating ilo as minor, this is the phrygian mode, so it will be in E. Unfortunately, alt-tuner's default lattice only has 3 ilo notes. You could lengthen the ilo row, but there's an easier way. You can turn tho into ilo by setting the tho slider to 1o6 = 44/27 = 845.5¢. In effect this adds another interval to the ilo row. To get your scale, you must use w5 as your tonic, so that A is the center note. Use tuning taps to get wE 3oF 1oG wA wB 1oC 1oD. (The black keys can be tuned any old way.)

You can turn ilo into tho by setting the ilo slider to L3o4 = 351/256 = 546.4¢. Then 1o3 becomes 3o3, 1o7 becomes 3o7, and 1o4 becomes L3o4. If you use w5 as your tonic, you will get 3o2 instead of L3o4.

Or you can split the difference of 3u1o1 = $2^{5} \cdot 3^{-3} \cdot 11^{1} \cdot 13^{-1}$ = 4.9¢ to get a pseudocolor that equates ilo and tho: set the ilo slider to 548.9¢ and the tho slider to 843.0¢. This isn't quite JI, but it's very close.

To explore extended 3-limit tunings, set the yo slider to Lw3 = 408.0¢ and the zo slider to Lw6 = 1019.6¢. The yaza notes now provide an unbroken chain of 23 fifths.

There are many approaches to non-JI-centric tempering. Freed from the constraints of providing simple strong just harmonies, the scale can instead be chosen to produce simple strong melodies. What makes a strong melody is debatable, but one criteria is the size of the melodic steps. It has been suggested that melodies with equal sized steps are more singable. They certainly could be called simpler.

One of the most popular approaches to non-JI-centric tempering is to divide the octave into equal sized steps, just like the familiar 12-ET, but with the number of steps being something other than 12. These types of tunings are called **edos** ("EE-dough"), short for "equal division of the octave". Once you choose the number of steps, everything else about your scale is completely determined. All the step sizes are exactly equal, which makes things considerably simpler than just intonation. Any chord or melody can be played in any key without adjusting the scale, and results in freedom to modulate. Edos are particularly popular among guitarists, because it makes fretting a guitar much easier. See chapter 5.18 for more on this.

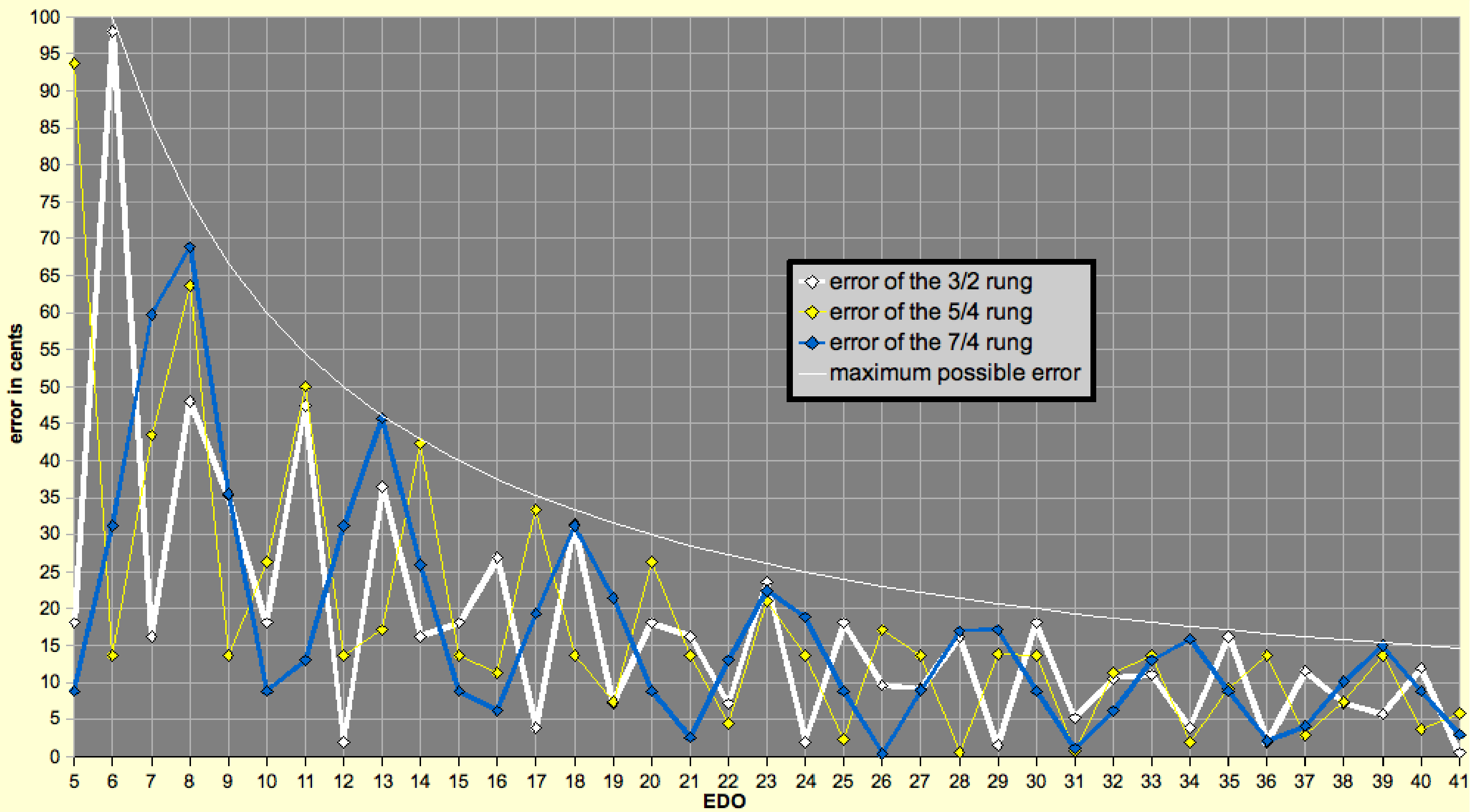
I've always liked this quote about edos from the xenwiki, http://xenharmonic.wikispaces.com/EDO:

> "What are EDO scales like? Very straightforward to work with, the step size being so even and all. Some find the monotony bland, others find it a safe stable footing for music-making. The only property shared by all of them is the equality of their step-sizes; otherwise, their individual properties are as different as can be. The lower-numbered EDOs, especially 5 to 24, possess very strong and unique "characters", which some composers have found to be inspiring in their own right."

Edos are named after the number of steps in them, as in 15-edo, 17-edo, etc. The familiar 12-ET is also called 12-edo. The equal-sized steps of an edo are called **edo-steps**. For example, a 10-edo-step is 120¢. Every interval in an edo can be written as a fraction of an octave. A backwards slash is used to differentiate this **octave fraction** from a frequency ratio. For example, 3\5 = "3 of 5" = 3 steps in 5-edo = 720¢, whereas 3/5 = "3 to 5" = descending yo 6th = -884¢.

Different edos will approximate the JI rungs with varying degrees of accuracy. The following chart shows the accuracy of each edo from 5-edo to 41-edo. The curved line represents the maximum possible discrepancy, which is half of one edo-step. For example, because the 12-edo-step is 100¢, the most a 12-edo interval can miss a rung by is 50¢. When comparing edos, remember that the wa rung is far more likely to be stacked and thus its accuracy is paramount. Those who want "JI, but simpler" may prefer highly accurate edos like 31-edo or 41-edo. Those who want fresh dissonances may be drawn to highly inaccurate edos like 11-edo or 13-edo.

Figure 4.1.1 – The discrepancy of the wa, yo and zo rungs in each edo from 5 to 41 (see also Figure 5.7.6)



Especially with the larger, more accurate edos, it can be useful to think of the edo as a special kind of temperament that just happens to conform to an edo. For example, if we sharpen the wa rung to 720¢, sharpen the yo rung to 400¢ and flatten the zo rung to 960¢, every interval in the yaza lattice will be contained in 15-edo. That's because all three rungs are multiples of the 15-edo-step of 80¢.

Each JI rung is mapped to an edo-step via an **edo-mapping**. There can be more than one edo-mapping for an edo. For example, in 12-edo, the ilo rung 1o4 = 551¢ could be tempered to either 500¢ or 600¢. The edo-mapping that most closely approximates JI is the **nearest edo-mapping**. Figure 4.1.1 shows all the nearest edo-mappings. If the graph included more distant edo-mappings, they would "poke through the roof" of the maximum discrepancy line.

Alt-tuner will let you set up your keyboard to have more than 12 notes per octave, allowing access to all the notes of an edo. But often those using edos work with only a subset of all the possible notes. For example, when working in 22-edo, it's convenient to use only 12 of the 22 notes and map them to the 12 keys of the standard keyboard. One way to fit 22 equal steps into 12 unequal semitones is with most of the semitones being two 22-edo steps wide, and two being only one step wide. The larger ones are abbreviated "L" and the smaller "s". One possible scale is LLLLsLLLLLLs. This use of L and s is distinguished from the use of large and small to describe the magnitude of ratios by context. In this context, L and s are interval sizes. In this particular 22-edo example, L = 109¢ and s = 55¢.

A more familiar example of edo subsets is the white keys of a piano tuned to 12-ET. Another example is the classical harp, with only 7 strings per octave, but with pedals that can sharpen or flatten each string. Both these examples have a 7 note subset of the 12 notes. The major scale is LLsLLLs, and the minor one is LsLLsLL. One is obviously a mode of the other. Here, L = 200¢ and s = 100¢.

The L and s notation can be used to describe any scale or tuning with only two step sizes. L is not always twice the size of s. For example, a major scale can be constructed from a 19-edo subset that also runs LLsLLLs. Here the large steps are three 19-edo-steps and the small ones are two. L = 3\19 = 189¢ and s = 2\19 = 126¢.

The property of having only two step sizes is good for melodic simplicity. If the large and small steps are spread out through the octave fairly evenly, the scale has another simplifying property: not just the 2nds but all the intervals come in only two sizes. Except the octave, which obviously comes in only one size. Mathematicians call this property

Myhill's property, but microtonalists call it **moment of symmetry**, abbreviated **MOS** and pronounced "moss". I call a scale that has this property **mossy**.

An example of a mossy scale is the LLsLLLs major scale in 12-ET. This table shows the size of every interval in the C major scale in semitones:

Table 4.1.1 – All interval sizes in the C major scale, in semitones

interval	from C	from D	from E	from F	from G	from A	from B	sizes
2nds	C to D = 2	D to E = 2	E to F = 1	F to G = 2	G to A = 2	A to B = 2	B to C = 1	1 & 2
3rds	C to E = 4	D to F = 3	E to G = 3	F to A = 4	G to B = 4	A to C = 3	B to D = 3	3 & 4
4ths	C to F = 5	D to G = 5	E to A = 5	F to B = 6	G to C = 5	A to D = 5	B to E = 5	5 & 6
5ths	C to G = 7	D to A = 7	E to B = 7	F to C = 7	G to D = 7	A to E = 7	B to F = 6	6 & 7
6ths	C to A = 9	D to B = 9	E to C = 8	F to D = 9	G to E = 9	A to F = 8	B to G = 8	8 & 9
7ths	C to B = 11	D to C = 10	E to D = 10	F to E = 11	G to F = 10	A to G = 10	B to A = 10	10 & 11
8ves	C to C = 12	D to D = 12	E to E = 12	F to F = 12	G to G = 12	A to A = 12	B to B = 12	12

The rightmost column summarizes each row. There are only two sizes of each non-octave interval, therefore the major scale is a mossy scale. Analyzing any mode of the major scale (minor, dorian, etc.) is just a matter of rearranging the columns in this table. Therefore all such modes are also mossy. More generally, every mode of a mossy scale is mossy, and every mode of a non-mossy scale is non-mossy.

The major pentatonic scale C D E G A, as well as all its modes, is also mossy. Compare the melodic minor scale C D E♭ F G A B C. The scale is LsLLLLs, and the 2nds only come in two sizes, 1 and 2 semitones. But in addition to the usual perfect and augmented 4ths, there is a diminished one B – E♭. So 4ths come in three sizes, and the melodic minor scale is not mossy. Such a scale is called a **MODMOS**, a modified MOS scale. The modification is raising or lowering one or more notes by the difference between s and L. Usually this means switching an s with an L, as in this case, but it can also mean creating other step sizes, "extra large" = L + L - s (as in the harmonic minor scale C D E♭ F G A♭ B C), or "extra small" = s + s - L (as in C D E F♯ G♭ A B C) or both (as in C D E F♭ G A B C).

Mossy scales generally produce simpler melodies, but remember, simplicity is not always desirable. The musical cultures of the Middle East relish melodic complexity and seem to have a distinct bias <u>against</u> mossy scales.

The mossy property depends only on the arrangement of the large and small steps, not on the actual size of those steps. Any scale that runs LLsLLLs will be mossy, and any scale that runs LsLLLLs won't be. For example, the 19-edo major scale discussed earlier is mossy.

To explore all the variations of LLsLLLs and its modes with alt-tuner, tap to the 7 ratios on the wa row. If you're in D, these 7 ratios will be for the 7 wa keys. Go to the graph view and tap the other 5 notes silent, to make the "mossiness" more apparent in the histogram on the right. This creates a 3-limit dorian JI scale, the LsLLLsL mode. Now move the wa slider around between 685.7¢ and 720¢. Every possible LLsLLLs scale lies in this range. Below 685.7¢ you'll get an ssLsssL scale, which is also mossy. At 685.7¢, s = L and you get 7-edo. At 720¢, s = 0¢ and you get 5-edo. Above 720¢, s < 0¢ and you get upside-down intervals.

Certain 3-limit JI scales are mossy, notably the diatonic, pentatonic and chromatic scales. No JI scales with a prime limit higher than 3 are exactly mossy, but they can come quite close, e.g. the ya major scale. JI scales based on the harmonic series are "anti-mossy" scales, with hardly any two steps anywhere of equal size! In the alt-tuner histogram, exactly mossy scales have two long lines of each color, near-mossy scales have two tight clusters of medium-length lines, and anti-mossy scales have numerous short lines. In chapter 6.9, the 19 keys per octave example is mossy and the harmonic series example is anti-mossy.

There are other approaches to non-JI-centric tempering besides edos, even among equal-sized-steps tunings. Another approach is to divide some interval instead of the octave up equally. Such a scale is called an **EDONOI**, equal division of a non-octave interval. For example, the music of the country of Georgia uses a scale that divides the fifth into 4 equal steps. The Georgian second is about 175¢, the neutral third is about 351¢, and the slightly sharp fourth is about 526¢. EDONOIs are even further from JI because they don't contain any just octaves. The Georgian octave is a fifth plus a fourth, which makes 1228¢. A quick way to set up EDONOIs in alt-tuner is to first set up an edo and then stretch or compress the octave until the non-octave interval is just. More on EDONOIs in chapter 4.8.

Going further from JI, one can select a step size that doesn't add up to any ratio at all, and construct a completely irrational scale. An example of this is the scale formed from an equal division of an interval who's "ratio" is an irrational number, like the golden ratio phi, which as a musical interval is 833¢. This would be a completely "anti-just" approach.

To complete this broad survey of tuning approaches, let's look at the other side of the spectrum. Beyond ya and yaza JI lies 11-limit JI. It magnifies the disadvantages of JI: more intervals, more commas, and more overall complexity. The progression continues with 13-limit JI, 17-limit, etc. Very high-limit JI tends to result in scales based on the higher overtones of the harmonic series, and could be called "ultra-just". Everything is perfectly in tune, but to bring out the subtle power of the harmonic series, the music tends to be drone-based and unmodulating. Just the opposite of the modulational freedom that edos encourage!

Table 4.1.2 – The harmony vs. melody spectrum, from "ultra-just" to "anti-just"

harmonic series scales	higher prime-limit JI	lower prime-limit JI	MOS scales	edos	EDONOIs

Finally, one could leave behind the entire spectrum and construct scales that have neither equal step sizes nor rational intervals. To do this in alt-tuner, one could use the keybend screen to create completely arbitrary scales. Or for guitars, use the bridge-shifting technique described in chapter 5.18.

In between JI and edos lie JI-centric temperaments, which are the subject of the following chapters.

Chapter 4.2 – JI-centric Tempering

One may wonder, if chords sound better when they're in tune, why do we use 12-ET instead? One reason is that certain chord progressions don't work in JI. They either tend to drive a choir's pitch flat or sharp, or they force a singer to adjust the pitch of a note. To see why, first let's look at some that don't, like the progression in Pachelbel's Canon, which goes D – A – Bm – F♯m – G – D – G – A. To translate this progression into JI, we'll make a few obvious assumptions. We'll stick to ya JI, as yaza JI was not in use at the time. And we'll assume that if a chord has any notes in common with the previous chord, those notes will be tuned the same in both chords. In ya JI, Pachelbel's Canon becomes Dy – Ay – yBg – yF♯g – Gy – Dy – Gy – Ay. This progression can be played in JI without difficulty. You can verify this by tracing the chord movement on the harmonic lattice in figure 2.3.1. This progression's "footprint" in the lattice is a compact scale without any duplicate notes, i.e. without any commas. Another progression that works in JI is the common I – V – VIm – IV, which translates to Iy – Vy – yVIg – IVy.

However, a chord progression like C – F6 – C – G forces a pitch adjustment. In ya JI, this would be Cy – Fy6 – Cy – Gy. The footprint contains two D notes. The F chord uses the yo D, but the G chord uses the wa D. The difference is the gu comma g1 = 81/10 = 22¢. In practice, a good string quartet or choir will perform this progression in JI with what microtonalists call a **comma shift**, which is one kind of **pitch shift**, or **shift** for short. They'll use a slightly sharper D for the G chord. Because the shift is by a gu comma, I call this a gu shift. Since there is an intervening chord between the F and G chords, the shift is not too noticeable. Voicing matters too, it helps if the yo D appears in a different octave than the wa D. But a keyboard instrument like the piano or the organ can't shift. If it's tuned to JI, there will always be a wolf interval in this progression. To avoid this, since the 16th century, such instruments have been tempered away from JI. The details of how this is done are covered in the next chapter.

Table 4.2.1 – Examples of chord progressions with a gu shift

progression	in ya JI	in C	note that shifts
I – IV6 – I – V	Iy – IVy6 – Iy – Vy	Cy – Fy6 – Cy – Gy	yD up to wD
Im – III – Im – IVm – VII	Ig – gIIIy – Ig – IVg – wVIIy	Cg – gE♭y – Cg – Fg – B♭y	gB♭ down to wB♭

Almost all of the examples in this chapter use the gu comma only. There are other commas that can arise, of course. But in conventional Western music, the gu comma is by far the most common one.

Now consider the progression C6 – Gadd9, which in JI might be Cy6 – Gy,9. In the C chord, the interval from G to A is y2 = 10/9. But in the G chord, it's w2 = 9/8. The interval must change by a comma, to avoid a wolf chord. I call this a **comma warp** because the interval is warped from one chord to the next. A comma warp requires that there be two or more common notes between two adjacent chords. The two chords are usually tetrads or pentads. See the last example in the following table for a rare triadic comma warp:

Table 4.2.2 – Examples of chord progressions with a gu comma warp

progression	in ya JI	in C	interval that is warped
I6 – Vadd9	Iy6 – Vy,9	Cy6 – Gy,9	G–A widens from y2 to w2
Iadd9 – IIm7	Iy,9 – wIIg7	Cy,9 – yDg7	C–D narrows from w2 to y2
Im7no5 – IVsus4	Ig7no5 – IV4	Cg7no5 – F4	C–B♭ narrows from g7 to w7

There are two possible mappings to JI. Either way, the footprint in the lattice contains a duplicated note. But the note that is duplicated changes depending on the mapping. For example, Iadd9 – IIm7 can be mapped to Cy,9 – yDg7, with C duplicated. But it could also be mapped to Cy,9 – Dg7, with D duplicated.

A comma warp can result in a comma shift. In C6 – Gadd9, the D note shifts upwards by g1. With no intervening chord, the shift is quite noticeable. In C – F6 – C – G, the D note disappears, then reappears shifted, less noticeable.

Next, consider the chord progression C – Am – Dm – G – C. This progression starts with a smooth two-notes-in-common chord change and then has three satisfying fourthward cadences in a row. Thumb through any fake book, and you'll find half the songs use this progression. In ya JI, this is Cy – yAg – yDg – yGy – yCy. On the lattice, this progression "travels" from C to the yo C off to the left. The footprint contains duplicate C, E and G notes. As a result, the last chord will be tuned g1 flatter than the first one. Microtonalists call this a **comma pump**, because the pitch of the song is pumped up by a comma. The progression is said to pump the comma, although in this case it would be more accurate to say it "deflates" the comma. Microtonalists call this pumping **tonic drift** or **drift** for short. I call this particular comma pump the gu comma pump, and this particular drift an ascending or descending gu drift.

Table 4.2.3 – Examples of chord progressions that pump the gu comma

progression	in ya JI	in C	drift of the tonic
I – IV – IIm – V – I	Iy – IVy – yIIg – yVy – yIy	Cy – Fy – yDg – yGy – yCy	wC down to yC
I – IV – ♭VII – V – I	Iy – IVy – wVIIy – yVy – yIy	Cy – Fy – B♭y – yGy – yCy	wC down to yC
Im – ♭III – ♭VII – IV – Im	Ig – gIIIy – gVIIy – gIVy – gIg	Cg – gE♭y – gB♭y – gFy – gCg	wC up to gC

The progressions that travel fourthward drift down, and the ones that travel fifthward drift up. A recent example of the last progression in the table is "Boulevard of Broken Dreams" by Green Day. It's one of the worst case scenarios for gu comma pumps. The comma is pumped every four seconds in the verses, and if this song were played in JI, the song would drift from E minor all the way up to A minor!

The earlier C – F6 – C – G example is a **partial comma pump**, because although two D notes a comma apart are used, the progression doesn't travel beyond these two D notes. Thus shifts can be used without causing tonic drift. In this example, the partial pump starts on F6 and ends on G. A partial pump becomes a full pump if the first chord is added on at the end, e.g. F6 – C – G – F6. For the progression C – F6 – C – G – C – F6 – C – G etc., there is an ascending partial pump from the F6 chord to the G chord, and a matching descending partial pump from the G chord to the F6 chord. The D note gets pumped up and down repeatedly. But in a full pump like C – F6 – G – C – F6 – G – C etc., all the notes get pumped down repeatedly. This tonic drift can be thought of as a "global" comma shift.

Another type of comma issue is the **broken pump**, which arises from changing the order of the chords in a comma pump. Consider the pump C – Am – Dm – G – C. Each chord is connected by a common note or notes only to its immediate neighbors. If the chords are rearranged, some connections are broken: C – Am – G – Dm – C. There are no common notes between the Am and G chords, nor between the Dm and C chords. This is not a pump, and the tonic needn't drift. Because two triads with roots a 3rd, 4th, 5th or 6th apart tend to have common notes, broken pumps usually have at least two root movements by 2nds/7ths.

Table 4.2.4 – Examples of chord progressions with a broken gu pump

progression	in ya JI	in C	note that shifts
I – IIm – IV – V – I	Iy – yIIg – IVy – Vy – Iy	Cy – yDg – Fy – Gy – Cy	yD up to wD
Im – ♭VII – Vm – IV – I	Ig – gVIIy – Vg – IVy – Ig	Cg – gB♭y – Gg – Fy – Cg	gF down to wF
I – ♭VII – IV – V – I	Iy – wVIIy – IVy – Vy – Iy	Cy – B♭y – Fy – Gy – Cy	yD up to wD

The last example is the outro of "Alison (My Aim Is True)" by Elvis Colstello. Like comma warps, there are two possible JI mappings for broken pumps, and two possible footprints. For example, the outro could be tuned with the B♭ and F chords having gu roots.

When determining the chord progression, one must take into account prominent melody notes. These notes tend to make triads become tetrads or pentads, and they tend to connect the chords, so true broken pumps are rare.

The final type of comma issue is the innate comma chord or **ICC**, covered in chapter 2.4.

Table 4.2.5 – Examples of chords with an innate gu comma

chord	in ya JI	in C	wolf interval
maj6add9	Iy6,9	Cy6,9 = wC yE wG yA wD	yA–wD = g4
min7add11	Ig7,11	Cg7,11 = wC gE♭ wG gB♭ wF	gB♭–wF = y5

If a pair of chords has <u>three</u> or more notes in common, it can cause a **triple warp**. For example, C9 – B♭9, with both chords tuned h9 or 4:5:6:7:9. The three common notes are B♭, C and D. The B♭–C interval narrows by r1, from r2 to w2. C–D narrows by g1, from w2 to y2. And B♭–D narrows by rg1 = 36/35, from r3 to y3.

The three common notes form a triad. The Ch9 chord contains a C2,z7no5 triad, and the B♭h9 chord contains a B♭y,9no5 triad. It would be possible to warp one triad directly into the other. But musically, there's not much point. The other two notes in the chord are needed to motivate the warp.

A single chord progression can contain multiple comma issues. For example: "Oh! Darling" by the Beatles goes Gaug (pickup chord) – C – G – Am – F – Dm7 – G7 – Dm7 – G7 – C – F – C – G7. Modifying the chords slightly for the sake of this example, and tuning everything to ya JI, so that G7 is Gy,w7, we get an ICC, a pump, several warps, and a partial pump:

> Gaug (ICC with $g^3 2$ = 128/125) – C6 (start of a downward g1 pump) – Gadd9 (G-A warped wider by g1 = 81/80) – Am7 (G-A warped narrower) – F – Dm7 – G7 (D-F warped narrower by g1) – Dm7 (D-F warped wider) – G7 (D-F warped narrower) – C (end of pump, tonic drifts downward by g1) – F6 (start of a g1 partial pump) – C – G7 (end of partial pump, D shifted up by g1)

A partial pump can be quite long. For example, if the classic Shona mbira chord progression (covered in chapter 5.12) is extended from dyads to triads, a long partial pump is created: C – Em – Am – C – F – Am – Dm – F – Am – C – Em – G – C. The major chords have wa roots, and the minor ones yo roots.

A comma pump usually requires at least three chord changes (e.g. C – F6 – G – C, four chords, but three changes). But some comma pumps are only two chord changes. For example, the "tritone swap" I7 – ♭V7 – I7 can be tuned Ih7 – zgVh7 – zzggIIh7, pumping the double ruyo comma rryy-2 = 50/49 downwards. But this is a special case, because generally one could simply return directly to the first chord from the second one (Ih7 – zgVh7 – Ih7, or alternatively Ih7 – ryIVh7 – Ih7). The zzggIIh7 chord would be preferred over the Ih7 chord only in certain contexts, for example to hide a comma shift in a prominent melody note. To pump rryy-2 unambiguously, more chords are required: I7 – ♭VIIm6 – ♭V7 – ♭IVm6 – ♭♭II7, which would be tuned as Ih7 – zVIIs6 – zgVh7 – zzgIVs6 – zzggIIh7.

Another example: C7 – B♭add9 – C7, tuned as Ch7 – zB♭y,9 – zCh7, which pumps the ru comma r1 = 64/63 = 27¢ downwards. But Ch7 – B♭y,9 – Ch7 would be simpler, and Ch7 – zB♭s6 – zFh7 – zCh7 would be less ambiguous. A two-changes comma pump always contains two comma warps, one for each chord change. Both warps warp the same interval. The two warps raise or lower first one note of the interval, then the next, causing the whole chord to shift.

The five comma issues can be ordered by the minimum number of chord changes they require:

Table 4.2.6 – Comma issues

issue	# of chord changes needed	necessary conditions
ICC	none	the chord must have at least three notes
warp	one	the two chords must have at least two common notes
triple warp	one	the two chords must have at least three common notes
partial pump	two	each chord must have at least one common note with its neighbors
pump	three (occasionally two)	as above, plus the last chord must be the same as the first chord
broken pump	four	the last chord must be the same as the first chord

Shifts and drifts needn't always be viewed negatively. A slight shift of a note higher at a dramatic point in the melody may add a feeling of intensity. Downward shifts are harder for Western ears to appreciate. As for tonic drift, consider the Beatles song "And I Love Her". The progression runs:

> verse: F♯m – C♯m – F♯m – C♯m – F♯m – C♯m – A – B7 – E
> chorus: C♯m – B – C♯m – G♯m – C♯m – G♯m – B7

This song translates easily to ya JI. The verse has a partial pump, implying a comma shift for F♯. The chorus has a full pump, implying an upwards tonic drift. In the original recording, the Beatles modulate upwards from E to F (or is it C♯ minor to D minor?) immediately after one of the choruses, presumably to give the song freshness before repeating more verses. That modulation has always struck me as cliched. One could achieving the same affect much more subtly and smoothly with an upwards drift of 22¢ after each chorus.

The worst comma problem I can imagine is a wolf unison, where a unison in two voices is mistuned by a comma of 20-30¢ (a slight mistuning would merely be chorusing). But even this is deliberately used to good effect by Michael Harrison on his "Revelations" CD. There really is no right or wrong in music. So rather than discussing comma problems, and better or worse solutions, let's discuss comma causes, and more noticeable or less noticeable effects.

"Pump" refers to tonic drift, which is an effect, not a cause. But a comma pump is a cause, and therefore would be better called an "innate comma progression". Unfortunately, "comma pump" is an established term with a long history, so we're stuck with it.

A pump can result in shifting instead of drifting, for example, Cy – Fy – yDg – Gy – Cy. Between the D and G chords, the D note shifts from yD up to wD. The shift is by a full comma, and there are no intervening chords, so it's quite noticeable. It's much easier to "sneak" a comma shift into a partial pump or a broken pump than a regular pump.

Often, a comma's effects can be avoided by arranging. For example, C – Am – Dm – G7 – C is not a comma pump if the 5th is dropped from the G7 chord: Cy – yAg – yDg – Gy,w7no5 – C. Another example: C6add9 is not an ICC if the 3rd is dropped and the 6th is wa, and especially if the C–A interval is voiced as a minor 3rd, not a major 6th. Also, as noted above, a pitch shift is much less noticeable if the note after shifting is voiced in a different octave. Timbre matters too: if a middle C note is sung by a soprano, then sung slightly sharper by a baritone, the shift is less noticeable than if the soprano sung both notes. Finally, spacial placement matters. If an ICC is tuned justly, with two notes creating a wolf interval, the wolf is less noticeable if the two notes are panned to opposite sides of the stereo field, or if the two singers are at the outer positions of the barbershop quartet lineup.

A comma's effects can appear horizontally (melodically), or vertically (harmonically), or both. (Obviously, an ICC can only cause vertical effects.) The horizontal effects include comma shifts and tonic drift. The vertical effects include mistuning one or more intervals by a comma, creating wolf intervals. Unless the comma is only a few cents wide, these effects are quite noticeable. Microtonalists have devised subtler tuning strategies with far less noticeable effects. Tempering (chapter 4.3) is a vertical effect that mistunes all or most intervals by only a fraction of a comma. Adaptive JI (chapter 4.7) is a horizontal effect in which the pitch of all or most notes shift upwards or downwards, usually by only a fraction of a comma. These are pitch shifts, but not comma shifts. Adaptive tuning (chapter 4.7) combines both strategies, creating very slight mistunings and pitch shifts.

Tempering adjusts the size of all the intervals in the lattice. Tempering out a comma means that the comma is adjusted down to zero cents. Tempering out the gu comma makes any two intervals differing by a gu comma sound identical. The wa 5th and the yo 5th both sound the same, as do the wa 2nd and the yo 2nd. The next chapter covers tempering out the gu comma, aka meantone temperament.

The various comma effects can be indicated with color notation. When using tempering or adaptive strategies, a comma pump is notated with an equals sign equating two chord roots. The equals sign should occur roughly halfway through the pump. For example, Cy – yAg – y=wDg – Gy – Cy. The D chord's root is yo because it's a 4th above yA, but wa because it's a 5th above wG. It's written "y=w", not "w=y", because the yA chord comes before the wG chord. The order is always old=new. A descending gu pump is always notated "y=w" or "w=g", and an ascending pump is always "w=y" or "g=w".

The next few tables list the various possible effects for each comma cause. The affected chords are bolded. Effects are listed in order of most noticeable to least noticeable.

Comma pumps: in the third example, the yy3 in the D chord lies between the gF and the yA.

Table 4.2.6 – Possible effects of a gu comma pump on the I – VIm – IIm – V – I chord progression

comma effect	in ya JI	in C	specific effect
wolf chord	Iy – **wVIw(y5)** – wIIg – Vy – Iy	Cy – **Aw(y5)** – Dg – Gy – Cy	the A chord has a w3 and a y5
	Iy – yVIg – **wIIw(y5)** – Vy – Iy	Cy – yAg – **Dw(y5)** – Gy – Cy	the D chord has a w3 and a y5
	Iy – yVIg – **wIIg(y5)** – Vy – Iy	Cy – yAg – **Dg(y5)** – Gy – Cy	the D chord has a yy3 and a y5
	Iy – yVIg – yIIg – **Vy(y5)** – Iy	Cy – yAg – yDg – **Gy(y5)** – Cy	the G chord has a w3 and a y5
tonic drift	**Iy** – yVIg – yIIg – yVy – **yIy**	**Cy** – yAg – yDg – yGy – **yCy**	the final C chord is g1 lower
comma shift	**Iy – wVIg** – wIIg – Vy – Iy	**Cy – Ag** – Dg – Gy – Cy	wC & yE shift up to gC & wE
	Iy – **yVIg – wIIg** – Vy – Iy	Cy – **yAg – Dg** – Gy – Cy	yA shifts up to wA
	Iy – yVIg – **yIIg – Vy** – Iy	Cy – yAg – **yDg – Gy** – Cy	yD shifts up to wD
	Iy – yVIg – yIIg – **yVy – Iy**	Cy – yAg – yDg – **yGy – Cy**	yG shifts up to wG
tempering	**Iy – yVIg – y=wIIg – Vy – Iy**	**Cy – yAg – y=wDg – Gy – Cy**	all chords are tempered
adaptive JI	"	"	all pitches shift
adaptive tuning	"	"	tempered plus pitch-shifted

It may seem potentially confusing that the last three strategies are all notated the same. But they all need further specification. Tempering can be quarter-comma vs. third-comma, and adaptive tunings need a strength percentage. This extra information is written at the top of the page, identifying the exact tuning.

Partial pumps: there are fewer strategies for dealing with these. Tonic drift isn't one of the effects. Adaptive JI simply creates a comma shift. This strategy can be less noticeable than tempering, if the intervening chord is held for long enough.

Table 4.2.7 – Possible effects of a gu partial comma pump on the I – IV6 – I – V chord progression

comma effect	in ya JI	in C	specific effect
wolf chord	Iy – **IVy,w6** – I – Vy	Cy – **Fy,w6** – Cy – Gy	the F chord has a w3 and a y5
	Iy – IVy6 – I – **Vy(y5)**	Cy – Fy6 – Cy – **Gy(y5)**	the G chord has a w3 and a y5
comma shift, adaptive JI	Iy – **IVy6** – I – **Vy**	Cy – **Fy6** – Cy – **Gy**	yD shifts up to wD
tempering	**Iy – IVy6 – I – Vy**	**Cy – Fy6 – Cy – Gy**	all chords are tempered
adaptive tuning	"	"	tempered plus pitch-shifted

Broken comma pumps: adaptive tuning splits the difference between the two possible JI mappings.

Table 4.2.6 – Possible effects of a broken gu comma pump on the I – IIm – IV – V – I chord progression

comma effect	in ya JI	in C	specific effect
wolf chord	Iy – **wIIw(y5)** – IVy – Vy – Iy	Cy – **Dw(y5)** – Fy – Gy – Cy	the D chord has a w3 and a y5
	Iy – wIIg – **gIVy(y5)** – Vy – Iy	Cy – Dg – **gFy(y5)** – Gy – Cy	the F chord has a w3 and a y5
	Iy – yIIg – IVy – **Vy(y5)** – Iy	Cy – yDg – Fy – **Gy(y5)** – Cy	the G chord has a w3 and a y5
comma shift	Iy – **yIIg** – wIVy – **Vy** – Iy	Cy – **yDg** – Fy – **Gy** – Cy	yD shifts up to wD
	Iy – wIIg – **gIVy** – Vy – Iy	**Cy** – Dg – **gFy** – Gy – Cy	wC shifts up to gC
tempering	Iy – y=wIIg – w=gIVy – Vy – Iy	Cy – y=wDg – w=gFy – Gy – Cy	all chords are tempered
adaptive JI	"	"	C & D shift by 1/2 comma
adaptive tuning	"	"	tempered plus pitch-shifted

Comma warps: in the tempered warp, the order of the colors in the equation (y=w vs. w=y) is arbitrary.

Table 4.2.8 – Possible effects of a gu comma warp on the Iadd9 – IIm7 chord progression

comma effect	in ya JI	in C	specific effect
wolf chord	**Iy,y9** – yIIg7	**Cy,y9** – yDg7	the C chord has a y5 (G–D), a w7 (E–D), and a y9 (C–D)
	Iy,9 – **wIIw7**	Cy,9 – **Dw7**	the D chord has 2 w3's (D–F and A–C), a Lw3 (F –A) and a w7
	Iy,9 – **wIIg,w7**	Cy,9 – **Dg,w7**	the D chord has a w3 (A–C), a y5 (F–C), and a w7 (D–C)
comma shift	**Iy,9 – yIIg7**	**Cy,9 – yDg7**	wD shifts to yD
	Iy,9 – wIIg7	**Cy,9 – Dg7**	wC shifts to gC
tempering	**Iy,9 – w=yIIg7**	**Cy,9 – w=yDg7**	all chords are tempered
adaptive JI	"	"	C shifts up and D shifts down by half a comma
adaptive tuning	"	"	tempered plus pitch-shifted

Innate comma chords: in a tempered ICC, the equals sign is applied to a note in the chord, not to the root. C6add9 is written Cy6,w=y9, because the 9th is both a 5th above wG, making it wa, and a 4th above yA, making it yo. The order (y=w vs. w=y) is arbitrary. ICCs are covered more fully in chapter 4.x.

Table 4.2.9 – Possible effects of an innate gu comma on the maj6add9 chord

comma effect	in ya JI	in C	specific effect
wolf chord	**Iy6,9**	**Cy6,9**	D–A is a y5
	Iy6,y9	**Cy6,y9**	G–D is a y5
	Iy,w6,9	**Cy,w6,y9**	A–E is a y5
tempering	**Iy6,w=y9**	**Cy6,w=y9**	the chord is tempered

If one is making music using midi and/or fretless instruments, the tuning is very flexible. A different tuning strategy can be selected for each song. But if using something more fixed, perhaps an acoustic piano, one must take into account the entire repertoire of songs to be played on that instrument. In effect, all the songs become one long song, and all five comma issues will almost inevitably arise. Drifts and shifts are not possible, and the only choice is between wolves and tempering.

Behind the concept of comma issues is the notion of two notes a comma apart being the "same" note. This depends on the keyspan, and the sizing framework used. Frameworks are covered in Part V. In 12-tone, the ru comma has a keyspan of zero, and w7 and z7 are the "same" note. The progression C7 – B♭– F7 – C7 becomes Ch7 – zB♭h7 – zFh7 – zCh7. But in 19-tone, z7 is a diminished 7th, not a minor 7th. As a result, r1 has a keyspan of one, and a ru comma pump isn't a pump but a modulation: Ch7 – B♭♭h7 – F♭h7 – C♭h7. Furthermore, Cy,add9 – Dh7 is not a warp, because the D chord contains not C but C♭, and there are no longer two common notes. Likewise, in 22-tone, the gu comma pump becomes a modulation.

Chapter 4.3 – Meantone Temperament

As mentioned in the previous chapter, it's impossible to play a progression like D – G – Em – A in JI on a 12-tone fixed-pitch instrument like the keyboard without playing a dissonant yo fifth. Starting around the Baroque era, musicians retuned their keyboards to avoid yo fifths, or to put it another way, to make a yo fifth sound more like a wa one. Unfortunately, this also causes a wa fifth to sound more like a yo one! They did this by tempering out the comma that is the difference between a yo fifth and a wa one, the gu comma g1 = 81/80 = 22¢. This is called meantone tempering.

The gu comma can be expressed in sum-of-rungs format as g1 = 4·w5 - y3 - 2·w8. Prefixing an interval with a capital T refers to the tempered interval. Tw8, Tw5 and Ty3 are the tempered rungs. In JI, g1 = 22¢, but meantone tempers out g1, and Tg1 = 0¢. Thus Tg1 = 4 · Tw5 - Ty3 - 2 · Tw8 = 0¢. Assuming untempered octaves, the wa and yo rungs can be related directly to each other with this equation. Ty3 = 4 · Tw5 - 2400¢ and Tw5 = Ty3 / 4 + 600¢.

In alt-tuner, cycle to the ya preset. Go to the linkages screen and OK the first comma, the gu comma. (If you don't see it, click the little yellow "1" in the upper left.) Because this comma uses the wa and yo rungs, OKing it creates a **linkage** between wa and yo. The two tempering sliders are linked, and when you move one of them, the other one reacts. To see this in action, see www.tallkite.com/images/alt-tuner/greenlinkage.gif.

In C, play a yo D, tap it up to wa, and play it again. It should sound identical. The major 2nd is averaged between the yo one and the wa one. Average = mean, major 2nd = whole tone, hence the term "meantone". Mathematically speaking, Ty2 = Tw2.

Play a wa fifth like wC – wG and see if you can hear a slight flatness. Play a yo fifth like wD –yA. It should sound equally flat, since Tw5 = Ty5. The slightly flat meantone fifth is certainly a huge improvement over the yo fifth!

Tempering blurs the difference between nearby JI intervals. Instead of a wa fifth or a yo fifth, there is only a meantone fifth. In chapter 3.3, "Paradoxical Intervals", I said that the ratio and cents are the reality and the quality and degree are the theory. Once you start tempering, the ratio becomes theoretical and the cents are the only reality.

The meantone 5th will appear in alt-tuner's display sometimes as a wa 5th and sometimes as a yo 5th. They look very different. The meantone minor 3rd is both a wa and gu 3rd, since Tw3 = Tg3. The harmonic lattice is now somewhat misleading. I find it helpful to imagine it wrapped around itself, so that fifthward wa is connected to fourthward yo, and fourthward wa is connected to fifthward gu. This creates many equivalences:

Tw5 = Ty5	Tw4 = Tg4
Tw2 = Ty2	Tw7 = Tg7
Tw6= Ty6	Tw3 = Tg3
TLw3 = Ty3	Tsw6 = Tg6
TLw7 = Ty7	Tsw2 = Tg2

Likewise fifthward yo connects to fourthward yoyo, fourthward gu to fifthward gugu, etc. If you select either of these ratios in alt-tuner, the other one will "light up" or "resonate" to indicate the equivalence. The ya lattice becomes a row of meantone fifths coiled around itself. This chain of fifths can be uncoiled like so:

... Tg2 – Tg6 – Tg3 – Tg7=Tw7 – Tw4 – **w1** – Tw5 – Tw2=Ty2 – Ty6 – Ty3 – Ty7 ...

In other words, meantone can be thought of as a tempered 3-limit tuning.

We've been talking about meantone as if it were one single tuning, but it's actually a whole spectrum of tunings. When you first OK the comma, the wa rung is flat and the yo rung is just. To explore the different varieties of meantone, move the wa and yo sliders around. As you do, you'll notice they react to each other. You may notice that the yo one is "jumpier". Because the gu comma has 4 wa rungs but only 1 yo rung, the yo rung has to "work harder" to temper out the comma. Moving the wa slider 1¢ moves the yo slider 4¢.

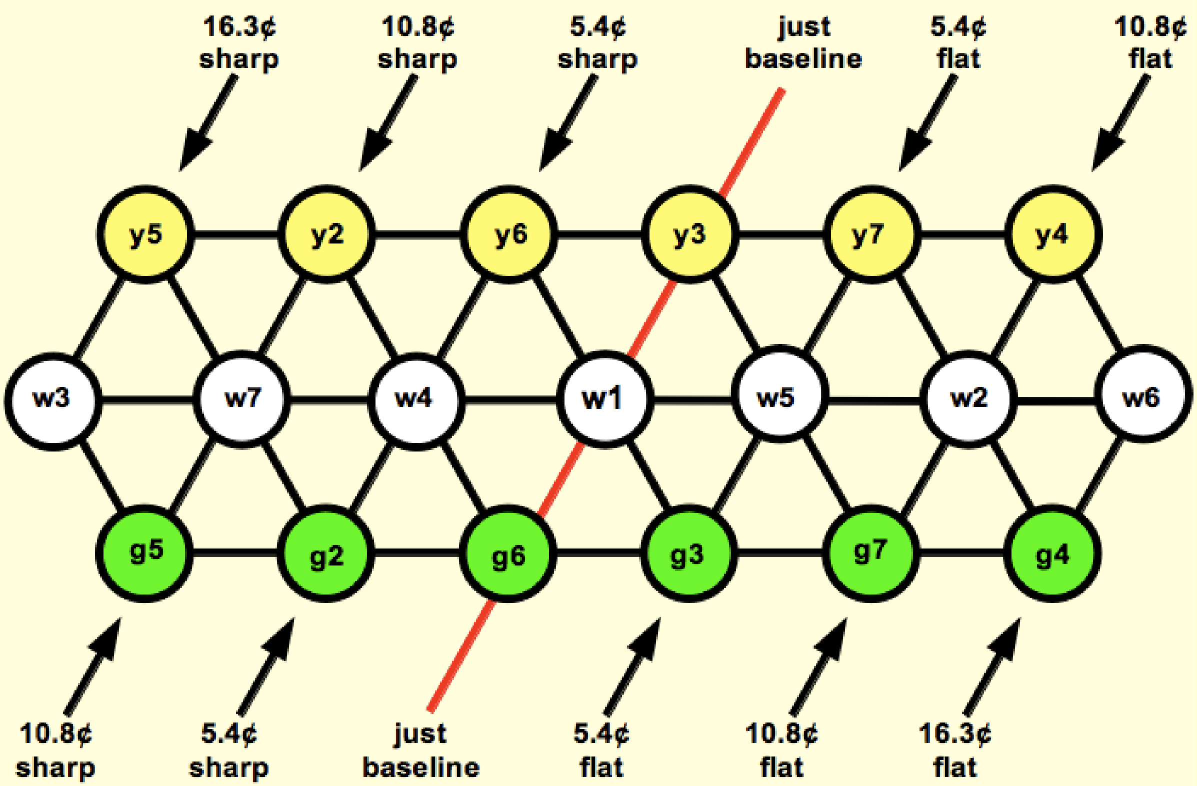
Double-click the yo slider to reset it to the center. This is quarter-comma meantone, in which y3 is just, and Ty3 = y3.

The wa slider is flat of just by one quarter of a gu comma = about 696.6¢. Move the yo slider so that it lines up exactly under the wa slider. This is third-comma meantone, in which g3 is just. You can verify this by playing a g3 and looking at the cents display above the lattice. Play a chord progression using y5, like I – VIm – IIm – V – I in both quarter-comma and third-comma meantone and compare. As noted in the previous chapter, such a comma pump is written in color notation as Iy – yVIg – y=wIIg – Vy – Iy. The equals sign is placed somewhat arbitrarily, roughly halfway through the pump.

Move the wa slider to 700¢ and the yo one will move to 400¢. The ya ratios will all be tuned to 12-ET, in which the major 3rds are 13.7¢ sharp. 12-ET could be thought of as 1/11 comma meantone. Double-click the wa slider to the default 702¢ to create an all-wa tuning with really sharp major 3rds! This is the pythagorean tuning, which could be thought of as zero-comma meantone.

Double-click the yo slider to return to quarter-comma meantone. The yo rung is centered, so Ty3 is just. So is its inverse, Tg6. Some intervals will be sharpened by the tempering and others will be flattened. Tw5 is a quarter-comma flat, which comes to about 5.4¢. Its inverse Tw4 will be sharp by the same amount. Tempering discrepancies add up, so Ty7 = Tw5 + Ty3 is also 5.4¢ flat, and Tw9 = Tw5 + Tw5 is 10.8¢ flat. This is one of the problems with meantone: the major 9th is quite flat of the just 9/4 interval, and 9 chords tend to sound bad. Especially if voiced widely, with Ww9 = 9/2 or WWw9 = 9/1. The next figure shows the discrepancies for a ya lattice. All the notes on any given upper-right/lower-left diagonal have the same discrepancy.

Figure 4.3.1 – The difference between tempered and just for quarter-comma meantone



The y3, w1 and g6 are all just, and the line through these three ratios is the just baseline. (Microtonalists would call y3 an eigenmonzo.) The further a ratio is from this baseline, the more it's tempered. Distance is measured in wa rungs. Each horizontal wa rung equals a quarter of a comma. For example, g7 is two wa rungs away from g6, which is on the baseline, so it's half a comma flat. Likewise y6 is a quarter-comma sharp.

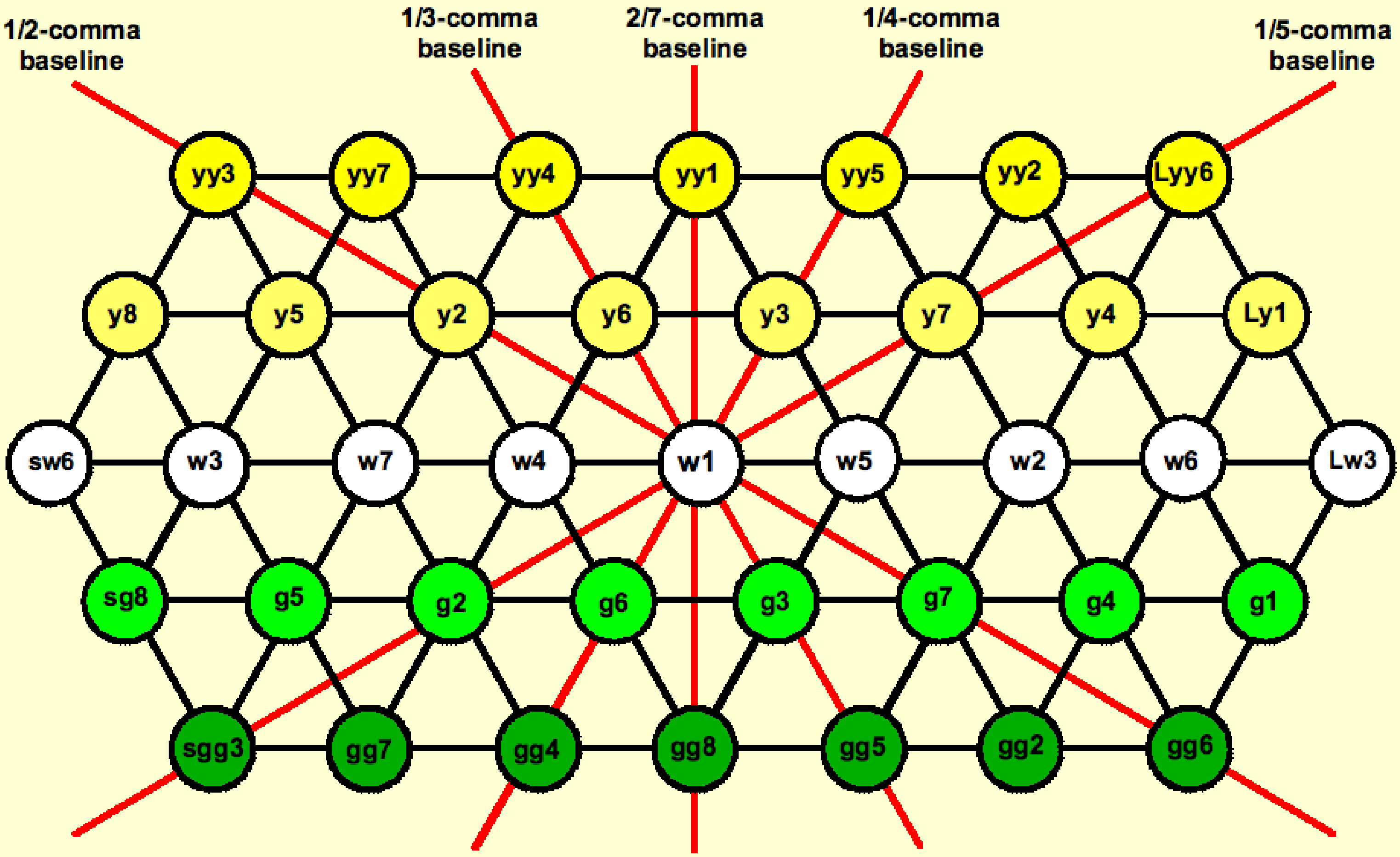
Each variety of meantone has a different baseline. What's the baseline for third-comma meantone? Search along the wa row for a ratio that's a whole number of commas flat. For example, Tw5 is 1/3 comma flat, Tw2 is 2/3 comma flat, and Tw6 is an entire comma flat. That means Tw6 = w6 - g1 = y6. So Ty6 is just, as is its inverse Tg3. The third-comma baseline runs through y6, w1 and g3. Each horizontal wa rung from the baseline equals a 1/3 comma discrepancy. The distance from the baseline can also be measured in diagonal yo rungs. Because the yo rung is 1/3 comma flat, each yo rung step in the northeast direction from the baseline adds 1/3 comma of flatness.

For fifth-comma meantone, set the wa slider to be as flat from just as the yo slider is sharp. To find the baseline, follow the wa row five rung steps to Lw7, which is one comma flat. TLw7 = Lw7 - g1 = y7. So Ty7 is just, as is its inverse Tg2. The fifth-comma baseline runs through y7, w1 and g2.

For two-sevenths-comma meantone, set the wa slider to be twice as flat from just as the third. 7 · Tw5 = TLw1 = Lw1 - 2 · g1 = yy1. The baseline runs vertically through yy1, w1 and gg8. One wa rung step = 2/7 of a comma. The y3 is half a wa rung step from the baseline = 1/7 of a comma.

For n/d-comma meantone, in which the fifth is flattened by n/d of a comma, the ratio d · w5 - n · g1 is just, as is its octave-reduced equivalent. Because g1 = 4 · w5 - y3 (octave-reduced), the just ratio is (d - 4 · n) · w5 + n · y3.

Figure 4.3.2 – The just baseline for various meantones



A just wa slider makes a horizontal baseline, with the entire yo row one comma sharp, the yoyo row two commas sharp, etc. Flattening the wa slider causes the baseline to rotate counter-clockwise, sweeping through the lattice. Our familiar 12-ET can be considered a type of meantone. It is very nearly 1/11-comma meantone, and the baseline passes very nearly through the large yo 3rd, Ly3. Thus almost any ya ratio can be made just (along with its inverse, and multiples of it or its inverse, and any octave transpositions of those). If that ratio in rung format has W wa rungs and Y yo rungs, the fifth needs to be flattened by Y / (W + 4 · Y) of a comma. The yo third will be sharpened by W / (W + 4 · Y) of a comma. The only ratio that can't be made just is the comma we have tempered out, g1. Obviously, if the linkage is defined as Tg1 = 0¢, Tg1 can't also equal 22¢!

As you experiment with this linkage, you'll find that while either slider can be set to just, you can't set both of them to just at the same time. How close to just can both sliders be? In other words, how can we minimize the largest discrepancy of the two rungs? Set the yo slider to 390.6¢, so that it's exactly as sharp as the wa slider is flat. They will both be off by 1/5 comma = 4.3¢. Any movement at this point will bring one slider closer to just and the other further away. Thus one of the sliders will always be off by at least 4.3¢. This is the minimum rung discrepancy of y3 and w5 in this linkage (assuming untempered octaves). It's a fifth of a comma because g1 has five rung steps.

But in practice, chords will often be mistuned by more than this amount. For example, in fifth-comma meantone, both g3 and w2 are two-fifths comma flat. So g chords and y,9 chords are mistuned by 8.6¢. And in a g7 chord, the 7th is three-fifths comma flat, or 12.9¢ flat. The *overall* minimum discrepancy depends on which intervals are considered consonant, that is, which ones actually turn up regularly in chords. The most commonly used intervals are g3, y3, w5, g7, y7, and w9. Since the octave is not tempered, their octave inverses y6, g6, w4, etc. will have similar discrepancies. Considering all these intervals, the g7 has the most discrepancy. Thus the overall minimum discrepancy for fifth-comma meantone is 12.9¢.

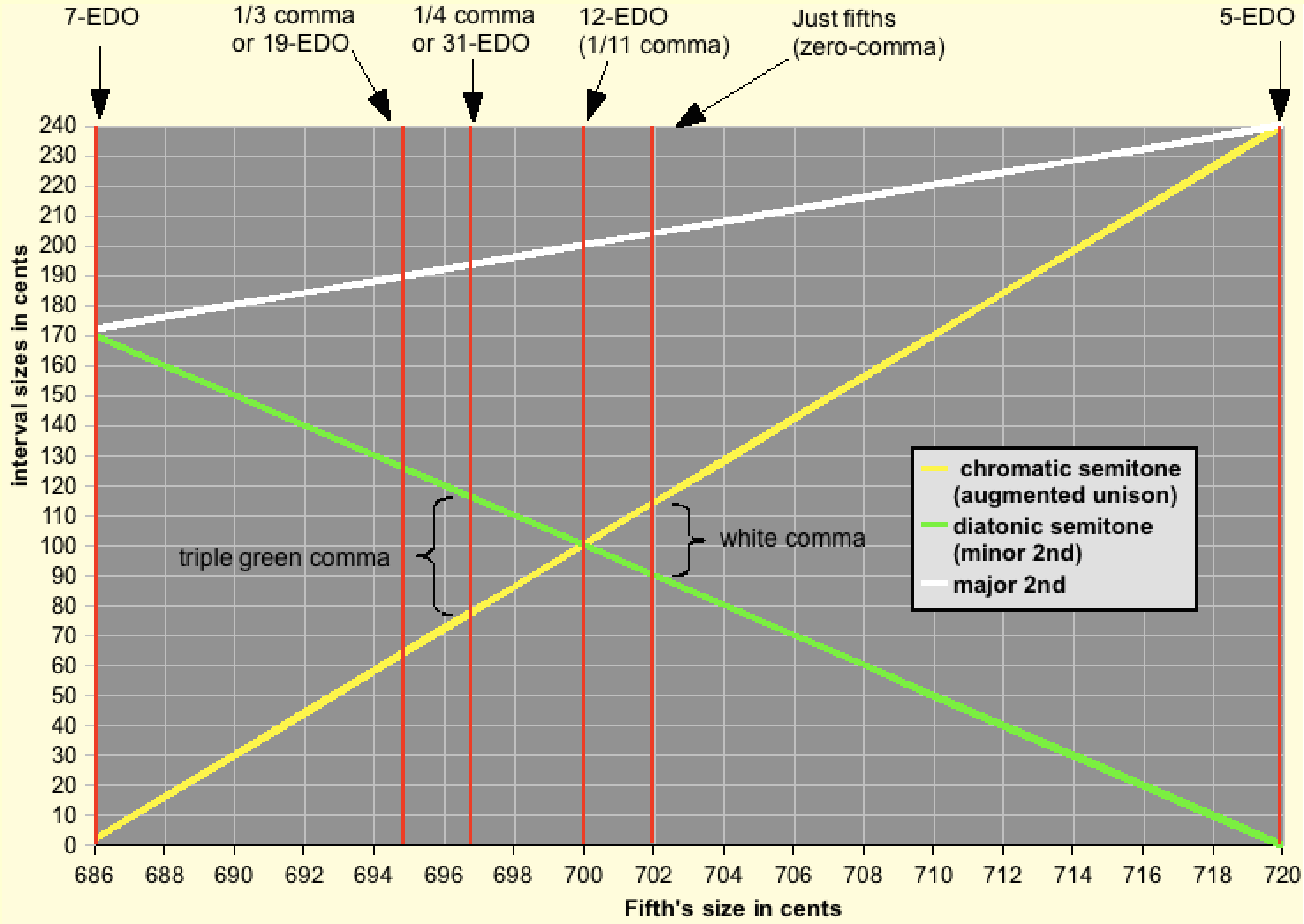
Different varieties of meantone will have different overall minimum discrepancies. The best meantone, the one with the *least* overall minimum discrepancy, turns out to be quarter-comma meantone, in which both g7 and w9 are a half-comma flat. The least overall minimum discrepancy in meantone is 10.8¢. To JI-accustomed ears, that's really noticeable. Still, that's not quite as bad as 12-ET, which approximates ya JI with an overall minimum discrepancy of 17.6¢, about 9/11 of a comma.

The line through y8, w1 and g1 in Figure 4.3.2 represents two unobtainable extremes of the rotation of the baseline. Another method for finding the best meantone is to choose the midpoint of this range of rotation, so that the just baseline is perpendicular to the comma. The calcuation becomes much simpler if a square lattice is used, as in Figures 1.3.8 and 1.3.9. The comma has the X-Y coordinates (4, -1), and the baseline goes through (1,4) = $Ly^4 6$. Adding 4 commas to this ratio converts it to wa without changing its pitch, and the point at (17,0), the TLLw6, is 4 commas flat. Thus a single wa 5th is 4/17 comma flat.

In general, for a comma (a, b, c), the perpendicular-baseline version of the temperament will have the wa 5th flattened by b/k comma and the yo 3rd flattened by c/k, where $k = b^2 + c^2$. Thus for meantone, Ty3 is 1/17 comma sharp. For a za comma (a, b, 0, c), the fifth is again flattened by b/k, and the zo 7th is flattened by c/k.

Besides analyzing meantone harmonically, we can also analyze it melodically. A major scale in meantone will be of the form LLsLLLs, where L = maj 2nd and s = min 2nd, and we can compare the sizes of L and s. Because meantone is basically tempered 3-limit, L = Tw2 and s = Tsw2. As mentioned near the end of chapter 4.1, all LLsLLLs scales lie in the wa slider range 686¢ (7-edo) to 720¢ (5-edo). As you move the wa slider to the right, L increases and s decreases. At 720¢, s becomes 0¢. At 685¢, s equals L. In the chart below, L is the white line and s is the green line. Several varieties of meantone are shown. Both quarter-comma and third-comma are very well approximated by edos. The fifth ranges all the way from 7-edo to 5-edo, even though meantone usually refers to a much narrower range of fifth size, perhaps half-comma to fifth-comma (about 691-698¢).

Figure 4.3.3 – Small intervals in various meantones



Chromatically, the octave contains seven diatonic semitones = Tsw2 = min 2nd and five chromatic ones = TLw1 = aug unison. When the wa slider is set to 700¢ (12-edo), they are equal. Moving the slider to the right makes the former narrower and the latter wider. When the fifth is just, the minor 2nd is smaller than the augmented unison by a wa comma LLw-2. With quarter-comma meantone, the minor 2nd is larger by the triple gu comma ggg2 = 128/125.

Consider the common melody over a V – I cadence, from the major 7th up to the tonic. In quarter-comma meantone, the semitone this melody spans is slightly widened from JI's 112¢ to 117¢. In third-comma meantone, the semitone is a very wide 126¢. This wide semitone makes the major 7th sound distressingly flat to modern ears. This flatness robs the cadential melody of its power.

When meantone first came into use, 1/3 or 2/7 comma meantone was common, with both the fifth and the major third flattened. Over the centuries, both the fifth and the major third have become sharper. The baseline has moved steadily clockwise to its present nearly-horizontal position. In terms of the parable: when Zarlino first imposed his corsets and harnesses on Tertia and Quintia, he made Tertia adapt more than Quintia. As time went by, Tertia balked at having to always crouch so low, and persuaded Duplius of further alterations that would allow her to stand straighter. The dress fitted Quintia less well, but as the junior wife, she had no choice.

Chapter 4.4 – Other Commas: Rank

The gu comma 81/80 is the most frequently pumped comma, due to its low odd limit and prime limit. But there are many other commas, and all the concepts in the last two chapters can be applied to them. Look at the lattice in Figure 2.3.2 and find two occurrences of the same letter. Also include in your search any two enharmonic equivalents like C# and D♭. Any two such notes will usually be less than 50¢ apart, and will imply a comma. Any chord progression that travels through the lattice from one note to the other implies tempering out that comma.

Some commas are easier to temper out than others. The smaller the better, and the more rungs it spans the better. At 22¢ and at least 4 rungs, the gu comma requires at most only 5.4¢ per rung of tempering. On the other hand, the rugu comma rg1 = 36/35 = 49¢ requires an enormous 25¢ per rung.

Let's look at the 3-limit commas first. The only one not too large or too remote to easily pump is the wa comma, LLw-2 = 3^{12} / 2^{19} = 24¢. This one is rarely pumped in actual chord progressions. The progression might go:

I – IV – ♭VII – ♭III – ♭VI – ♭II – ♭V=♯IV – VII – III – VI – II – V – I

Such a progression sounds to me more like a keyboard exercise than a song. However, the Yes song "Awaken" does pump it. Moving on to ya commas, there is the triple gu comma $g^3 2$ = 128/125 = 41¢. This one occasionally gets used, for example in Schubert's String Quartet #15, or in John Coltrane's "Giant Steps", or in the intro to The Doors' "Light My Fire":

wVIIy – IVy – gVIy – gIIy – [ggIV=yIII]y – yVIIy – Iy

As with the meantone comma pump, the equals sign is place roughly halfway through the pump. Here it's placed to avoid using double colors. Because the comma is a dim 2nd, not a unison, gg4 (a dim 4th) is the same note as y3 (a major 3rd). Brackets are used to indicate that the 5th chord is a yo chord with a root of either ggIV or yIII. This equivalence could also be written as ggIVy=yIIIy.

Another possibility is the yo minicomma Ly-2 = 2¢. This comma is so small that it can often be ignored, as a 2¢ tonic drift after eight chord changes isn't very noticeable. There's also the gugu comma sgg2 = 2048/2025 = 19¢. This comma implies a tritone of exactly 600¢, more on this in chapter 4.6.

The main yaza comma is the ru one, r1 = 64/63 = 27¢. This one arises from such progressions as I7 – ♭VII7 – IV7 – I7. Other yaza commas are the zozo comma, zz2 = 49/48 = 36¢, the double ruyo comma rryy-2 = 50/49 = 35¢, the ruyoyo minicomma ryy-2 = 225/224 = 7.7¢, and the deep purple microcomma pp1 = 2401/2400 = 0.7¢.

We can temper out several commas at once. We can link the zo slider to yo and wa if we temper out another comma that has either zo or ru in it. Tempering out both g1 and ryy-2 creates septimal meantone and relates the wa, yo and zo sliders to each other.

In alt-tuner, OK both g1 and ryy-2 to hear this linkage. When you move any of the first three tempering sliders, the other two respond. They move at different speeds, with the zo one being the jumpiest. But in a sense, they move as one, because the position of any one of them dictates the position of the other two.

What's the minimum discrepancy of this linkage? In quarter-comma meantone, yaza consonances like z3, zg5 and z7, and even r3, z11 and ry8 are all tempered less than g7 and w9, so the best yaza meantone tuning remains quarter-comma meantone.

As we saw earlier, the g1 linkage equates 5thwd wa with yo and 4thwd wa with gu. It also equates 4thwd zo with zogu and 5thwd ru with ruyo. The minicomma linkage creates even more equivalences: zo with yoyo, yo with zogu, and ruyo with gu:

Tz6 = Tyy5
Tz3 = Tyy2
Tz7 = Tyy6
Ty4 = Tzg5
Tg2 = Try1
Tg6 = Try5

All these equivalences cause the chain of fifths to cover most of the yaza lattice. Proceeding fifthward from 1/1:

w1 – Tw5 – Tw2=Ty2 – Ty6 – Ty3 – Ty7 – Ty4=Tzg5 – Tzg2 – Tzg6=Tz6 – Tz3 – Tz7 – Tz4 ...

Proceeding fourthward from 1/1:

w1 – Tw4 – Tw7=Tg7 – Tg3 – Tg6 – Tg2=Try1 – Try4 – Try7=Tr7 – Tr3 – Tr6 – Tr2 – Tr5 ...

If you can tolerate the mistuning, septimal meantone greatly opens up the harmonic lattice. In chapter 4.7 we'll see how to reduce the mistuning considerably.

The "invisible rung", aka the octave rung or the clear rung, can also be tempered. Alt-tuner won't let you temper this one independently of the other rungs. Instead it will stretch or compress all the rungs proportionally. Stretching the octave spreads all the notes out from each other uniformly, like raisins in a rising loaf of bread. This stretch slider defaults to a locked state for technical reasons involving midi, however you can unlock it by setting the output mode in prefs/misc to non-octave.

Recall from chapter 1.3 that the harmonic lattice has a certain number of dimensions, depending on what prime subgroup is used. The 3-limit lattice, which consists of just the wa row, has two dimensions, counting the hidden octave dimension. The ya lattice is three dimensional, and the yaza lattice is four dimensional. The za lattice uses the 2.3.7 subgroup, and is three dimensional.

Each comma that is tempered out reduces the number of dimensions of the lattice by one. For example, ya meantone is two-dimensional. All possible notes are found on a single chain of tempered fifths. If the clear rung were visible, that chain would be seen to be replicated in other octaves, creating a two-dimensional lattice. Both ya meantone and yaza septimal meantone temperaments are two-dimensional, and moving the stretch slider plus just one tempering slider will cover all possible variations of the temperament. The dimensionality of the lattice is called the **rank** by microtonalists. Meantone is two dimensional and hence is a rank-2 temperament.

The rank equals the number of primes minus the number of commas

Alt-tuner will tell you what rank your lattice is. Alt-tuner defaults to having six rungs (2.3.5.7.11.13), although not all of them may be used in the current scale. The messages on the linkages screen refer to all 6 rungs, but also refer to the lattice created by only the rungs that are linked. In other words, it assumes that the only rungs you're interested in are the ones that are linked. The clear rung is assumed to be of interest, even for linkages like yy1 = 25/24 that don't include the clear rung.

A primary color comma like r1 or g1 will only link two sliders, and tempering it out will reduce a three-rung lattice to a rank-2 temperament. However, a compound color comma like the ruyoyo minicomma usually links three sliders. If we temper out the minicomma, the four-dimensional yaza lattice becomes three-dimensional, and we get a rank-3 temperament. This type of linkage is looser than a rank-2 one. If you move the wa slider, the yo one will react. If you move the yo one, the zo one reacts. If you move the zo one, the wa one reacts. You must move at least two of the sliders to explore all the possibilities, as well as the stretch slider. Three sliders, hence rank-3.

If you temper out enough commas, the dimensionality gets reduced so much that the tempering sliders get locked into a single fixed position. This creates an edo. All edos are always rank-1, and the only variations possible are stretching or compressing them. For the rank-3 ya lattice, it only takes two commas to create an edo. Tempering out both g1 and LLw-2 makes 12-edo. Not surprising, as tempering out LLw-2 makes the wa row loop around on itself, creating the familiar circle of twelve fifths. For the za lattice, tempering out r1 and zz2 creates 5-edo. Alt-tuner displays a message letting you know which edo you're in. It takes three commas to reduce the full yaza lattice to an edo. In general, the smaller the commas, the larger the edo, and the closer to JI the edo is. If you temper out the minicommas Ly-1, ryy-2 and pp1, you get 41-edo.

Tempering out as many commas as there are rungs "crashes" the tuning. If you temper out any two wa commas, for example both sw2 = 256/243 and Lw1 = 2187/2048, you're asking that the 5th be both three-fifths of an octave and four-sevenths of one. Alt-tuner resolves this contradiction by setting both the wa and stretch sliders to zero cents. This is a rank-0 tuning, in which the entire tuning is condensed into one note, without even any octaves. Every key on the keyboard has the same exact pitch. Tempering out three ya commas or four yaza comma also creates rank-0. This 1-limit tuning, jokingly called the "OM temperament", is the simplest tuning possible.

Chapter 4.5 – Temperament Names Part I

Color notation provides an easy way to identify various temperaments. Meantone is called the gu temperament, after the gu comma it tempers out. Tempering out the ru comma creates the ru temperament. Let's review the commas we've named so far. Recall that minicommas are commas under 10¢. Here are some tables from chapters 2.2 and 3.2:

Table 2.2.1 – Commas

ratio	cents	quality & degree	name	
225/224	7.7¢	desc dim 2nd	ryy-2	the ruyoyo minicomma
81/80	22¢	perf unison	g1	the gu comma
$3^{12} / 2^{19}$	23¢	desc dim 2nd	LLw-2	the wa comma
64/63	27¢	perf unison	r1	the ru comma
50/49	35¢	desc dim 2nd	rryy-2	the double ruyo comma
49/48	36¢	min 2nd	zz2	the zozo comma
36/35	49¢	perf unison	rg1	the rugu comma

Table 3.2.4 – More commas

ratio	cents	name		quality	class	derivations	
$2^{-15}\cdot 3^{8}\cdot 5^{1}$	1.95¢	yo minicomma	Ly-2	desc dim 2nd	10	Ly-2 = LLw-2 - g1	
$2^{10}\cdot 3^{-6}\cdot 5^{1}\cdot 7^{-1}$	5.8¢	ruyo minicomma	sry1	perf unison	9	sry1 = r1 - g1 = ry1 - Lw1	
$2^{11}\cdot 3^{-4}\cdot 5^{-2}$	19.5¢	gugu comma	sgg2	dim 2nd	8	sgg2 = g1 - Ly-2 = r1 - ryy-2	
$2^{7}\cdot 5^{-3}$	41¢	triple gu comma	ggg2	dim 2nd	6	g1 + g1 + g1 - LLw-2 = rg1 - ryy-2	
$2^{-5}\cdot 3^{-1}\cdot 5^{-2}\cdot 7^{4}$	0.72¢	double zozogu microcomma	z^4gg3	double-dim 3rd	12	zz2 - rryy-2	aka the deep purple microcomma
$2^{-1}\cdot 3^{-7}\cdot 5^{4}\cdot 7^{1}$	0.40¢	zo quadyo microcomma	y^4z1	aug unison	13	zz2 + sry1 - ggg2	

We've been omitting the magnitude for brevity's sake: the wa comma is actually the double large wa comma, and the yo minicomma is actually the large yo minicomma. But when naming temperaments, the magnitude must be included. Thus tempering out the yo minicomma produces the large yo temperament.

Recall from chapter 3.2 that a ratio's magnitude is calculated from the prime exponents. Add up all the exponents except the first, divide by 7, and round off. 0 = central, 1 = large, -1 = small, 2 = double large, -2 = double small, etc. For example, 135/128 = (-7, 3, 1). 3 and 1 add up to 4, which divides to 4/7, which rounds off to 1, so 135/128 is large.

Any color refers to a lattice row, and any combination of a color and a magnitude refers to a seven-note segment of that row, and the smallest note in this seven-note segment is the one referred to by the temperament of that color and magnitude. For example, the yo temperament tempers out the smallest (least cents) interval in the central (neither large nor small) part of the yo row. These seven are y8, y5, y2, y6, y3, y7 and y4. The smallest of these is y2, thus the yo temperament tempers out 10/9, a rather unlikely temperament! There is no y1 interval, because y1 would be a descending interval 80/81. Half of all the seven-note segments are lacking a "1" interval. Some segments, like the small gu and small zo ones, are also lacking a 2nd, and have a 9th instead.

The temperament can be abbreviated by the magnitude and color, followed by a capital "T" for temperament. Whereas Tg1 means tempered 81/80, gT means the temperament in which Tg1 = 0¢. In the central wa 7-note segment, the smallest ratio is w1 = 1/1. But tempering out 1/1 makes no sense, so instead "wa temperament" is an alternate name for tempering out the pythagorean comma LLw-2. This avoids referring to this important temperament by the lengthy term "double large wa temperament". However, it is still written "LLwT".

Table 4.5.1 – Various single-comma temperaments

temperament name	shorthand	comma tempered out	conventional microtonal name
wa temperament	LLwT	LLw-2 = (-19, 12) = 23.5¢	"pythagorean"
yo temperament	yT	y2 = 10/9 = 182.4¢	
large yo temperament	LyT	Ly-2 = (-15, 8, 1) = 2.0¢	"schismatic"
gu temperament	gT	g1 = 81/80 = 21.5¢	"meantone"
zo temperament	zT	z2 = 28/27 = 63.0¢	"trienstonic"
ru temperament	rT	r1 = 64/63 = 27.3¢	"archy"

Historically, meantone has implied that the fifth is flattened by 1/5 to 1/3 comma. However, gu temperament has fewer connotations. The fifth can be flattened by a full comma, or can be perfectly just, or can be anywhere in between.

In the large gugu segment, the smallest interval is Lgg1 = 6561/6400 = (-8, 8, -2) = 43¢. But Lgg1 happens to be the sum of two gu commas. If a comma is a multiple of another comma, tempering it out is the same as tempering out the other comma. Thus there is no large gugu temperament, because it would be identical to the gu temperament. There is however a large gugu 2nd temperament, see Table 4.5.2.

Because 6561/6400 = $(81/80)^2$, it's called not the large gugu comma but the **squared** gu comma. Squared, **cubed**, etc., are general terms for intervals that break down into two or more identical ratios. Squared usually refers to commas, but it also includes larger intervals like the wa ninth, the yoyo aug 5th, the large wa 3rd, etc. Squared, cubed, etc., intervals can also be called doubled, tripled, etc., but this is potentially confusing, because the <u>tripled</u> gu comma = Lg^31 = 65¢ is different from the <u>triple</u> gu comma = g^32 = 41¢.

If the interval that is tempered out is not the smallest of the seven-note segment, the temperament name must include the degree as well as the color and magnitude. See the first two and the last two entries in the table below.

If the interval is narrower than sw2 = 256/243 = 90.2¢, the degree isn't needed. If it is wider than w2 = 9/8 = 203.9¢, the degree must be used. If it falls in between, add up all the prime exponents except for the first one. If that number mod 7 is 4 or 5, the degree is required. For example, 16/15 = (4, -1, -1), the sum is -2, and -2 mod 7 is 5, so the degree is required.

Table 4.5.2 – More single-comma temperaments

temperament name	shorthand	comma	conventional name
gu 2nd	g2T	g2 = 16/15 = 111.7¢	"father"
large yo unison	Ly1T	Ly1 = 135/128 = 92.2¢	"mavila" or "pelogic"
zogu	zgT	zg2 = 21/20 = 84.5¢	"septisemi"
small ruyo	sryT	sry1 = (10, -6, 1, -1) = 5.7¢	"hemifamity"
ruyoyo	ryyT	ryy-2 = 225/224 = 7.7¢	"marvel"
rugu	rgT	rg1 = 36/35 = 48.8¢	"mint"
large ru	LrT	Lr-2 = (-13, 10, 0, -1) = 50.7¢	"harrison"
yoyo	yyT	yy1 = 25/24 = 70.7¢	"dicot"
gugu	ggT	gg2 = 27/25 = 133.2¢	"bug"
small gugu	sggT	sgg2 =(11, -4, -2) = 19.6¢	"diaschismic" or "shrutal"
zozo	zzT	zz2 = 49/48 = 35.7¢	"semaphore"

double ruyo	rryyT	rryy-2 = 50/49 = 35.0¢	"pajara"
zo triple gu	zg^3T	zg^32 = 126/125 = 13.8¢	"starling"
triple yo	y^3T	y^31 = 250/243= 49.2¢	"porcupine"
triple gu	g^3T	g^32 = 128/125 = 41.1¢	"augmented"
large triple zo	Lz^3T	Lz^32 = (-10, 1, 0, 3) = 8.4¢	"slendric"
large triple ru	Lr^3T	Lr^3-3 = (-9, 11, 0, -3) = 15.0¢	"lee"
small quadyo	sy^4T	sy^41 = (5, -9, 4) = 27.7¢	"tetracot"
quadgu	g^4T	g^42 = (3, 4, -4) = 62.6¢	"dimipent"
large quintyo	Ly^5T	Ly^5-2 = (-10, -1, 5) = 29.6¢	"magic"
sixfold yo	y^6T	y^6-2 = (-6, -5, 6) = 8.1¢	"hanson" or "kleismic"
large sevenfold yo negative 2nd	Ly^7-2T	Ly^7-2 = (-13, -2, 7) = 100¢	
quintru negative 2nd	r^5-2T	r^5-2 = (11, 2, 0, -5) = 160¢	

It's possible but unlikely that a temperament name would contain a negative degree, as in the last two entries. Such commas are always remote and difficult to pump, and such temperaments usually have a large discrepancy. In general, the longer the temperament name, the less musically useful the temperament is.

We can glean a lot of information about the temperament directly from the name. The prime subgroup is obvious from the colors used. Wa = 3-limit, yo or gu = ya, zo or ru = za, and zogu, ruyo, etc. = yaza. Wa = rank-1, but yo, gu, ru or zo = rank-2, and compound colors = rank-3. More about this in chapter 4.8. In comparison, the conventional names, except for "augmented", have no readily apparent musical meaning.

As noted above, the wa comma's temperament, although written LLwT, is called wa temperament for short. Other wa temperaments are small wa temperament for sw2 and large wa temperament for Lw1. All other wa commas and temperaments are identified by the number of fifths they span, and thus the edo that all wa intervals are constrained to, see the table below. (A few unlikely exceptions: tempering out w2, w3, etc. would create the wa 2nd, wa 3rd, etc. temperament.) Thus LLwT can also be written w-12T, the "wa 12 temperament". Other rungs may or may not be used in these tunings; if they aren't, the tuning would be referred to not as a wa temperament but simply as an edo. This issue is addressed further in Table 4.8.5.

Table 4.5.3 – Various single-comma wa temperaments

temperament name	shorthand	comma	conventional name	implied edo
small wa	swT	sw2 = 256/243 = (8, -5)	"blackwood"	5-edo
large wa	LwT	Lw1 = (-11, 7)	"apotome"	7-edo
wa or wa-12	LLwT or w-12T	LLw-2 = (-19, 12)	"pythagorean"	12-edo
wa-19	w-19T	L^3w-2 = (-30, 19)		19-edo
wa-41	w-41T	s^6w5 = (65, -41)		41-edo
wa-53	w-53T	L^8w-6 = (-84, 53)	"mercator"	53-edo

All these temperaments temper out a single comma. Multiple-comma temperaments, like septimal meantone, are trickier to name. These temperaments, as well as those using unusual prime subgroups like 2.5.7 or 3.5.7, are covered in chapter 4.8, "Temperament Names Part II".

Chapter 4.6 – More Commas: Periods and Generators

Periods and generators were introduced back in chapter 1.2. All possible intervals are generated by adding or subtracting periods and generators to/from the tonic. Mathematically, the period is simply another generator. But the period has a special musical significance, because the tuning repeats periodically within it. The period is usually the octave, because the principle of octave equivalence is ingrained in most musical cultures.

In untempered JI, the period and generators are simply the lattice rungs. The period is the invisible rung w8 = 2/1, and the generators are the other rungs w5 = 3/2, y3 = 5/4, z7 = 7/4, etc. The sum of a generator and the period is also a generator. Many microtonalists use an alternative set of generators: w8, Ww5 = 3/1, WWy3 = 5/1, WWz7 = 7/1, etc. Furthermore, the octave inverse of a generator is also a generator; w4 could be used instead of w5. This is simply the same lattice rung, pointing in the opposite direction.

The sum or difference of any two generators is also a generator. Ya JI is usually thought of as generated by {w8, w5, y3}. But it could be generated by {w8, w5, g3}, or by {w5, w4, g3}, or even by {w3, w2, g2}. But adding a generator to itself won't make a generator. Thus double colors wouldn't work; {w8, w5, yy1} can't generate y3. Squared or cubed intervals wouldn't work; {w8, w9} can't generate w5, and neither could {w8, w2}. And of course the generators must be linearly independent; {w5, y3, g3} wouldn't work because w5 = y3 + g3.

In general, the preferred set of generators is the simplest, and JI is usually thought of as generated by {w8, w5, y3, z7...}. However, it can be useful to think of it as generated by {w8, w5, g1, r1...}. This is, after all, the basis of color notation: 3-limit intervals modified by commas.

Periods and generators are often named as JI intervals, even when discussing temperaments. It would be more precise to say {Tw8, Tw5} than {w8, w5}. Sometimes the generator is written as ~w5, with the tilde meaning approximate.

The total number of generators, including the period, equals the rank, which equals the number of primes minus the number of commas. Sometimes as commas are tempered out, the set of JI generators is simply reduced. For example, ya JI is generated by {w8, w5, g1}, and meantone is generated by {w8, w5}. But tempering out a double or triple comma tends to creates a non-rung generator. For example, y^3T has {w8, y2}. When this happens, the question arises whether the generator shoud be inverted, e.g. y2 could be g7. Also, an alternate generator can be found by adding or subtracting the comma(s) tempered out, e.g. y2 could be gg2. More on this later in the chapter.

Despite these issues, a rank-2 temperament's generator is fairly well defined. But a rank-3 temperament has two generators besides the period. There are many valid pairs of generators. Unless one pair happens to be a pair of JI rungs, the choice of generators is arbitrary.

Rank-1 tunings are edos and have no generators other than the period. The period is the edo-step. Mathematically speaking, the only essential interval in 12-ET is the semitone, because every 12-ET interval can be thought of as a stack of semitones. However, musically speaking, people often think of edos as special cases of rank-2 tunings. For example, 12-edo is generally thought of as meantone, with a period of an octave and a generator of a fifth.

The **ru temperament** rT tempers out the ru comma r1 = 64/63 = 27¢. This allows chord progressions like Ih7 – zVIIgr – IVh7 – Ih7 or Ih7 – IVh7 – Vh7. The zo 7th equals two wa 4ths. Like meantone, there are different versions, described by the fraction of a comma that the fifth is tempered by. Unlike meantone, the fifth is sharpened, not flattened. The zo 7th is equated to two flattened fourths. The ru comma is not much larger in cents than the gu one, but it spans only three rungs, so the amount of mistuning created is higher.

Alt-tuner defaults to half-comma ru temperament, in which the fifth is sharpened by 14¢ and the zo seventh is just. As you move the wa slider towards the center, Tz7 is sharpened, and the zo slider moves to the right. The closest to center that the two sliders can be is third-comma ru temperament, with Tw5 and Tz7 both 9¢ sharp. Third-comma because the ru comma spans three rung-steps. Because both sliders are equally sharp, z3 and r6 are just, and the just baseline passes through these two ratios.

In chapter 4.3 we talked about the overall minimum discrepancy for meantone, looking at all intervals likely to be used in chords, such as g3, y3, w5, g7, y7, w9 and their octave inverses. We can do the same for the ru temperament, expanding our list to include z3, r3, zg5 and z7. Assuming the yo rung is untempered, we can determine which type of ru temperament is best, that is, which has the least overall minimum discrepancy. The answer is quarter-comma ru temperament, which has a maximum discrepancy of half a comma = 13.6¢, which is what both Tz7 and Tw9 are sharp by. Its just baseline passes through z6 and r3. If w9 and g7 are excluded from the list of intervals, the best ru temperament is third-comma ru temperament, with a maximum discrepancy of 9.1¢. This is slightly better than quarter-comma meantone's least overall minimum discrepancy of 10.8¢.

If we assume a za tuning, with no yo or gu intervals, ru temperament becomes a rank-2 tuning. Like all rank-2 tunings generated by the fifth, it can be analyzed melodically with Figure 4.3.3. Because the ru temperament's fifth is sharp, the chromatic semitone is always larger than the minor 2nd. In other words, unlike meantone, C♯ is sharper than D♭. For quarter-comma ru temperament, the difference between them is the triple ru comma r^3-2 = 729/686 = 105¢. This makes C – D♭= 56¢ and C – C♯ = 161¢. The latter is about three times as large. If it were exactly three times as large, we would have an edo. Which edo? The major 2nd C – D interval would be four times the size of the minor 2nd. The octave contains five major 2nds and two minor 2nds. One M2 equals four m2s, thus five M2s equals twenty m2s, thus one octave equals 22 m2s. Thus quarter-comma ru temperament is closely approximated by 22-edo. Third-comma ru temperament has semitones of 45¢ and 177¢, the latter about four times as large, and is approximated by 27-edo. Ru temperament's tiny minor 2nd is very distinctive melodically.

Ru temperament with an untempered yo rung creates a rank-3 tuning, with all ya commas still present, but altered. All fifthward commas on the right of the lattice, like gu and wa, are sharpened. All fourthward ones like the gugu one are flattened. **Three-less** (no wa rungs, in other words, the 3-exponent is zero) ya commas like triple gu are unchanged.

We can temper out an additional comma to reduce the rank to 2. Because the gu comma is on the fifthward side of the lattice, meantone flattens the fifth. But the ru comma is fourthward, and the ru temperament sharpens the fifth. As a result, tempering out both the gu and ru commas doesn't work very well. The best tuning leaves the fifth little changed, with the yo and zo rungs bearing the brunt of the tempering, with each one about 25¢ sharp.

Instead of the gu comma, let's try tempering out the ruyoyo minicomma ryy-2 = 225/224 with the ru comma. Even though the minicomma is also fifthward, its smaller size makes this temperament much more accurate. The minimum rung discrepancy remains 1/3 of a ru comma, about 9¢. This temperament makes possible not only those chord progressions that pump the ru comma but also those that pump the minicomma, like Ih7 – yIIIs6 – ryI=gIIh7 – IVh7 – Ih7.

Exploring this temperament by moving alt-tuner's sliders, something unusual happens: the yo and zo sliders move back and forth as one. Sharpening one by 5¢ sharpens the other one as well by exactly 5¢. The minicomma equates the zogu 5th = 7/5 with the yo 4th = 45/32. The ru comma equates the yo 4th with the ruyo 4th = 10/7. Thus Tzg5 = Try4. Since zg5 is the octave inverse of ry4, it follows that Tzg5 is exactly half an octave. Assuming untempered octaves, Tzg5 is locked at exactly 600¢. The wa, yo and zo sliders aren't locked, but no matter how you move them, Tz7 is always 600¢ larger than Ty3. Why did this happen? Because this temperament involves squared intervals.

Tempering out any two commas also tempers out any combination of them. The sum of the ru comma and the minicomma is the double ruyo comma rryy-2 = 50/49. While rryy-2 is not a squared interval, the <u>wide</u> double ruyo comma Wrryy-2 = rryy7 = 100/49 is, because it equals $(10/7)^2$ = two ruyo fourths. Tempering out rryy-2 equates Trryy7 to the octave, thus Try4 equals half an octave. As does its octave inverse, Tzg5.

This same phenomenon can happen in ya temperaments too. The gugu comma sgg2 = 2048/2025 creates the small gugu temperament sggT. Widened, Wsgg2 = sgg9 = 4096/2025 = $(64/45)^2$ = two gu fifths = one octave. Thus both Tg5 and its inverse Ty4 are always exactly half an octave.

The period is usually the same as the **interval of equivalence**, which is almost always the octave. But in sggT, the tuning repeats periodically every half-octave, and the period is the half-octave, equivalent to the tempered gu 5th or the tempered yo 4th. The period of rryyT is the Tzg5 or the Try4. The period is always the interval of equivalence or some fraction of it, a half, a third, etc. Such periods are **fractional periods**, because they are some fraction of the traditional period, the octave. A temperament that creates a fractional period is said to **split** the octave. The color of the period always matches the color of the comma, or its compliment: the double <u>ruyo</u> comma splits the octave into two <u>ruyo</u> 4ths, or two <u>zogu</u> 5ths.

If a line drawn in the lattice from the center to the comma passes through another note or notes, the comma splits the octave, and one of the other notes is the period. For example, a line drawn to the gugu comma passes through the gu

5th. This rule doesn't apply if the comma tempered out is simply another comma squared.

The **color depth** of a comma is the GCD of all the exponents in the monzo except the first two. A depth of one makes it single, a depth of 2 is double, etc. Any comma which isn't single is **deep**. A comma's color depth can be found directly from its name. For a comma to be double, all its colors must be double. For example, both yy1 and rryy-2 is double. But ryy-2 is single, even though it has yoyo in its name.

The mathematical requirement for a half-octave period is that all the comma's rung factors be even numbers, except for the octave factor. The color name somewhat indicate this. If a color's factor is even, that color will be double (or quadruple, or sixfold, or some multiple of 2). But neither the wa factor nor the octave factor is directly indicated by the color name. So there is only a loose correlation between the color name and a fractional period. Tempering out a double comma may or may not produce a half-octave period, but tempering out a non-double comma never will. There's one exception: tempering out a wa comma always splits the octave. This follows from the line-drawing rule.

Likewise, tempering out a triple comma may or may not produce a third-octave period. Tempering out the triple gu comma produces g^3T, with a a third-octave period of the yo 3rd. For a comma to be triple, all its colors must be triple (or sixfold, or ninefold, etc.).

If a comma is double, and it doesn't split the octave in half, it will split either the wa 4th or the wa 5th in half. If it's a rank-2 temperament, i.e. if the comma is primary-color, this smaller interval will be the generator. It's a **fractional generator**, in the sense that it's a fraction of the traditional generator, the wa 4th or 5th. For example, tempering out the yoyo semitone yy1 = 71¢ splits the 5th into two yo 3rds. Subtracting the double comma from the fractional generator produces an alternate generator with the complimentary color, in this case the gu 3rd.

A double, triple, etc. comma splits either the octave or some voicing of some other wa interval (or rarely, both). The former creates a fractional period, and the latter creates a fractional generator. This fractional interval's color will either match the color of the comma, or be its compliment. See the zozo temperament below for an example of a fractional generator.

A triple comma splits one of these intervals into three parts: the octave, the 4th, the 5th, or the wide 5th = 3/1. For example, the triple gu comma splits the octave into three yo 3rds, and the triple yo comma splits the 4th into three yo 2nds. The small triple lu comma 4096/3993 splits Ww5 into three lu 5ths. The fractional generator is 1u5, or its octave inverse 1o4, which splits Ww4.

A quadruple comma usually splits the octave or the re-voicied fifth into four parts, a quintuple one five parts, etc. But there are exceptions. Multi-comma temperaments can cause both the period and the generator to be fractional. A double comma and a triple comma may split the octave or the re-voicied 5th into six parts, or it may split one into halves and the other into three parts. Fractional periods and generators are used for notating rank-2 tempers, see chapter 5.16, "Notating Rank-2 Tunings, Part I: Triple Yo".

Even when a temperament doesn't split the octave or the fifth, it will split some other interval into a stack of another interval. For example, meantone splits the yo 3rd into two wa 2nds: y3 = 2 · w2. Splits suggests melodic possibilities, assuming that equally-sized melodic steps are desirable for melody. More meantone splits: y3 = 2 · y2 and g7 = 2 · w4. These splits can be derived from the first one, so we only need to know one split. Which split best describes the temperament? Preferably the two intervals will have low prime limits, low odd limits, and reasonably small sizes. Preferably one of the intervals will be wa and the other a primary color. For meantone, the best split is y3 = 2 · w2. Multiple-comma temperaments have multiple splits. More examples:

Table 4.6.1 – Various temperaments and their descriptive splits

comma(s) tempered out	descriptive splits	comments
g1 = 81/80	y3 = 2 · w2	
ryy-2 = 225/224	r2 = 2 · g2	rank-3
g1 and ryy-2	y3 = 2 · w2 and z6 = 4 · w2	thus z6 = 2 · y3
r1 = 64/63	r3 = 2 · w2	
r1 and ryy-2	w2 = 2 · g2 and r3 = 2 · w2	thus r3 = 4 · g2
sgg2 = (11, -4, -2)	w2 = 2 · g2	also splits the octave, w8 = 2 · g5

LLw-2 = (-19, 12)	w8 = 12 · sw2	rank-1, splits the octave into an edo

The **zozo temperament** zzT tempers out zz2 = 49/48 = 36¢. This is a big comma spanning only a few rungs, so the tuning isn't very accurate. It equates r2 and z3, and makes them both equal to half a fourth. Thus w4 = 2 · z3. The note that represents Tr2 and Tz3 is best tuned to be midway between them, to minimize the discrepancy for both z3 and z7. The overall minimum discrepancy for the zozo temperament is half a comma = 18¢, from the "zero-comma" temperament with a just fifth. In this tuning, all wa intervals are just, all zo intervals are a half-comma flat, and all ru intervals are a half-comma sharp. Example comma pump: Iz7 – zIIIz7 – [zVII=rVI]z7 – Iz7. Melodically, the major 2nd and minor 3rd fuse to create an interval of about 250¢, reminiscent of 5-edo. In the pentatonic notation introduced in chapter 5.3, this would be a neutral subthird.

The interval from the tonic to the note representing both Tr2 and Tz3 is the (fractional) generator of the temperament. Unlike all the other temperaments we've looked at so far, the fifth is not a generator. No combination of fifths and octaves will generate the Tr2/Tz3 interval. However, one octave minus two such intervals will generate the fifth.

The octave inverse of a generator is also a generator. Thus Tr6 and Tz7 are alternate generators of zzT. However, octave inverses aren't usually listed as generators. For example, meantone is thought of as generated by fifths, not fourths. The general rule for microtonalists is to choose the smaller of the two possible generators. However, there is a strong historical precedence for choosing the fifth over the fourth, so this is an exception to the rule.

When the fifth isn't a generator, it affects how the temperament is notated. Conventional notation is based on a chain of fifths, modified with sharps and flats. This notation works well with any rank-2 temperament that is generated by the 5th. But to notate the zozo temperament, other accidentals are needed. Also, unlike all the other temperaments so far, the zozo temperament's comma has a nonzero keyspan, because it's a minor 2nd. We've seen a nonzero stepspan comma with the triple gu comma. Notating a comma pump when the comma has a nonzero stepspan requires an enharmonic adjustment midway through. An example of this is the intro/outro of "Light My Fire" by the Doors. But nonzero keyspans are even harder to notate, because D is enharmonically equivalent to E♭. More on this in Part V.

What additional comma can we temper out along with the zozo one? Tempering out the ru comma creates 5-edo, because both commas are za, and a tuning with two commas and only three rungs is rank-1. Tempering out the gu comma creates a large discrepancy. It mistunes zogu and ruyo intervals by about 40¢! Tempering out zozo and the minicomma also creates a large discrepancy. However, either the zo triple gu comma or the small gugu comma would work.

The double ruyo temperament rryyT tempers out rryy-2 = 50/49 = 35¢. This comma is three-less, so the fifth can remain just. It flattens the y3 and/or sharpens the z7. It splits the octave, and also splits Tw2 into two Try1. It equates Tzg5 and Try4, and makes them both equal to half an octave. The discrepancy for Tzg5 and Try4 is always exactly half a comma. The best tuning for rryyT is quarter-comma, making an overall minimum discrepancy of half a comma = 17¢. In this tuning, all wa intervals are just, all yo and ru intervals are a quarter-comma flat, and all gu and zo intervals are a quarter-comma sharp. All zogu intervals are a half-comma sharp, and all ruyo intervals are a half-comma flat. Example chord progression: Ih7 – Is7 – zgV=ryIVh7 – ryIVs7 – Ih7.

Another chord progression requiring tempering out rryy-2 is the tritone swap: I7 – ♯IV7 (or I7 – ♭V7). In C, this would be C7 – F♯7. This is a comma warp, so rather than avoiding tonic drift, we're avoiding pitch shifts. The E and B♭ of the C chord must be the same as the E and A♯ of the F♯ chord. Taking the dom7 chords as h7 chords, we have Ch7 – ryF♯[zzg4=y3]z7. In the other comma pumps, one chord root changes color and/or degree. Instead, here the 3rd of the 2nd chord changes color from purple to yo. However, in quarter-comma rryyT, the purple microcomma is actually about 55¢, and zozogu and ruruyo are no longer almost identical. Purple really only makes sense in untempered JI, so we refer to the purple 3rd by its proper name, the zzg4. The notation ryIV[zzg4=y3]z7 indicates that the zzg4 of this chord, which is the z7 of the I chord, is equated with the y3. This chord could also be written as ryIV[(zzg4)z7=h7] or as ryIV(zzg4)z7=ryIVh7. The root movement could also be written as by a dim 5th, such as Ih7 – zgVy[ryy6=z7] or Ih7 – zgV[y,ryy6=h7].

Tempering out both the zozo and double ruyo commas creates a very inaccurate temperament, because these two commas add up to the yoyo semitone yy1 = 71¢, which equates the gu 3rd with the yo 3rd. The tuning closest to JI is the third-semitone temperament. But this is so far from just that many prefer to keep the 5th just and flatten the yo 3rd down to a neutral 3rd of 351¢. This puts the zo 7th at 951¢, halfway between a major 6th and a minor 7th.

This temperament tempers out both the zozo and yoyo commas. The former splits the 4th and the latter splits the 5th, and together they split the octave! The generator is the yo rung, Ty3 = 5/4. For any temperament, alternate generators can be found by adding or subtracting any comma. Subtracting yy1, we find Tg3 = 6/5 is also a generator.

A few paragraphs earlier I said that the octave inverse of a generator is also a generator. It's more accurate to say the period inverse. In this case the period is a half-octave equivalent to Tzg5 or Try4, so alternate generators are Try4 - Ty3 = Tr2 and Tzg5 - Tg3 = Tz3.

Furthermore, alternate generators can be created by adding periods on. For example, any generator plus an octave is still a generator. For this temperament, this gives alternate generators of Tg3 + Try4 = Tr6 and Ty3 + Tzg5 = Tz7, as well as Tz3 + Try4 = Ty6 and Tr2 + Tzg5 = Tg6.

Out of all these possibilities, microtonalists usually choose the smallest interval, here Tr2, as the generator. Alt-tuner lists all the generators, excluding octave inverses, in order from lowest odd limit to highest: Ty3, Tg3, Tz7 and Tr6.

For more about periods and generators, see chapter 4.10. For notating these tempers, see chapter 5.16.

Chapter 4.7 – Adaptive Tuning With Alt-Tuner *

(<u>Very</u> rough draft of an unfinished chapter!)

As we saw in chapter 4.2, when confronted with a comma pump, a choir or a string quartet uses adaptive tuning, in which the notes of the scale are shifted slightly sharp or flat to keep the tonic from drifting. Vocalists do this solely by pitch memory of the tonic. Violinists and cellists are also aided by their open strings.

Fixed pitch instruments like the keyboard have lacked the ability to do this, until the advent of alt-tuner. Alt-tuner distinguishes between <u>modulating</u> intervals, which are always fully tempered, and <u>sounding</u> intervals, which are tempered fully, partially, or not at all according to the tempering strength slider. For example, suppose you want to play C – Am – Dm – G – C. Let's look at six ways to do this: with a wolf chord, with a comma shift, with a tonic drift, with tempering, with adaptive JI, and with adaptive tuning.

<u>Wolf method</u>: our progression is Cy – yAg – Dw,y5 – Gy – Cy. The D chord sounds awful!

4 screenshots, one for each chord

<u>Comma shift method</u>: our progression is Cy – yAg – yDy – Gy – Cy. The D note shifts awkwardly on the G chord.

4 screenshots, one for each chord

<u>Tonic drift method</u>: Cy – yAg – yDg – yGy – yCy. We avoid the wolf by modulating on each chord change, so that the root of the current chord is always the center note of the lattice. We end up on the yo C, causing the pitch to drift flat.

C chord, A chord screenshots

This A is yo, even though it appears wa in the lattice, because it's a yo 6th from the wa C we started on.

3 screenshots. Annotate last one "we drifted flat" by A-440 -22¢

<u>Tempering method</u>: Cy – yAg – y=wDg – Gy – Cy. First temper out the gu comma, then adjust your tuning to quarter-comma or fifth-comma or whatever. No modulating required. There will be no wolves or drifts, but the chords will be mistuned. The 5ths and minor 3rds will be too flat, and the 4ths and major 6ths will be too sharp.

4 screenshots, annotate first one "tempered fifth" by wa slider

<u>Adaptive JI method</u>: In adaptive JI, we modulate by tempered intervals, but the scale at any moment is made up of just intervals. Move the tempering strength slider from 100% down to 0%, and modulate on each chord change as before. All chords will be in tune, but there will be pitch shifts. We'll assume quarter-comma tuning for now. Because the tempering strength is 0%, all sounding intervals are just. The first chord Cy is justly tuned.

screenshot

Modulate to A before playing yAg. In the tonic drift method, we modulated by a just interval y6. Here we modulate by a tempered interval, Ty6 = y6 + 5.4¢. The cents offset is one quarter-comma = 5.4¢ sharper than in the tonic drift method. This causes all notes common to both lattices (B♭, F♯, and the 7 natural notes) to shift 5.4¢ sharper. Since A, C and E all shift equally, the Ag chord is still justly tuned.

screenshot, Am chord, annotate cents offset with an arrow

Modulate to D before playing y=wDg. We're modulating by Tw4 = w4 + 5.4¢. Again, all notes common to both lattices (all notes except D♯, which becomes E♭) shift up 5.4¢. Again, D, F and A all shift equally, and Dg is in tune.

screenshot

Modulate a Tw4 to G before playing Gy.

screenshot

Finally, modulate a Tw4 to C and play Cy. The cents offset returns to zero.

screenshot

Adaptive tuning method: Tempering allows mistunings but not pitch shifts. Adaptive JI allows pitch shifts but not mistunings. Adaptive tuning allows both, minimizing both. Adaptive tuning is done by setting the tempering strength greater than 0% (no tempering) and less than 100% (no shifts). We modulate as before. The lattice looks the same as with the adaptive JI method.

In adaptive tuning, we modulate by tempered intervals, but each chord is made up of **detempered** intervals. For quarter-comma meantone at 33% strength, the just fifth is 702.0¢, the tempered fifth = Tw5 is noticeably off at 696.6¢ (flattened by 1/4 comma), and the detempered fifth or sounding fifth is a very acceptable 700.2¢ (flattened by 33% of 1/4 = 1/12 comma = only 1.8¢). Various intervals at various strengths, with the mistuning shown as a fraction of a comma:

Table 4.7.1 – Various detempered intervals in adaptive quarter-comma meantone

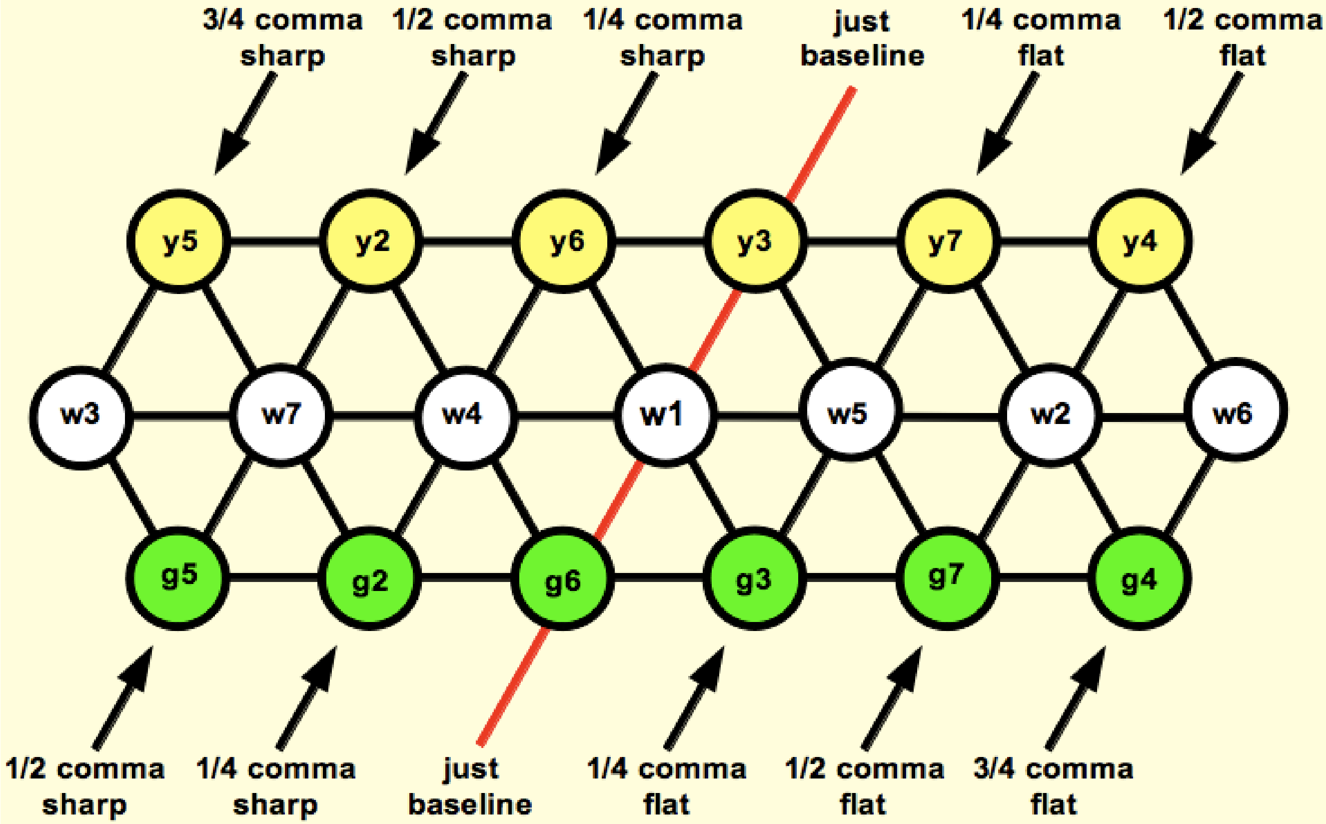
strength:	0%	25%	33%	50%	100%
g3	just	-1/16	-1/12	-1/8	-1/4
y3	just	just	just	just	just
w4	just	+1/16	+1/12	+1/8	+1/4
w5	just	-1/16	-1/12	-1/8	-1/4
y6	just	+1/16	+1/12	+1/8	+1/4
w7	just	+1/8	+1/6	+1/4	+1/2
g7	just	-1/8	-1/6	-1/4	-1/2
w9	just	-1/8	-1/6	-1/4	-1/2

What strength is best? Lower settings favor harmonies and higher settings favor melodies. The JND (just noticeable difference, with "just" meaning "barely", not "well-tuned") is the smallest perceptible change of something. For melodic pitch, it's about 6¢. But for harmonic pitch, it's much less, as little as a fraction of a cent. Ask a tamboura player about this! However, many find a slight tempering of 1-2¢ to be quite pleasant. It adds a very slow vibrato. My personal rule of thumb is for the shift to be about twice as large as the tempering. One might think a 33% strength would ensure this. As we'll see, that's not always the case.

Which version of meantone (quarter-comma, third-comma, etc.) should be used? It depends on the chord progression. Remember, some progressions sound fine in JI. For example, I – V – VIm – IV is best tuned justly. I – IV – V7 – I can be tuned justly, as Iy – IVy – Vy,w7 – Iy. The w7 in the V chord provides the necessary tension to drive the cadence, as noted in earlier chapters. See the discussion near figures 1.3.10 and 2.2.5.

Let's review figure 4.3.1, which shows deviations from just for quarter-comma meantone:

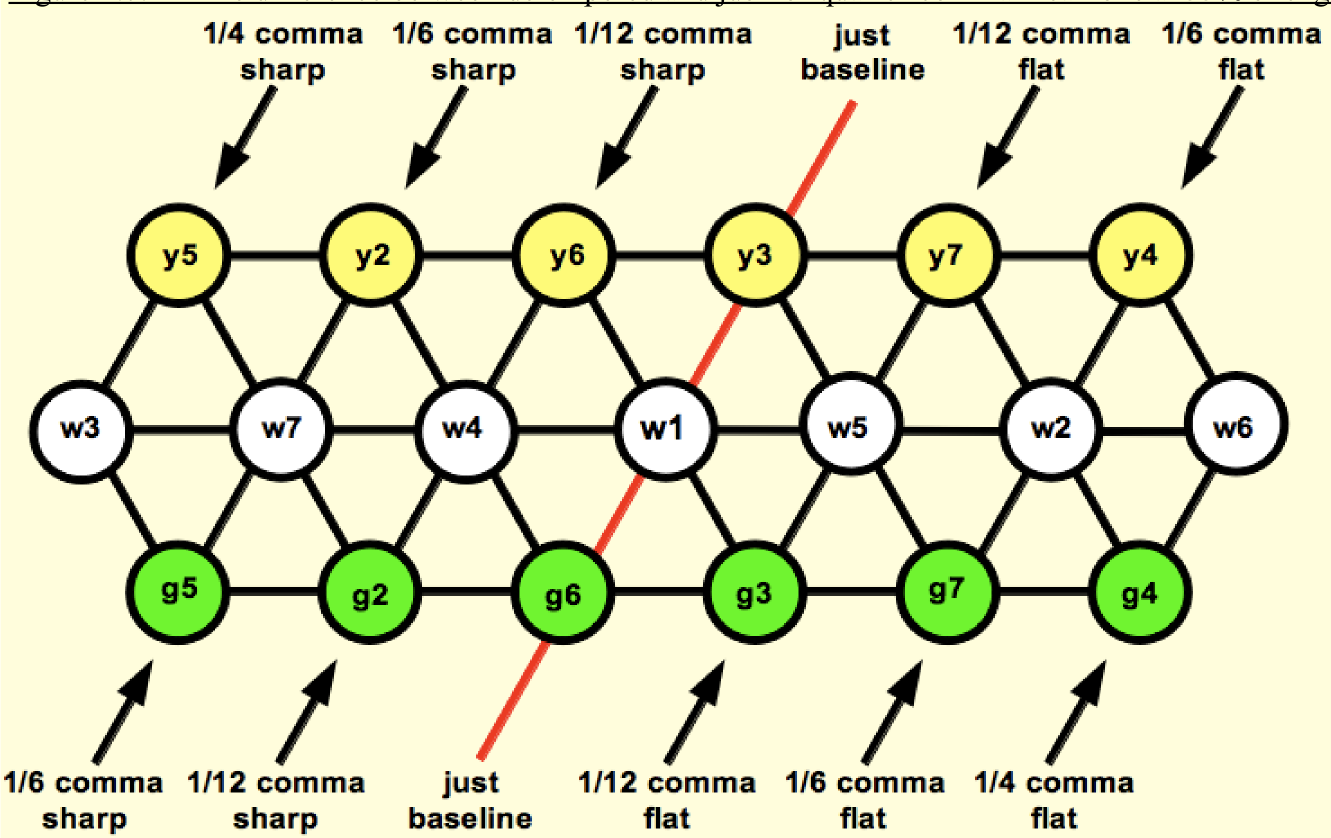
Figure 4.7.1 – The difference between tempered and just for quarter-comma meantone



For our example progression C – Am – Dm – G – C , this is the best version of meantone. It spreads the shifting equally among the four chord changes, so that each chord change causes a quarter-comma shift. The root movements are all by major 6ths and perfect 4ths, both of which correspond to a quarter comma in the chart above.

The chart above shows the tempered intervals that we modulate by. The next chart shows the detempered intervals that are used to construct chords, assuming 33% strength.

Figure 4.7.2 – The difference between detempered and just for quarter-comma meantone at 33% strength

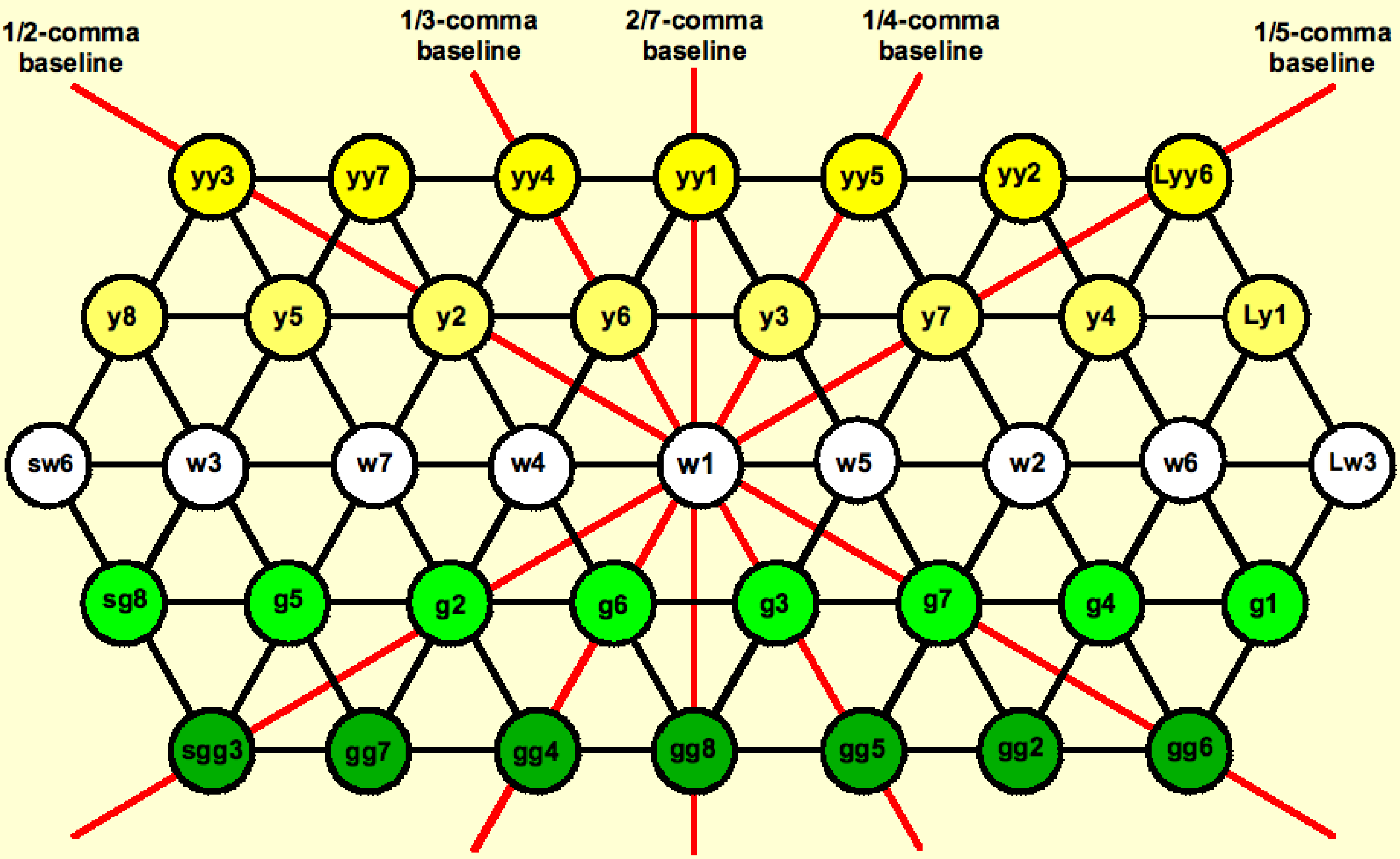


The chords used are all major and minor, and the intervals from the root (g3, y3 and w5) all lie on either the just baseline or the 1/12-comma flat line. The maximum tempering is therefore 1/12 comma. The modulations are all by either y6 or w4. The intervals of modulation are tempered, not detempered, so we use figure 4.7.1. Both y6 and w4 lie on the 1/4-comma sharp line, which seems to indicate a 1/4-comma shift. But the shift is actually the difference between the current note (at a detempered interval) and the future note (at a fully tempered interval). The shift is therefore 1/4 comma minus 1/12 comma, which is 1/6 comma. Thus for 33%, the shift is twice the tempering, following the rule of thumb.

To find the best tuning for a song, first determine the chords, considering prominent melody notes as well. For example, a sustained A note over a C major chord creates what is in effect a C6 chord. Then determine the JI interpretation, including any commas pumped.

To find the best version of meantone for any chord progression, consult Figure 4.3.2, reproduced below. For any red baseline, picture a series of parallel lines, each running through one of the wa notes. Each chord change should move the root to the next parallel line. Often but not always, for N chord changes, the best tuning is 1/N-comma.

Figure 4.3.2 – The just baseline for various meantones



Consider C – Dm7 – G7 – C, tuned as Cy – yDg7 – y=wGy,g7 – Cy. Because there are three chord changes, an obvious choice is third-comma meantone. The third-comma baseline runs y6 – w1 – g3. The parallel lines run y3 – w5 – g7, y2 – w4 – g6, y5 – w7 – g2, etc. In this example, the chord roots are w1, y2, y5=w5 and w1. Each chord change does in fact move the root to the next line, and third-comma meantone is the best tuning. If quarter-comma were used, there would be a large half-comma shift between the C and D chords.

The best strength is ...

The root of the 2nd chord is y2, not w2, because the JI interpretation requires this to avoid a pitch shift. For the progression C(9) – Dm – G7 – C, the JI interpretation is Cy,w9 – Dg – Gy,g7 – Cy, and the roots are w1, w2, w5 and w1. The gu comma is not actually pumped, and the chords can be played in JI. However, the g4 may sound odd in the melody. ...

Consider C(9) – F6. Tuned as Cy,w9 – Fy6, this creates a comma warp. The best tuning is half-comma meantone. At 0% strength, both the wC and the wD shift by half a comma = 11¢. They shift towards each other to narrow the w2 to a y2. At 100%, C is fixed at wC and D is fixed at yD. At S%, ...

But at 100%, each chord's w5 is 11¢ flat, and each y3 is 22¢ flat. At 20%, the tempering is half of the shifting. The 5ths are 1/10 comma = 2.2¢ flat and the major 3rds are 1/5 comma = 4.4¢ flat, and both the C and the D shift by 2/5 comma = 8.7¢.

The melody over C – Am – Dm – G – C might be D-D-D-D-C. The progression becomes in effect C(9) – Am(11) – Dm – G – C. The first two chords cause a comma warp. Because the Am(11) chord is sharpened by 1/4 comma, the shift is only 3/4 of a comma, but this is still noticeable. If the tempering strength is 33%, this is further reduced to a half-comma. Better to tune this song with half-comma meantone at 20%, so that the shift is only 8.7¢.

Consider Am – Dm – G – CM7 – FM7 – Bm7(♭5) – Esus4 – E7 – Am ("I Will Survive"). The JI chords are Ag – Dg – w=gGy – gCy7 – gFy7 – y=wBg7(gg5) – E4 – Ey,w7 – Ag. This progression pumps <u>two</u> commas in <u>seven</u> chord changes (counting both E chords as one), so an obvious choice, and in fact the best tuning, is 2/7-comma. If quarter-comma were used, there would be a large half-comma shift from the F chord to the B chord, not as good.

Best strength?

Could switch at F to a custom tuning centered on E, and use it for the B and the 1st E chord, then switch back?

Now consider C – Dm – F – G – C ("Young Americans", David Bowie). This is a broken comma pump, because there are no common notes between the C and Dm chords, or between the F and G chords. One possible tuning strategy uses untempered JI with a comma shift for the D note: Cy – yDg – Fy – Gy – Cy. The upward shift will not be very noticeable, because there is a full measure of an F chord between the yD and the wD. Another possible tuning is Cy – Dg – gFy – Gy – Cy. The gFy chord has a gC, so the C shifts instead of the D. The pitch shift can be lessened with adaptive tuning. The root of the Dm chord is tuned midway between yD and wD, notated as either y=wDg or w=yDg. The root of the F chord is the 3rd of the Dm chord.

Cy – y=wDg – w=gFy – Gy – Cy

In this case, it's debatable if this is an improvement. But it would be if the chord changes were quicker, or if the melody over the F chord used D prominently (making the F chord in effect F6). *However if the G chord has a 7th, The root movements by a major 2nd are best tuned with third-comma: Cy – yDg – Fy – y=wGy,w7 – Cy.*

Meantone: if in C, stay in the "valley" created by the black key "mountain range". If you cross the range, the tonic drifts by a triple gu comma. A problem if there's root movement by two major 3rds, in effect pumping the triple gu comma. Or if there's root movement by a tritone, as in Bruno Mars' "When I Was Your Man". The solution here is to have the correct tritone selected, y4 vs. g5.

Move the rest of the chapter to a new chapter? Make it an appendix, since it's so technical?

In theory, adaptive meantone (meantone using either adaptive JI or adaptive tuning) is a rank-5 tuning. In practice, it's usually a rank-3 tuning. In theory, it's generated by the tempered octave and 5th, and the detempered octave, detempered 5th and detempered yo 3rd. In practice, the octave is usually not stretched, so both the tempered and the detempered octave are just. The detempered octave is not needed as a generator, and the temperament becomes rank-4. In practice, the 5th is usually tempered by some simple fraction of the comma (1/4, 1/3, etc.), which makes whichever yo interval lies on the baseline just. This yo interval is the same detempered as tempered, and is therefore generated by some number of tempered fifths. The detempered yo third is some number of detempered fifths from this interval, and is therefore generated by some combination of tempered and detempered fifths. The detempered third is not needed as a generator, and the temperament is rank-3.

For example, unstretched adaptive QC meantone is generated by the just octave, the tempered 5th, and the detempered 5th. The major 3rd, tempered or detempered, is just. It's generated by four tempered 5ths minus two octaves.

In any rank-3 tuning, the sum or difference of the two generators can replace either generator. For example, ya JI is usually thought of as generated by the octave, 5th and major 3rd. It could instead be thought of as generated by the octave, 5th and <u>minor</u> 3rd. Or the octave, major 3rd and minor 3rd. Which generators are chosen is merely a matter of convenience. For adaptive meantone, it's more convenient to replace the detempered 5th with the difference between the tempered 5th and the detempered 5th. The difference will be some small fraction of the gu comma. This fraction is the **wa shift**, the amount of pitch shift caused by moving in the lattice by one fifth. (Moving by other intervals creates other pitch shifts.) The wa shift depends on the type of meantone and the strength slider. For 1/4-comma meantone at 0%, it's a descending 1/4 comma. At 50%, it's 1/8 comma. At 75%, it's 1/16 comma. At 100%, the tuning is no longer adaptive, the wa shift is zero, and the rank changes from 3 to 2. The formula is:

wa shift = tempered wa 5th - detempered wa 5th = T · (1 - S)

where T is the amount of tempering (here w5 is flattened by a quarter-comma, so T is -5.4¢) and S is the strength setting as a decimal (e.g. 25% = 0.25).

Unstretched, adaptive third-comma meantone is also generated by the just octave, the tempered 5th, and the wa shift. The yo 6th, tempered or detempered, is just. It's generated by three tempered fifths, octave-reduced. The detempered yo 3rd is generated by four tempered 5ths plus a wa shift, octave-reduced.

For 2/7-comma meantone, the baseline passes through the yoyo semitone yy1. The generator is half of the wa shift. The detempered fifth equals the tempered fifth plus two half-shifts. The detempered yo 3rd equals four tempered fifths plus a half-shift. For 3/8-comma meantone, the generator is a third-shift, and so forth.

If the fifth is just, the wa shift is zero. The third generator is the yo shift, which is the difference between the tempered and detempered yo 3rds. The detempered y3 equals four tempered w5's plus the yo shift.

For example, suppose you have quarter-comma meantone, with strength 33%. The three generators are the octave, the tempered fifth of 696.6¢, and the wa shift of 1/6 comma = 3.6¢.

Alt-tuner has a display which indicates how much the fifth is tempered. What if your tempering is not quite an exact fraction of a comma? For example, what if alt-tuner displays "Tw5 - w5 = -0.199 g1"? Mathematically, it would be a rank-4 temperament. However, the fourth generator can be expressed as the difference between the actual tempered fifth and the nearby 1/5-comma tempered fifth, which is 0.0215¢, an inaudible amount even if stacked ten times. Musically, it's still a rank-3 temperament.

The next table contains all the possible pitches, written as a JI interval plus a fraction of a gu comma. The bold entries are a chain of tempered fifths from unadapted quarter-comma meantone. Each column is a chain of wa shifts. There are many more columns and rows than what is shown.

Table 4.7.1 – Adaptive quarter-comma meantone with 33% tempering strength (wa shift = 1/6 comma)

g6	g3 - 1/12	g7 - 1/6	g4 - 1/4	w1 + 2/3	w5 + 7/12	w2 + 1/2	w6 + 5/12	Lw3 + 1/3
g6 - 1/6	**g3 - 1/4**	g7 - 1/3	g4 - 5/12	w1 + 1/2	w5 + 5/12	w2 + 1/3	w6 + 1/4	Lw3 + 1/6
g6 - 1/3	g3 - 5/12	**w7 + 1/2**	w4 + 5/12	w1 + 1/3	w5 + 1/4	w2 + 1/6	w6 + 1/12	Lw3
g6 - 1/2	w3 + 5/12	w7 + 1/3	**w4 + 1/4**	w1 + 1/6	w5 + 1/12	w2	w6 - 1/12	Lw3 - 1/6
sw6 + 1/3	w3 + 1/4	w7 + 1/6	w4 + 1/12	**w1**	w5 - 1/12	w2 - 1/6	w6 - 1/4	Lw3 - 1/3
sw6 + 1/6	w3 + 1/12	w7	w4 - 1/12	w1 - 1/6	**w5 - 1/4**	w2 - 1/3	w6 - 5/12	y3 + 1/2
sw6	w3 - 1/12	w7 - 1/6	w4 - 1/4	w1 - 1/3	w5 - 5/12	**w2 - 1/2**	y6 + 5/12	y3 + 1/3
sw6 - 1/6	w3 - 1/4	w7 - 1/3	w4 - 5/12	w1 - 1/2	y5 + 5/12	y2 + 1/3	**y6 + 1/4**	y3 + 1/6
sw6 - 1/3	w3 - 5/12	w7 - 1/2	w4 - 7/12	w1 - 2/3	y5 + 1/4	y2 + 1/6	y6 + 1/12	**y3**

When alt-tuner's lattice is centered on one of the bolded notes, the available sounding notes are the ones on the same row of the table, plus those 4 rows above or below. The row 4 above is for the gu intervals, and 4 below is for yo ones. For example, the sounding wa 6th is w6 - 1/4, and the sounding yo 6th is y6 + 1/12. With a 12-tone keyboard, only one note per column is available. One must choose between the wa 6th and the yo 6th. When the lattice is centered on w1, the perfect 5th must be either w5 + 7/12 (Lg5) or w5 - 1/12 (w5) or y5 + 1/4 (y5). Note that the just w2 is sometimes available, but the just w5 is never available.

If the tempering strength is changed from 33% to 25%, all the unbolded intervals change slightly. The just w2 is no longer available, but the just w6 is.

Table 4.7.2 – Adaptive quarter-comma meantone with 25% tempering strength (wa shift = 3/16 comma)

g6	g3 - 1/16	g7 - 1/8	g4 - 3/16	w1 + 3/4	w5 + 11/16	w2 + 5/8	w6 + 9/16	Lw3 + 1/2
g6 - 3/16	**g3 - 1/4**	g7 - 5/16	g4 - 3/8	w1 + 9/16	w5 + 1/2	w2 + 7/16	w6 + 3/8	Lw3 + 5/16
g6 - 3/8	g3 - 7/16	**w7 + 1/2**	w4 + 7/16	w1 + 3/8	w5 + 5/16	w2 + 1/4	w6 + 3/16	Lw3 + 1/8
sw6 + 7/16	w3 + 3/8	w7 + 5/16	**w4 + 1/4**	w1 + 3/16	w5 + 1/8	w2 + 1/16	w6	Lw3 - 1/16
sw6 + 1/4	w3 + 3/16	w7 + 1/8	w4 + 1/16	**w1**	w5 - 1/16	w2 - 1/8	w6 - 3/16	Lw3 - 1/4
sw6 + 1/16	w3	w7 - 1/16	w4 - 1/8	w1 - 3/16	**w5 - 1/4**	w2 - 5/16	w6 - 3/8	Lw3 - 7/16
sw6 - 1/8	w3 - 3/16	w7 - 1/4	w4 - 5/16	w1 - 3/8	w5 - 7/16	**w2 - 1/2**	y6 + 7/16	y3 + 3/8
sw6 - 5/16	w3 - 3/8	w7 - 7/16	w4 - 1/2	w1 - 9/16	y5 + 3/8	y2 + 5/16	**y6 + 1/4**	y3 + 3/16
sw6 - 1/2	w3 - 9/16	w7 - 5/8	w4 - 11/16	w1 - 3/4	y5 + 3/16	y2 + 1/8	y6 + 1/16	**y3**

In third-comma meantone, available notes in alt-tuner's lattice include those three rows above or below. The detempered yo 3rd is y3 - 1/9.

Table 4.7.2 – Adaptive third-comma meantone with 33% tempering strength (wa shift = 2/9 comma)

g6 + 1/3	g3 + 2/9	g7 + 1/9	g4	w1 + 8/9	w5 + 7/9	w2 + 2/3	w6 + 5/9	Lw3 + 4/9
g6 + 1/9	**g3**	g7 - 1/9	g4 - 2/9	w1 + 2/3	w5 + 5/9	w2 + 4/9	w6 + 1/3	Lw3 + 2/9
g6 - 1/9	g3 - 2/9	**g7 - 1/3**	g4 - 4/9	w1 + 4/9	w5 + 1/3	w2 + 2/9	w6 + 1/9	Lw3
g6 - 1/3	g3 - 4/9	w7 + 4/9	**w4 + 1/3**	w1 + 2/9	w5 + 1/9	w2	w6 - 1/9	Lw3 - 2/9
sw6 + 4/9	w3 + 1/3	w7 + 2/9	w4 + 1/9	**w1**	w5 - 1/9	w2 - 2/9	w6 - 1/3	Lw3 - 4/9
sw6 + 2/9	w3 + 1/9	w7	w4 - 1/9	w1 - 2/9	**w5 - 1/3**	w2 - 4/9	y6 + 4/9	y3 + 1/3
sw6	w3 - 1/9	w7 - 2/9	w4 - 1/3	w1 - 4/9	y5 + 4/9	**y2 + 1/3**	y6 + 2/9	y3 + 1/9
sw6 - 2/9	w3 - 1/3	w7 - 4/9	w4 - 5/9	w1 - 2/3	y5 + 2/9	y2 + 1/9	**y6**	y3 - 1/9
sw6 - 4/9	w3 - 5/9	w7 - 2/3	w4 - 7/9	w1 - 8/9	y5	y2 - 1/9	y6 - 2/9	**y3 - 1/3**

Let's look at how the tempered and detempered (sounding) intervals are generated. C is the comma in cents. X is the difference between Tw5 and w5, as a fraction of C. For quarter-comma, X = -1/4. X is negative because the 5th is flattened. S is the strength as a fraction. For 33%, S = 1/3. WS is the wa shift. Dw5 is the detempered wa 5th, and Dy3 is the detempered yo 3rd. The three generators are in bold.

Tw8 = w8 (assume unstretched octaves)

Tw5 = w5 + X · C

Ty3 = 4 · **Tw5** - 2 · **Tw8** = y3 + (4 · X + 1) · C

WS = -X · (1 - S) · C

Dw5 = **Tw5** + **WS**

To express the detempered yo 3rd in terms of the three generators takes a bit of math:

Ty3 = TLw3 = Lw3 + 4 · X · C = (y3 + C) + 4 · X · C = y3 + (4 · X + 1) · C

Dy3 = y3 + S · (4 · X + 1) · C = Ty3 - (4 · X + 1) · (1 - S) · C

Dy3 = Ty3 + (4 + 1/X) · [-X · (1 - S) · C] = Ty3 + (4 + 1/X) · **WS**

Dy3 = 4 · **Tw5** - 2 · **Tw8** + (4 + 1/X) · **WS**

Thus all intervals, tempered and detempered, can be expressed as the sum or difference of the generators. For quarter-comma meantone, 1/X = -4, and Dy3 = Ty3. For third-comma, Dy3 = Ty3 + WS. But for two-sevenths, Dy3 = Ty3 + WS / 2. But every interval should be some whole number of genenerators. Therefore for this meantone, the third generator must be half of WS. If S = 3/10 (30%), WS is 1/5 comma, and the generator is 1/10 comma.

Table 4.7.3 – Adaptive two-sevenths-comma meantone with 25% tempering strength

g6 + 1/3	*g3 + 2/9*	*g7 + 1/9*	*g4*	*w1 + 8/9*	*w5 + 7/9*	*w2 + 2/3*	*w6 + 5/9*	*Lw3 + 4/9*
g6 + 1/9	*g3*	*g7 - 1/9*	*g4 - 2/9*	*w1 + 2/3*	*w5 + 5/9*	*w2 + 4/9*	*w6 + 1/3*	*Lw3 + 2/9*
g6 - 1/9	*g3 - 2/9*	*g7 - 1/3*	*w4 + 5/9*	*w1 + 4/9*	*w5 + 1/3*	*w2 + 2/9*	*w6 + 1/9*	*Lw3*
g6 - 1/3	*g3 - 4/9*	*w7 + 4/9*	*w4 + 1/3*	*w1 + 2/9*	*w5 + 1/9*	*w2*	*w6 - 1/9*	*Lw3 - 2/9*
sw6 + 4/9	*w3 + 1/3*	*w7 + 2/9*	*w4 + 1/9*	**w1**	w5 - 1/14	w2 - 1/7	w6 - 3/14	Lw3 - 2/7
sw6 + 2/9	*w3 + 1/9*	*w7*	*w4 - 1/9*	w1 - 3/28	w5 - 5/28	*w2 - 4/9*	*y6 + 4/9*	*y3 + 1/3*
sw6	*w3 - 1/9*	*w7 - 2/9*	*w4 - 1/3*	w1 - 3/14	**w5 - 2/7**	*y2 + 1/3*	*y6 + 2/9*	*y3 + 1/9*
sw6 - 2/9	*w3 - 1/3*	*w7 - 4/9*	*w4 - 5/9*	w1 - 9/28	*y5 + 2/9*	*y2 + 1/9*	*y6*	*y3 - 1/9*
sw6 - 4/9	*w3 - 5/9*	*w7 - 2/3*	*w4 - 7/9*	w1 - 3/7	*y5*	**y2 + 3/7**	*y6 - 2/9*	*y3 - 1/3*
							y6 + 1/7	
				w1 - 3/4				y3 - 1/28
								y3 - 1/7

For 3/8-comma meantone, or any 3/N-comma meantone, the generator would be one-third of WS. In theory, the meantone's comma fraction could be some irrational number like 1/pi, and the temperament would be rank-4. (The fourth generator would be the sounding yo 3rd.) But the best meantone for any chord progression is of the form 1/N if the chord progression pumps a single gu comma, or 2/N if it pumps two commas, as in "I Will Survive". Pumping more than two commas is possible but unlikely.

50/49 tritone sub: set to 1/4-comma, set slider to 50%, makes a shift twice as big as the tempering.

(unfinished chapter)

Chapter 4.8 – Temperament Names Part II

Multiple-comma temperaments require multiple commas in their name. Unlike single-comma temperaments, there's more than one way to name them. For example, the gu and ru temperament could also be called the rugu and ru temperament, or the rugu and gu temperament, because any two of these commas imply the third one.

Any two commas can be added and subtracted from one another to generate an infinite number of other commas. Any two commas from this infinite set will define the exact same temperament, as long as one comma isn't a multiple of the other. Because r1 minus g1 is sry1, gu and ru temperament could also be called gu and small ruyo temperament, ru and small ruyo temperament, etc. Since the temperament name is based on the commas, there are infinitely many names for any multi-comma temperament! To avoid confusion, a consistent method is needed to choose which commas to use.

There are two obvious methods. One is to minimize the number of colors in each comma (smallest possible prime subgroup), and the other is to minimize the size of the numbers in each comma's ratio (smallest possible odd limit).

The first method starts with a matrix in which each column is a color and each row is any tempered-out comma in rung-sum format. The matrix is then hermite-reduced (see en.wikipedia.org/wiki/Hermite_normal_form), but with the diagonal running up from the lower right, not down from the upper left as usual. The matrix rows become the commas used. The commas are listed lowest prime-limit first. Our example becomes gu and ru temper, abbreviated g&rT:

g1 = 81/80 = (-4, 4, -1, 0)	becomes	g1 = 81/80 = (-4, 4, -1, 0)
rg1 = 36/35 = (2, 2, -1, -1)		r1 = 64/63 = (6, -2, 0, -1)

This is the **prime-limit name**, or **prime name** for short. For most rank-2 tunings, the commas that result are primary-color commas, like gu or zozo, as opposed to compound colors like zogu or rugu. A rank-2 tuning will have a compound-color comma only if it also has a wa comma, e.g. sw&rgT.

The second method chooses the commas that have the lowest odd limit. All commas must be linearly independent (no redundant commas). If two possible commas have the same odd limit (e.g. rryy-2 = 50/49 and zz2 = 49/48), a tie-breaker is needed. Use the **double odd limit (DOL)**, which is found by factoring out all twos and listing the larger number first: the DOL of 50/49 is (49, 25) and the DOL of 49/48 is (49, 3). Since 3 is smaller than 25, zz2 has a lower DOL and is preferable to rryy-2. The commas are listed lowest double odd limit first. Our example becomes the rugu and ru temperament = rg&rT. This is the **odd-limit name**, or **odd name** for short. The odd-limit commas can be hard to determine; see the "diminished" example in Table 4.8.1. Combining $g^4$2 and r^4-2 to produce rg1 and rryy-2 is not obvious. Alt-tuner finds these commas automatically.

A possible third method would choose the commas with the smallest Tenney height, which is the product of the ratio's numerator and denominator. When the Tenney height is less than 100, it accurately sorts ratios by their relative consonance. However, commas have a much larger Tenney height. Furthermore, the comma represents a chord progression's path through the harmonic lattice. It's a modulation, not an interval, and its consonance is irrelevant. Also, since you can't modulate by an octave, the ratio's 2-factors are also irrelevant. Finally, it's possible for two ratios to have the same Tenney height. For these reasons, the double odd limit is preferable to the Tenney height.

Ideally, a temperament name should be short and informative, and use commas that are primary-color, low odd-limit, and familiar. Ideally, one of these two methods of choosing commas would always create the better name. Unfortunately, sometimes one method gives a better name, and sometimes the other does, and there are always two possible names.

Table 4.8.1 – Various two-comma temperaments (all yaza and rank-2)

prime-limit name	shorthand	odd-limit name	shorthand	conventional name
gu and ru	g&rT	rugu and ru	rg&rT	"dominant meantone"
gu and large ru	g&LrT	**gu and zo triple gu**	g&zg^{3}T	"septimal meantone"
gu and zozo	g&zzT	zozo and gu	zz&gT	"godzilla"
quadgu and quadru	g^4&r^4T	**rugu and double ruyo**	rg&rryyT	"diminished"

small gugu and ru	sgg & rT	**double ruyo and ru**	rryy & rT	"pajara"
large quadyo and zozo	Ly^4 & zzT	**zozo and ruyoyo**	zz & ryyT	"negri"
triple yo and ru	y^3 & rT	ru and triple yo	r & y^3T	"porcupine"
double large sixfold yo and large triple zo	LLy^6 & Lz^3T	**ruyoyo and large triple zo**	ryy & Lz^3T	"miracle"
small yo and ru	sy & rT	ru and zozoyo	r & zzyT	"superpyth"
large yo semitone and large ru semitone	Ly1 & Lr1T	**rugu and large yo semitone**	rg & Ly1T	"armodue"

The preferred names are bolded. Often the two names are identical except for order, in which case either name is acceptable, but the prime name is preferred, if only because it's easier to remember what order the commas are in.

Each naming method has its advantages. The prime name better indicates the relationships between temperaments. When a temperament is extended to a higher prime limit (or a larger prime subgroup) by tempering out an additional comma, microtonalists call it a **child** temperament. For example, both dominant meantone and septimal meantone are children of meantone, and all three are in the meantone **family**. Dominant meantone's prime-limit name indicates this clearly: gT becomes g & rT. The odd-limit name doesn't: gT becomes gr & rT. On the other hand, for diminished, negri, miracle and armodue, the prime-limit name is very long, the commas are very obscure, and the odd-limit name is preferable.

How to choose between the two names? Rank the commas by the remoteness class (see chapter 3.1), which reflects both the prime limit and the odd limit. For each name, find the class of every comma, and add up all the classes. Don't include wa commas in the total, because these are desirable, since they indicate the implied edo (see below). Choose the name that minimizes the sum of the classes. If both names have the same sum, the prime-limit name is preferred. Alt-tuner chooses the best name automatically.

Sometimes both methods miss the obvious name. For example, septimal meantone's prime name is gu and large ru, and its odd-limit name is gu and zo triple gu. Unfortunately neither name references ryy-2, which is more familiar than either Lr-2 or $zg^3$2. However, septimal meantone shouldn't be called "gu and ruyoyo", because more than two possible names is too confusing.

If a temperament uses a wa comma, mathematically, the other commas in the prime name should be three-less commas (no wa rungs). For example, tempering out LLw-2 and ryy-2 would create "wa and large sixfold ruyoyo temperament", with the very obscure Lr^6y^{12}-6 negative sixth! To avoid such awkward names, the three-less requirement is relaxed, to make "semi-reduced" commas. The wa comma is combined with each comma in the prime name in such a way as to minimize the double odd limit. The new comma has the same color (although usually not doubled), preserving the essential nature of the prime name: the commas are ordered by color. In our example, LLw-2 is added to Lr^6y^{12}-6 to make $L^3r^6y^{12}$-7, which is $(ryy\text{-}2)^6$, which reduces to ryy-2. The prime-limit name becomes wa and ruyoyo temperament.

Table 4.8.2 – Various rank-2 temperaments that use wa commas

prime-limit name	shorthand	odd-limit name	shorthand	implied edo
wa and ruyoyo	LLw & ryyT	ruyoyo and wa	ryy & LLwT	12-edo
large wa and ruyoyo	Lw & ryyT	ruyoyo and double zogugu	ryy & zzg^4T	7-edo
small wa and ruyoyo semitone	sw & ryy1T	ruyoyo semitone and ruyoyo	ryy1 & ryyT	5-edo
wa and rugu	LLw & rgT	rugu and wa	rg & LLwT	12-edo
large wa and rugu	Lw & rgT	rugu and large wa	rg & LwT	7-edo
wa-nineteen and rugu	w-19 & rgT	rugu and wa-nineteen	rg & w-19T	19-edo

Using semi-reduced commas, the prime name is often identical to the odd-limit name, except for the comma order. The prime name is usually preferable because it always includes the wa comma, which indicates the implied edo.

All of the above generalizes to higher primes. For example, the ilo temperament (1oT) tempers out 1o1 = 33/32 = 53¢.

Table 4.8.3 – Various single-comma yazala and yazalatha temperaments (mostly rank-3)

temperament name	**shorthand**	**comma**	**conventional name**
lologu	1oogT	1oog1 = 121/120 = 14¢	"biyatismic"
lozogugu	1ozggT	1ozgg2 = 77/75 = 46¢	(rank-4)
luyo	1uyT	1uy1 = 45/44 = 39¢	
loyo	1oyT	1oy1 = 55/54 = 32¢	"telepathmic"
loruru	1orrT	1orr-2 = 99/98 = 18¢	"mothwellsmic"
lulu	1uuT	1uu1 = 243/242 = 7¢	"rastmic"
thoyo	3oyT	3oy1 = 65/64 = 27¢	"wilsormic"
thogugu	3oggT	3ogg = 26/25 = 68¢	
double thogu	3ooggT	3oogg2 = 676/675 = 3¢	"island"
thulo	3uloT	3u1o1 = 352/351 = 5¢	

Thulo temperament is the pseudocolor of chapter 4.1 that merges ilo and tho. The next table lists gu and ru temperament and some of its children. The odd-limit name is mostly preferred, unless comparing the different children.

Table 4.8.4 – Various multi-comma 11-limit and 13-limit temperaments in the gu and ru family

prime-limit name	shorthand	odd-limit name	shorthand	conventional name
gu & ru	g&rT	rugu & ru	rg&rT	"dominant"
gu & ru & lulu	g&r&1uuT	**rugu & ru & lologu**	rg&r&1oogT	
gu & ru & large lu	g&r&L1uT	**rugu & luyo & ru**	rg&1uy&rT	"domineering"
gu & ru & small ilo	g&r&s1oT	**rugu & ru & lozogugu**	rg&r&1ozgg	"domination"
gu & ru & small lu & small tho	g&r&s1u &s3oT	**thogugu & rugu & luzogu & ru**	3ogg&rg& 1uzg&rT	"dominion" (3ogg2 = 26/25)

Chapter 3.7 covered alternate mappings for higher primes. Alternate keyspans (e.g. whether the ilo 4th is a perfect 4th or an augmented 4th) have no effect on temperament names. Alternate stepspans (e.g. whether 19/16 is a 2nd or a 3rd) only affect those tempers with a degree in their name, which temper out "commas" of at least 90¢.

Mathematically, two-less (all odd numbers) or three-less (nowa) commas wouldn't include wa and clear, but usually these rungs are included for musical reasons. For example, the triple gu comma 128/125 is three-less, and g^3T is technically a rank-1 temperament of the 2.5 prime subgroup. But that would create a very boring 3-edo "scale" consisting of a single unchanging augmented chord. So fifths would usually be used, creating a rank-2 temperament of the 2.3.5 subgroup.

The term **plus** is used to include primes not part of any comma. **Wa and clear are always assumed to be present, even if the commas lack those rungs.** "No" is used to explicitly exclude wa and/or clear, as in **nowa**, **noca** or **nowaca**. (Recall that "clear" refers to the prime number 2, but 2-limit ratios like 1/1, 2/1 and 4/1 are called wa for simplicity.)

The shorthand for plus and no is "+" and "-". For example, the boring 3-edo scale on the 2.5 subgroup would be triple gu nowa temperament, g^3-wT. (In practice this tuning would simply be called 3-edo.) The 2.3.5.7 subgroup with g1 = 81/80 tempered out is the gu plus za temperament, g+zT. The "all" color is always used, so there is never a "plus zo" or a "plus ru" temperament.

The minus sign is also used for negative intervals, as in Ly-2, and as a hyphen in remote wa commas and temperaments, as in w-19T (see Table 4.5.3 above). The meaning of "-" will always be clear from context.

Any edo can become a "plus" temperament by adding one untempered rung. For example, Blackwood is 5-edo+y. Blackwood could also be called sw+yT, but the edo-based name is preferred.

Table 4.8.5 – Various "plus" and "no" temperaments

temperament	**shorthand**	**prime subgroup**	**rank**	**commas**	**alternate names**
gu plus za	g+zT	2.3.5.7	rank-3	g1= 81/80	
ru plus ya	r+yT	2.3.5.7	rank-3	r1 = 64/63	
triple gu (3-edo plus wa)	g^3T	2.3.5	rank-2	$g^3$2 = 128/125	"augmented", 3-edo+w
3-edo (triple gu nowa)	3-edo	2.5	rank-1	"	g^3-wT
3-edo plus za	3-edo+z	2.5.7	rank-2	"	g^3-w+zT
triple gu plus za	g^3+zT	2.3.5.7	rank-3	"	
5-edo (small wa)	5-edo	2.3	rank-1	sw2 = 256/243	swT
5-edo plus ya	5-edo+y	2.3.5	rank-2	"	"blackwood", sw+yT
5-edo plus za	5-edo+z	2.3.7	rank-2	"	sw+zT
5-edo plus yaza	5-edo+yz	2.3.5.7	rank-3	"	sw+yzT
double ruyo	rryyT	2.3.5.7	rank-3	rryy-2 = 50/49	"pajara"
double ruyo nowa	rryy-wT	2.5.7	rank-2	"	
6-edo (triple gu and double ruyo nowa)	6-edo	2.3.5.7	rank-1	$g^3$2, rryy-2	g^3 & rryy-wT

The triple gu temperament's period is a third of an octave, and quadgu temperament's period is a quarter of an octave. They're both rank-2 tunings on the 2.3.5 prime subgroup. However, while g^3T "locks" the yo rung to exactly 400¢, g^4T doesn't lock either the yo or the wa rung. Either one can be tuned to any cents size within reason, and like meantone, there is an infinite spectrum of g^4T tunings. While g^3T is identical to 3-edo+w, 4-edo+w represents only one specific g^4T tuning, and 4-edo+y represents another. Most g^4T tunings do not contain just intervals.

Obviously, a nowa temperament's commas must be three-less. Otherwise wa would be intrinsic to the temperament and couldn't be removed. For example, a gu nowa temperament would be impossible.

A noca temperament is a non-octave scale. If it's rank-1, it's an **EDONOI**, an equal division of a non-octave interval. Every temperament has a noca version, except those with fractional periods. One would think a noca temperament would require two-less commas. It doesn't, because the rungs can be "voiced" (octave-reduced or -increased, and/or inverted) in such a way as to avoid using the 2/1 rung. For example, the comma yy1 = 25/24 isn't two-less. But yy1 can be derived from the 3/2 and 5/4 rungs, and yy-cT is an EDONOI in which the possibly stretched fifth w5 = 3/2 is divided into two equal parts, each one equal to a possibly flattened 5/4. Assuming the 5th isn't stretched, this EDONOI is written 2-ED(3/2) or 2-ED(w5), meaning that the 5th is divided into two equal parts. If not, it's 2-ED(Tw5). Either way, this creates a scale with rather large steps, consisting of stacked neutral 3rds.

Here are several EDONOIs, with the usual 3/2, 5/4 and 7/4 rungs often in alternate voicings:

Table 4.8.6 – Various single-comma noca rank-1 temperaments (single-comma EDONOIs)

temperament	shorthand	comma	scale step	rungs	EDONOI name
gu noca	g-cT	g1 = 81/80	w4 = 4/3	4/3 & 16/5	4-ED(Wg6)
ru noca	r-cT	r1 = 64/63	w4 = 4/3	4/3 & 7/4	2-ED(z7)
yoyo noca	yy-cT	yy1 = 25/24	y3 = 5/4	3/2 & 5/4	2-ED(w5)

gugu noca	gg-cT	gg2 = 27/25	g3 = 6/5	4/3 & 8/5	2-ED(w4) or 3-ED(g6)
zozo noca	zz-cT	zz2 = 49/48	r2 = 8/7	4/3 & 8/7	2-ED(w4)
triple yo noca	y^3-cT	$y^3$1 = 250/243	y2 = 10/9	4/3 & 8/5	3-ED(w4) or 5-ED(g6)
triple zo noca	z^3-cT	$z^3$3 = 343/324	z3 = 7/6	3/2 & 7/4	3-ED(w5) or 4-ED(z7)
large triple zo noca	Lz^3-cT	L$z^3$2 = 1029/1024	r2 = 8/7	3/2 & 8/7	3-ED(w5)
quadyo noca	y^4-cT	$y^4$1 = (5, -9, 4)	y2 = 10/9	3/2 & 5/2	4-ED(w5) or 9-ED(Wy3)

When a EDONOI is created from a rank-2 temperament by removing clear, the 2/1 period is removed, and the generator becomes the new period. The generator can be voiced in many ways by inverting and/or adding octaves on. Every alternate voicing is also a generator. For example gT's generator is w5, which can be voiced as Ww5, WWw5, w4, Ww4, etc. Each voicing generates a unique EDONOI. w5 generates 4-ED(WWy3), w4 generates 4-ED(Wg6), and Ww5 generates 4-ED(W^6y3). Since the period is also the scale step, the most musically useful EDONOI is the one that has the smallest period. For g-cT this is 4-ED(Wg6), with a w4 period. The EDONOI's comma-based name always refers to the EDONOI with the smallest scale step, thus "gu noca temperament" is always 4-ED(Wg6) and never 4-ED(WWy3).

A "scale" with step size of about a 4th or a 5th isn't very melodic, so any rank-2 temperament generated by the 5th doesn't yield a useful EDONOI. If the scale step is y3 = 5/4, the "scale" still sounds more like a chord than a scale, not very melodic. But if the generator is z7 = 7/4, the scale step is r2 = 8/7, much more melodic. For example, the zozo noca temperament, listed above, sounds like a stretched 5-edo. Double or triple commas can create small non-rung scale steps like y2, very melodic.

When edos are created by commas, the octave is assumed to be just, with the other intervals tempered to fit neatly into the octave. Likewise, with EDONOIs, the **NOI** (non-octave interval being divided) is assumed to be just, and the scale step is assumed to be tempered.

As with edos, the EDONOI-based name is preferred over the comma-based name, because it describes the tuning more directly. The various EDONOI names indicate the scale steps per rung. Each rung is a possible NOI. For example, y^3-cT could be either 3-ED(w4) or 4-ED(g6). There are many other NOIs, for example y^3-cT could also be 2-ED(g3). These other NOIs are found by adding or subtracting commas from some multiple of the scale step.

Only one of the NOIs can be just, and this is the one the EDONOI is named after. Often it will be the wa rung, as this one tends to have the simplest ratio, making it most adversely affected by tempering. If none of the NOIs are just, but the scale step is, the EDONOI is named after the scale step. For example, 1-ED(y2) indicates a scale of stacked just y2's. If all the NOIs plus the scale step are tempered, the EDONOI is named after the wa NOI, e.g. 3-ED(Tw4).

To find the often-revoiced rungs of an EDONOI, express the rungs in the form $3 \cdot 2^{-b}$, $5 \cdot 2^{-c}$, $7 \cdot 2^{-d}$, etc. The rung may need to be inverted, for example if b = 2, the rung is not 3/4 but 4/3. Choose b, c and d so that the rungs can be added or subtracted to make the comma that's tempered out, and also added up to make the scale step. This involves solving simultaneous equations.

For example, the y^3-cT EDONOI has a comma of $y^3$1 = 250/243 = (1, -5, 3) and a period of y2 = 10/9 = (1, -2, 1). This directly yields the equations 1 - 5b + 3c = 0 and 1 - 2b + c = 0, which in turn yields b = 2 and c = 3. The rungs are $3 \cdot 2^{-2} = 3/4$ and $5 \cdot 2^{-3} = 5/8$. Both rungs need inverting, making 4/3 = w4 and 8/5 = g6.

For multi-comma EDONOIs, there are three or more simultaneous equations to solve. The most popular EDONOI is equal-tempered Bohlen-Pierce, which divides Ww5 = 3/1 into 13 equal steps. This is written as 13-ED(Ww5) or 13-ED(3/1). When the non-octave interval being divided is of the form N/1, ED(N/1) is written as EDN. Thus Bohlen-Pierce is written as 13-ED3, and ED2 is an alternate name for edo. Bohlen-Pierce's commas, scale step and rungs are completely two-less (no even numbers).

Table 4.8.7 – Various multi-comma noca rank-1 temperaments (multi-comma EDONOIs)

temperament	commas	scale step	rungs	EDONOI names
zzy&r^{3}y^5-cT	zzy2 & r^{3}y^5-3	gg2 = 27/25	3/1, 5/1 & 7/1	13-ED3 (equal-tempered Bohlen-Pierce)
g&zz-cT	g1 & zz2	r2 = 8/7	4/3, 16/5 & 8/7	2-ED(w4) or 8-ED(Wg6)
y^3&r-cT	y^{3}1 & r1	y2 = 10/9	4/3, 8/5 & 7/4	3-ED(w5) or 5-ED(g6) or 6-ED(z7)
y^3&zz-cT	y^{3}1 & zz2	zg2 = 21/20	4/3, 8/5 & 8/7	6-ED(w4) or 10-ED(g6) or 3-ED(r2)
zz&ryy-cT	zz2 & ryy-2	g2 = 16/15	4/3, 5/4 & 8/7	4-ED(w4) or 3-ED(y3) or 2-ED(r2)
g&Lz^3-cT	g1 & Lz^{3}2	r2 = 8/7	3/2, 5/1 & 8/7	3-ED(w5) or 12-ED5

EDONOIs can be notated as stretched or compressed EDOs. EDO notation is covered in Part V.

When clear is removed from a rank-3 temperament, a rank-2 tuning is created that is an unequal division of a NOI. Both the period (the NOI) and the generator can be just. The tuning may or may not be a MOS scale. The period is no longer the scale step, so there is no requirement that the period be as small as possible. The rungs must still add up to the comma. But the rungs can't be uniquely determined from the comma, as shown below by the multiple versions of ryy-cT. In order for the name to specify an exact tuning, some convention must be adopted, such as the Bohlen-Pierce convention that all commas and rungs must be two-less, as in the last entry:

Table 4.8.8 – Various single-comma noca non-octave rank-2 temperaments

temperament	shorthand	comma	rungs	period	generator
ruyoyo noca	ryy-cT	ryy-2 = 225/224	3/1, 8/5 & 7/2	Ww5 = 3/1	g6 or y7
"	"	"	3/1, 16/5 & 8/7	Ww5 = 3/1	g2 = 16/15
"	"	"	3/2, 5/4 & 7/2	w5 = 3/2	y3 or g3
"	"	"	4/3, 5/2 & 7/2	w4 = 4/3	Ly1 = 135/128
B-P zozoyo noca	zzy-cT	zzy2 = 245/243	3/1, 5/1 & 7/1	Ww5 = 3/1	r3 = 9/7

Every edo also has many alternate comma-based names. These names include both a prime-limit name and an odd-limit name, and depend on the prime subgroup the edo is considered to be derived from. For example, 19-edo derived from 2.3.5 has a prime-limit name w-19 & gT and an odd-limit name g & Ly^{5}T. If derived from 2.3.7, it would be w-19 & zzT or zz & LrT. If derived from 2.3, it's a single-comma temperament with only one name, w-19T. Since all these temperaments sound exactly the same, the simplest name is used, 19-edo. Likewise 19-edo+z derived from 2.3.5.7 could be w-19 & g+zT or g & Ly^5+zT, but 19-edo+z is preferred.

An edo's comma-based name also depends on the edo-mapping. For example, in 12-edo, the ilo rung (551¢) can be rounded up to 600¢. Or it can be rounded down to 500¢, slightly more distant from just. The prime-limit name for 12-edo will have a different la comma in each case. If an edo-mapping is not the nearest edo-mapping, the edo name reflects this with a **tweak**, a letter added to the edo indicating which rung or rungs are not the nearest edo-mapping. Conventional microtonal notation calls these "warts", and uses "b" for wa, "c" for yo, "d" for zo, etc. 12-edo with a 500¢ ilo rung is "12e-edo". If a rung is even more distant, the letters are repeated. For example, in 72-edo, the nearest zo edo-mapping is 967¢. Zo's next nearest edo-mapping at 983¢ is 72d-edo, 950¢ is 72dd-edo and 1000¢ is 72ddd-edo. Since the nearest edo-mapping is flat of just, the 2nd nearest is sharp of just, the 3rd is flat, the 4th is sharp, etc.

Table 4.8.9 – Various alternate names for 12-edo

prime subgroup	edo-mappings	edo name	prime-limit name	odd-limit name
2.3.5	nearest	12-edo	w-12 & gT	g & g^{3}T
2.3.5.7	"	"	w-12 & g & rT	gr & rryy & rT
2.3.7	"	"	w-12 & rT	r & r^{3}T
2.3.5.7.11	"	"	w-12 & g & r & 1uu2T	gr & 1uy & rryy & 1uzgT
2.3.5.7.11	1o4 = 5\12	12e-edo	w-12 & g & r & 1oT	1or & 1o & gr & rryyT

Distant edo-mappings are created and tweaks are needed when forcing an edo to temper out a comma that isn't normally tempered out. For example, g1 = 81/80 isn't tempered out in 22-edo, so tempering out g1 while in 22-edo tweaks yo from 7\22 to 8\22 and creates 22c-edo.

For rank-2 temperaments, microtonalists use brackets to indicate a scale formed by a **generator chain**, also called a **genchain**: meantone[7] indicates the 7 note scale generated by 6 meantone fifths, or some mode of that scale. This can be extended to color names: meantone[7] is gu[7] or g[7], "gu heptatonic". An alternate name is "gu seven", not to be confused with the gu seventh = 9/5. Porcupine[8] is triple yo octotonic, or y^3[8]. Multi-comma temperaments are written g & r[7].

To indicate a specific mode, number the modes by their position in the genchain, somewhat analogous to harmonica positions. For meantone[7], Lydian is "first gu heptatonic", written 1st g[7]. Major or Ionian is 2nd g[7], Mixolydian is 3rd g[7], etc. There can be ambiguity in mode numbering, because the octave inverse of a generator is also a generator. Therefore the generator is defined as the smaller of the two, except that 3/2 is preferred over 4/3 for historical reasons.

MODMOS scales are indicated with chromatic alterations. The harmonic minor is "fifth gu heptatonic, sharp seven", written 5th g[7] ♯7, and the melodic minor is 5th g[7] ♯6 ♯7. This scale could also be named 2nd g[7]♭3, but mode names usually avoid altering the 3rd.

There can be ambiguity in note numbering for non-heptatonic scales. If the note names are drawn from the seven letters A–G, then the numbering is based on these seven letters, even if not all seven are used. A B C E F A is named A 1st g[5]♭3♭6. The F is referred to as a 6th, even though it's the 5th note of the scale. The meaning of sharp and flat can also be ambiguous. Here the sharp sign indicates moving 7 steps forward on the genchain, not 5. If the scale used other names, perhaps J–K–L–M–N, then N would be the 5th, not the 6th, and sharp would indicate 5 steps.

Non-MOS scales are written the same way as MOS scales, e.g. C 2nd g[8] for C D E F F♯ G A B C. But modified non-MOS scales, with a discontinuous genchain, must be written as MOS or MODMOS scales with added or removed notes. A B C D E F G G♯ A is named A 6th g[7] add ♯7. A B C E F G A is named A 5th g[7] no 4.

Specific tunings of a temperament can be indicated in fractions of a comma, as in the third-comma ru temperament. Thus the full name of a tuning might be "quarter-gu chromatic starting on E♭" for quarter-comma meantone ranging from E♭ to G♯.

Alt-tuner displays the prime name, the odd name and if applicable the edo name. A box is automatically placed around the preferred name, based on remoteness classes. When you create an edo with alt-tuner's edo slider, alt-tuner offers to derive linkages from the current edo-mapping using either the prime-limit name or the odd-limit name. This lists the commas in the names more fully, and shows any edo tweaks in color notation.

Adaptive JI can be indicated by adding back in rungs that have already been tempered. Meantone-based adaptive JI is g+w+yT. This is a rank-4 tuning generated by the octave, the wa fifth, the yo third, and the tempered wa fifth Tw5. (Ty3 isn't a generator because it can be derived from Tw5.) If octaves are not just, it would be g+c+w+yT, a rank-5 tuning. 12-edo-based adaptive JI is 12-edo+w+y, or 12-edo+y if you use 12-edo 5ths instead of just 5ths.

Temerament names can even be extended to include untempered JI, which is named as a "plus" tuning without any commas. Ya JI is the "plus ya tuning", written +yT, with "T" here meaning tuning. Yaza JI is +y+zT, za JI is +zT, and wa JI is +wT. Bohlen-Pierce 3.5.7 JI (yaza noca) is -c+y+zT, and 2.5.7 JI is -w+y+zT.

The temperament name, whether prime-limit or odd-limit, directly indicates both the prime subgroup and the rank. The prime subgroup is the rungs referenced in the name, including clear and wa unless expicitly excluded. Pairs of inverse colors like yo and gu both reference the same rung. The rank is the number of rungs in the name minus the number of commas. Count the word "edo" as a comma. Include rungs added with a plus, and include wa and clear unless excluded. The prime-limit name also shows the relationship between parent and child temperaments.

As noted in chapter 4.6, there is a loose correlation between a comma's depth and fractional periods. If the comma is double, the period may or may not be half of an octave. If the comma is triple, the period may or may not be a third of an octave. For multi-comma tempers, the prime name is more indicative that the odd name. If there are no double or triple commas in the prime name, the period is an octave, with one exception: wa commas always produce fractional periods. A comma that is double or triple that doesn't split the octave will always split either the 4th or the 5th or the 2nd.

There is also a loose correlation between a comma's depth and the generator. If all of a rank-2 temperament's commas have a depth of 1, the generator is always a fifth.

The temperament name indicates what general types of chord progressions would need that temperament. For example the triple yo temperament requires modulating by 3rds or 6ths at least three times. A large or small temperament requires modulating by 4ths or 5ths at least four times, possibly up to ten times.

Chapter 4.9 – Tuning Innate Comma Chords *

(<u>Very</u> rough draft of an unfinished chapter!)

Chapter 3.9 discussed using higher primes to tune certain chords. With many primes come many minicommas, and a very slight tempering can improve the chord by bringing all of the component intervals closer to just.

Often a comma implies an innate comma chord. The trick is to advance through the lattice from the 1/1 to the comma by chordal steps. These are steps that are likely to occur in chords. They must be of low odd limit, and not too narrow.

http://xenharmonic.wikispaces.com/marvel+chords

http://xenharmonic.wikispaces.com/Linear+chord

comma	ratio	chord quality	chord structure	intervals	chord notes
g1	81/80	C6add9	Cy6,w=y9	y3, w4, w4=g4, w4, w4	wC, yE, yA, y=wD, wG
			(alt. voicing)	w=y2, y2, g3, y2, g3	wC, w=yD, yE, wG, yA
		Cmin7add11	Cg7,w=g11	w=g4, w4, w4, y3, w4	wC, w=gF, gB♭, gE♭, wG
			(alt. voicing)	g3, w=y2, w2, g3, y2	wC, gE♭, g=wF, wG, gB♭
g^{3}2	128/125	Caug	Cy(yy5=g6)	y3, y3, y3=gg4	wC, yE, yyG♯=gA♭
sgg2		Caug7	Cy,yy5=sw6,w7	y3, y3=sg4, w2, w2	wC, yE, yyG♯=wA♭, wB♭
g^{4}2		Cdim7	Cg,y6(gg5=yy4)	g3, g3=y^{3}2, g3, g3	wC, gE♭, ggG♭=yyF♯, yA
r1	64/63	Csus4	C[w=z4]	w4, r2, w4	wC, w=zF, wG
		Cdom7sus4	C4[w=z7]	w4, w2, w3=z3, w2	wC, wF, wG, w=zB♭
ryy-2	225/224	Caug	Cy(yy5=z6)	y3, y3=zg4, r3	wC, yE, yyG♯=zA♭
			Cy(ry5=g6)	y3, r3=gg4, y3	wC, yE, ryG♯=gA♭
			Cr(ry5=g6)	r3, y3=gg4, y3	wC, rE, ryG♯=gA♭
zz2	49/48	C,^6 = C,v7	Cy[r6=z7]	y3, g3, r2=z3, r2	wC, yE, wG, rA=zB♭
y^{3}1	250/243	C.^m = C.vv	C[g=yy]y9	y2, y2, y3, w4	wC, yD, yyE=gE♭, wG
	(y,zg5)	Cmajor(♭5)		y3, w2=zgg3, ry4	wC, yE, yF♯=zgG♭
	(h7,no5)	Cdom7no5		y3, ry4, w2=zgg3	wC, yE, ryyA♯ =wB♭
		Cdom7(♭5)		y3, w2=zgg3, y3, r2	wC, yE yF♯=zgG♭, zB♭
		Caug7		y3, y3=zg4, w2, r2	wC, yE, yyG♯=zA♭, zB♭

The yo yoyo-5 chord y(yy5) (an augmented chord) divides the octave into two yo thirds and a gugu 4th = gg4 = 32/25 = 428¢. The gg4 is quite close to the much more singable ru 3rd = 9/7 = 435¢. This chord could be tuned as a combination of two yo 3rds and a ru one, with the ryy-2 minicomma 225/224 = 8¢ tempered out. In practice, all three 3rds would be tuned a few cents flat. Unlike other chords, this tempering doesn't worsen the chord, because the yaza augmented chord is an innate comma chord, and in any reasonably compact voicing, some interval will always beat.

Because the tempering equates several JI ratios, each chord has several names. Using "L" for Tr3 and "s" for Ty3, here are the three "inversions" of a tempered yaza augmented chord, each written out two ways. The first of each pair of names is preferred.

ssL:	yo yoyo-5 chord	y(yy5)	w1 – y3 – yy5	1/1 – 5/4 – 25/16	16:20:25
	yo zo-6 no-5 chord	y,z6no5	w1 – y3 – z6	1/1 – 5/4 – 14/9	36:45:56
sLs:	yo ruyo-5 chord	y(ry5)	w1 – y3 – ry5	1/1 – 5/4 – 45/28	28:35:45
	yo gu-6 no-5 chord	y,g6no5	w1 – y3 – g6	1/1 – 5/4 – 8/5	20:25:32
Lss:	ru ruyo-5 chord	r(ry5)	w1 – y3 – ry5	1/1 – 9/7 – 45/28	28:36:45
	ru gu-6 no-5 chord	r,g6no5	w1 – r3 – g6	1/1 – 9/7 – 8/5	35:45:56

[*change p to purple here, avoid confusion with po*]

The close proximity of purple, ilo and tho suggest another approach, using septimal approximations. The purple 4th is a good approximation of the ilo fourth 1o4 = 11/8 = 551¢. They differ by only the purple-lu minicomma, p1o1 = (9√3) / (11√2) = 3.57¢. The purple 5th matches 16/11 just as well. Other la ratios can be derived from p4 & p5:

1og2 = 11/10 = 11/8 ÷ 5/4 ≈ p4 - y3 = pg2
1u2 = 12/11 = 16/11÷ 4/3 ≈ p5 - w4 = p2

The 11-limit intervals can be made even better by tempering out p1o1, for example by flattening the fifths by about 1¢.

Furthermore, the purple 6th approximates the tho 6th 13/8 = 841¢ by the purple-thu minicomma, (16√2) / (13√3) = 8.49¢. Every ratio of 11 or 13 is closely approximated by a purple interval. Here's several versions of a 4:5:6:7:9:11:13 chord in yaza JI using purple intervals:

root	third	fifth	seventh	ninth	eleventh	thirteenth
w1	y3	w5	z7	w9	p11	p13
1/1	5/4	3/2	7/4	9/4	9/8·√3	4·√(2/3)
r2	ry4	r6	w8	r10	zg13	zg15
8/7	10/7	12/7	2/1	18/7	63/20	56/15
z3	zy5	z7	zz9	z11	ry12	ry14
7/6	35/24	7/4	49/24	21/8	45/14	80/21

The rugu 4th 48/35 = 547¢ approximates 11/8 almost as well, narrower by only the lozoyo minicomma 1ozy1 = 385/384 = 4.50¢. And the small rugu 6th srg6 = 512/315 = 841.0¢ is <u>very</u> close to 13/8. The difference is the tthurugu microcomma s3urg1 = 4096/4095 = 0.42¢.

Using the rugu approximations for both 11/8 & 13/8 on a y3 root gives us the h13 chord yIIIy,z7,9,rg11,rg13:

root	third	fifth	seventh	ninth	eleventh	thirteenth
y3	yy5	y7	zy7	y11	r13	r15
5/4	25/16	15/8	35/16	45/16	24/7	256/63

(this chapter is unfinished)

Chapter 4.10 – Constructing MOS scales *

Chapter 4.11 – Case Study: "Central Park West"

Let's examine a complex song with bold modulations, John Coltrane's "Central Park West". Here are the chords:

C♯m7 F♯7 (half-bar pickup)
bar 1 BM7 – Em7 A7
bar 2 DM7 – B♭m7 E♭7
bar 3 A♭M7 – Gm7 C7
bar 4 FM7 – C♯m7 F♯7
bar 5 BM7 – Em7 A7
bar 6 DM7 – C♯m7 F♯7
bar 7 BM7 – – –
bar 8 C♯m7/B – – –
bar 9 BM7 – – –
bar 10 C♯m7/B – C♯m7 F♯7
bar 11 (same as bar 1)

A dash means hold the chord for an extra beat. The standard jazz progression ii7 – V7 – IM7 is used to quickly modulate to remote keys. How to translate ii7 – V7 – IM7 to JI? The IM7 chord is obviously Iy,y7. The V7 chord is either Vy,g7 or Vy,w7 or Vh7. The 7th determines the ii7 chord's 3rd, and thus its 7th, which is a wa fifth above the 3rd. The three possibilities are:

minor is gu: wIIg7 – Vy,g7 – Iy7
minor is wa: wIIw7 – Vy,w7 – Iy7
minor is zo: wIIz7 – Vh7 – Iy7

If minor is gu, the song pumps a quintgu 2nd = $g^5 2$ = 6561/6250 = 84¢ in only 4 bars:

C♯g7 F♯y,g7
1 By7 – Eg7 Ay,g7
2 Dy7 – yyA♯g7 yyD♯y,g7 (A♯ not B♭ because it has yC♯ in common with the previous chord)
3 yyG♯y7 – yyyF^x=ggGg7 ggCy,g7
4 ggFy7 – C♯g7 F♯y,g7
5 By7

Then it pumps the gu comma twice in the next 6 bars:

5 By7 – Eg7 Ay,g7
6 Dy7 – y=wC♯g7 F♯y,g7
7 By7 – – –
8 yC♯g7/wB – – –
9 By7 – – –
10 yC♯g7/wB – y=wC♯g7 F♯y,g7
11 By7

Since all pumps are towards yo not gu, they're all descending, and they all add up. The whole 10 bars pumps a large sevenfold gu 2nd = $Lg^7 2$ = 127¢. This comma's ratio is so large that we actually need to use commas in the comma! $Lg^7 2$ = 43,046,721/40,000,000.

C♯g7 F♯y,g7
1 By7 – Eg7 Ay,g7
2 Dy7 – yyA♯g7 yyD♯y,g7
3 yyG♯y7 – y^3F^{x}g7 y^3B♯y,g7
4 y^3E♯=g^4Fy7 – ggC♯g7 ggF♯y,g7

5 ggBy7 – ggEg7 ggAy,g7
6 ggDy7 – gC♯g7 gF♯y,g7
7 gBy7 – – –
8 C♯g7/gB – – –
9 gBy7 – – –
10 C♯g7/gB – C♯g7 F♯y,g7
11 By7

The tonic in bar 11 hasn't drifted from bar 1. But in bars 7 & 9, the tonic has drifted a gu comma sharper. And in bar 5, it's two commas sharper. But tempering out Lg^{7}2 also tempers the gu comma, and fortunately it happens to make it smaller. Flattening 3/2 and sharpening 5/4 about equally results in 1/25 comma temperament, which makes Tg1 = -5.3¢, making bar 5 be 10.6¢ flat. This is about the best that can be done if using a fixed tuning. (Tempering out both Lg^{7}2 and g1 makes 12-edo.)

If using alt-tuner, we can use adaptive tuning to improve the chords (see chapter 4.7). We can do even better with a "divide-and-conquer" approach. Temper out g^{5}2 for the first four bars, then switch to a meantone tuning for the next six bars. There will be absolutely no tonic drift.

If minor is wa, bars 1-4 pump g^{3}2 = 128/125 downwards.

C♯w7 F♯y,w7
1 By7 – Ew7 Ay,w7
2 Dy7 – yA♯w7 yD♯y,w7
3 yG♯y7 – yyF^x=gGw7 gCy,w7
4 gFy7 – C♯w7 F♯y,w7
5 By7

Bars 5-6 pump g1 downwards as before, and bars 9-10 don't pump anything. The total pump is downwards by g^{4}2 = 648/625 = 63¢.

C♯w7 F♯y,w7
1 By7 – Ew7 Ay,w7
2 Dy7 – yA♯w7 yD♯y,w7
3 yG♯y7 – yyF^x=ggGw7 ggCy,w7
4 ggFy7 – gC♯w7 gF♯y,w7
5 gBy7 – gEw7 gAy,w7
6 gDy7 – C♯w7 F♯y,w7
7 By7 – – –
8 C♯w7/wB – – –
9 By7 – – –
10 C♯w7/wB – C♯w7 F♯y,w7
11 By7

Again, the overall tonic drift is zero, but there is drift mid-song. In bar 5, the tonic is flat by Tg1. The best fixed temperament is to set 3/2 flat by around 9¢, making 5/4 sharp by 9¢ and Tg1 = -14.1¢.

If using alt-tuner, temper out g^{3}2 in bars 1-4, switch to a meantone tuning for bars 5-6, then switch to JI for bars 7-10.

If minor is zo, the first 4 bars pump upwards a ruru triple-yo negative 2nd = rry^3-2 = 4000/3969 = 13¢.

C♯z7 F♯h7
1 By7 – Ez7 Ah7
2 Dy7 – ryA♯z7 ryD♯h7
3 ryG♯y7 – ryyF^x=zgGz7 zgCh7
4 zgFy7 – C♯z7 F♯h7

5 By7

The next 6 bars go down a gu comma and up a ru comma.

5 By7 – Ez7 Ah7
6 Dy7 – y=wC♯z7 F♯h7
7 By7 – – –
8 rC♯z7/wB – – –
9 By7 – – –
10 rC♯z7/wB – r=wC♯z7 F♯h7
11 By7

All 10 bars pump downwards a small triple-ru quadyo negative 2nd = sr^3y^4-2 = 20,480,000/20,253,807 = 19¢.

C♯z7 F♯h7
1 By7 – Ez7 Ah7
2 Dy7 – ryA♯z7 ryD♯h7
3 ryG♯y7 – ryyF^{x}z7 ryyB♯h7
4 ryyE♯y7 – rry^{3}B^x=zgC♯z7 zgF♯h7
5 zgBy7 – zgEz7 zgAh7
6 zgDy7 – zC♯z7 zF♯h7
7 zBy7 – – –
8 C♯z7/zB – – –
9 zBy7 – – –
10 C♯z7/zB – C♯z7 F♯h7
11 By7

In bar 5, the tonic is flat by Tsry1, and in bars 7 & 9, the tonic is flat by Tr1. The best fixed tuning would minimize both Tsry1 and Tr1 for the sake of tonic drift, as well as minimize damage to the chords. Since this is yaza with one comma, the temperament is rank-3, not rank-2. This gives us more freedom of choice. Because slight mistuning of chords is more noticeable than slight tonic drift, prioritize the chords somewhat. The best result is with 5/4 and 7/4 both tempered about 8¢ sharp, making sry1 = -11¢ and r1 = 13.6¢.

If using alt-tuner, temper out rry^3-2 for the first four bars, then switch to meantone for the next two bars, then to JI for three bars, then temper out the ru comma for the last bar.

What's the point of all this analysis? If your jazz choir wants to sing this song as smoothly as possible, they need to consciously decide on the tuning of the dom7 chords, and then learn how to pump the associated commas. If you can play a keyboard at choir rehearsal that has been tuned with alt-tuner, you can guide them through the unfamiliar pumps.

Here are all six tunings in relative notation. The bass notes of the slash chords are relative to the chord's root, not the scale's tonic (see the end of chapter 2.5). First in conventional notation:

ii7 V7 (half-bar pickup)
1 IM7 – iv7 ♭VII7
2 ♭IIIM7 – vii7 III7
3 VIM7 – ♭vi7 ♭II7
4 ♭VM7 – ii7 V7
5 IM7 – iv7 ♭VII7
6 ♭IIIM7 – ii7 V7
7 IM7 – – –
8 ii7/♭7 – – –
9 IM7 – – –
10 ii7/♭7 – ii7 V7

11	IM7

With minor as gu, first pumping $g^{5}2$ downwards, then pumping g1 downwards twice:

	wIIg7 Vy,g7
1	Iy7 – IVg7 wVIIy,g7
2	wIIIy7 – yyVIIg7 yyIIIy,g7
3	yyVIy7 – yyyV=ggVIg7 ggIIy,g7
4	ggVy7 – wIIg7 Vy,g7
5	Iy7 – IVg7 wVIIy,g7
6	wIIIy7 – y=wIIg7 Vy,g7
7	Iy7 – – –
8	yIIg7/7 – – –
9	Iy7 – – –
10	yIIg7/7 – y=wIIg7 Vy,g7
11	Iy7

With minor as gu, pumping $Lg^{7}2$ downwards.

	wIIg7 Vy,g7
1	Iy7 – IVg7 wVIIy,g7
2	wIIIy7 – yyVIIg7 yyIIIy,g7
3	yyVIy7 – y^{3}Vg7 y^{3}Iy,g7
4	y^{3}IV=Lg^{4}Vy7 – LggIIg7 LggVy,g7
5	LggIy7 – LggIVg7 LggVIIy,g7
6	ggIIIy7 – LgIIg7 LgVy,g7
7	gIy7 – – –
8	wIIg7/7 – – –
9	gIy7 – – –
10	wIIg7/7 – wIIg7 Vy,g7
11	Iy7

With minor as wa, first pumping $g^{3}2$ downwards, then pumping g1 downwards.

	wIIw7 Vy,w7
1	Iy7 – IVw7 wVIIy,w7
2	wIIIy7 – yVIIw7 yIIIy,w7
3	yVIy7 – yyV=gVIw7 gIIy,w7
4	gVy7 – wIIw7 Vy,w7
5	Iy7 – IVg7 wVIIy,g7
6	wIIIy7 – y=wIIg7 Vy,g7
7	Iy7 – – –
8	yIIg7/7 – – –
9	Iy7 – – –
10	yIIg7/7 – y=wIIg7 Vy,g7
11	Iy7

With minor as wa, pumping $g^{4}2$ downwards.

	wIIw7 Vy,w7
1	Iy7 – IVw7 wVIIy,w7
2	wIIIy7 – yVIIw7 yIIIy,w7
3	yVIy7 – yyV=ggVIw7 ggIIy,w7
4	ggVy7 – LgIIw7 LgVy,w7
5	gIy7 – gIVw7 gVIIy,w7
6	gIIIy7 – wIIw7 Vy,w7
7	Iy7 – – –
8	wIIw7/7 – – –

9	Iy7 – – –
10	wIIw7/7 – wIIw7 Vy,w7
11	Iy7

With minor as zo, first pumping rry^3-2 upwards, then g1 downwards, then r1 upwards.

	wIIz7 Vh7
1	Iy7 – IVz7 wVIIh7
2	wIIIy7 – ryVIIz7 ryIIIh7
3	ryVIy7 – ryyV=zgVIz7 zgIIh7
4	zgVy7 – wIIz7 Vh7
5	Iy7 – IVz7 wVIIh7
6	wIIIy7 – y=wIIz7 Vh7
7	Iy7 – – –
8	rIIz7/7 – – –
9	Iy7 – – –
10	rIIz7/7 – r=wIIz7 Vh7
11	Iy7

With minor as zo, pumping sr^{3}y^4-2 upwards:

	wIIz7 Vh7
1	Iy7 – IVz7 wVIIh7
2	wIIIy7 – ryVIIz7 ryIIIh7
3	ryVIy7 – ryyVz7 ryyIh7
4	ryyIVy7 – rry^{3}I=LzgIIz7 LzgVh7
5	LzgIy7 – LzgIVz7 LzgVIIh7
6	zgIIIy7 – LzIIz7 LzVh7
7	zIy7 – – –
8	wIIz7/7 – – –
9	zIy7 – – –
10	wIIz7/7 – wIIz7 Vh7
11	Iy7

(Part IV is unfinished)

Part V – Alternative Frameworks

Conventional music theory is based on two fundamental assumptions: 7 scale steps to an octave, and 12 keys per octave on the keyboard. In Part V we'll go beyond both assumptions. (Part V is unfinished.)

Chapter 5.1 – Frameworks

The year was 1991. I had just arrived in Ghana after spending a year in Zimbabwe studying Shona music. I was eager to study the gyil, the local version of the marimba. I contacted Kakraba Lobi, a famous gyil player and a great musician, and arrived at his house for my first lesson. I had learned quite a few Zimbabwean songs on marimba and considered myself fairly accomplished. But when I sat down and tried to learn what Kakraba was showing me, I couldn't do it. Even the simplest patterns were extraordinarily hard to memorize and repeat. I felt like I had never played an instrument before in my life!

The problem was that the gyil uses a pentatonic scale. But not the usual black-keys scale, with two large steps that feel like they span "missing notes". This scale was made up of five roughly equal-sized steps to the octave. As a result, every interval sounded quite different. What looked like a third on the gyil sometimes sounded like a major third and sometimes sounded like a fourth. What looked like a fifth was either a major sixth or a minor seventh, I couldn't quite decide. My mental map was scrambled. As a musician, I was used to conceptualizing songs and melodies in terms of intervals, but that was suddenly impossible. I left Kakraba's house that day very humbled.

After much time and effort, I learned to hear and understand gyil music on its own terms. I stopped trying to insert the two "missing notes," and I learned to think pentatonically. It wasn't easy. It felt like learning a foreign language after being monolingual all my life. This was my first encounter with an alien framework.

A **framework** is the mental map we use to categorize intervals. The framework most used in Western music is the diatonic or heptatonic one of 7 steps to the octave. In northern Ghana, as in other parts of Africa and of the Far East, it's the pentatonic framework. Most of the world has adopted one of these two frameworks.

But there's another layer to all this. If you ask a Westerner "how many notes are there to an octave?", you'll get the answer "seven". (Or "eight", if they count the final note that completes the octave.) But if you press the person, perhaps asking "how many <u>tones</u> are there?", and if they are a musician, you'll get the answer "well, twelve, really". There's two answers to the question because we use <u>two</u> frameworks simultaneously, one framework for naming intervals (7 names, A-B-C, do-re-mi, etc.) and another for categorizing intervals by size (12 semitones per octave). I call them the **naming** framework and the **sizing** framework. I call the two Western frameworks the 7-<u>note</u> framework and the 12-<u>tone</u> framework. We measure the size of intervals two different ways. An interval has a degree (3rd, 4th, etc.) via the naming framework, but also a keyspan (# of semitones it covers) via the sizing framework.

As a culture, we Westerners have gone to great lengths to keep this twelve-fold nature hidden. I've seen several musicians actually resort to counting frets to answer the tones-per-octave question! Our system of assigning qualities to intervals (major, minor, etc.) is actually our way of reconciling these two frameworks. It allows us to compare the exact size of intervals without using 12-tone terminology, in other words, without having to count semitones. We memorize two sequences ("diminished – minor – major – augmented" and "diminished – perfect – augmented"), learn some relationships ("the minor sixth is next to the perfect fifth"), do a little subconscious math ("perfect to augmented means one semitone wider"), and get our answer ("augmented fifth equals minor sixth").

These two frameworks combine to make a **system**. Because there are two answers to the "how many notes?" question, the West uses a **dual-framework** system. But many cultures have only one answer, and thus use a **single-framework** system. In northern Ghana, the answer is five. In Zimbabwe, the answer is seven. In fact, every African culture I know of is single-framework, either heptatonic or pentatonic. In my opinion, a single-framework system lends a certain directness and simplicity to the music that's quite appealing.

Of course, in the modern world, people are not as geographically isolated as they used to be and cultures borrow heavily from each other. It's best to say "Northern Ghana is *traditionally* pentatonic." But these traditions have a lot of resilience. Consider a typical Afro-pop band playing conventional chromatic instruments like keyboard or guitar. They will play different songs in different keys, and over the course of a gig they may use every note in the octave. But in the course of a single song, they never go outside of the seven note scale. That's because they are reflecting their culture's single-framework system. Even though they have adopted the West's 12-tone instruments, they have not adopted the West's 12-tone framework.

Indonesians also use two frameworks, but instead of one dual-framework system, they use two single-framework systems. Each one of their gamelans (large tuned-percussion orchestras) uses either a pentatonic tuning or a heptatonic one. They are a **dual-system** culture. It's been said that their heptatonic tunings are a subset of a larger 9-tone framework, just like our diatonic flute is tuned to a subset of a larger 12-tone framework. If this is so, Indonesians have both a single-framework system and a dual-framework system, all together using three frameworks!

A hallmark of dual-framework cultures is that their scales have large and small steps. The inverse is sometimes true: there's a tendency for single-framework cultures to have equidistant or near-equidistant scales, that is, scales with equal-sized steps. When there is only one framework, the naming framework <u>IS</u> the sizing framework. So it follows that an interval's name would correspond very closely to its size. In northern Ghana, the five steps of a gyil's scale are all roughly equal in size. In Zimbabwe, the mbira dzavadzimu (Shona kalimba) is traditionally tuned with seven roughly equal-sized steps.

However, there are plenty of single-framework cultures with unequal scale step sizes. The most dramatic example is the Wagogo people of Tanzania. Their music is pentatonic and always uses the exact same scale: w1 – w2 – y3 – w5 – z7 (1/1 – 9/8 – 5/4 – 3/2 – 7/4). This scale is a fragment of the harmonic series and runs from the 5th harmonic up to the 10th harmonic. This particular scale has 5 steps of 5 different sizes! They range fairly evenly from 182¢ to 316¢. Just intonation scales never have equal step sizes.

There are two opposing principles in music, melody and harmony. Each culture has reconciled these two principles in its own way. The desire for simple strong melodies and uncomplicated systems tends towards equidistant scales. But the desire for consonant harmonies tends towards just intonation, with its unequal step sizes. These unequal step sizes tend to create the sensation of "missing notes", which often leads to a dual-framework system. The larger sizing framework in turn creates an equidistant or near-equidistant scale. All this assumes the octave as given. Although octaves can be stretched or compressed, the just octave is the only interval that doesn't conflict with either principle.

But JI can only exert its pull if the overtones of the harmonic series are in a position to either clash or to agree. They must be clearly audible in the sound of several simultaneous notes. The main circumstance for this is vocal harmonies, defined very broadly as group singing in anything other than unison or octaves. Another circumstance is with instruments with a harmonic timbre such as string instruments and wind instruments. The cultures that have resisted the "JI pull" have largely avoided those circumstances. They tend to avoid vocal harmonies other than the octave and sometimes the fifth. They also tend to use inharmonic idiophones like the gamelan, the mbira and the gyil. If they use string instruments, they only use monophonic ones like fiddles. And if there's more than one instrument, they tend to play in unison.

In the Mande region of West Africa, there is a balafon (marimba) tradition going back to the 13th century that fits this description. It uses roughly 7-edo tunings, inharmonic instruments and mostly solo vocals. If there is more than one singer, they sing in unison or in octaves. But in the 19th century, the kora (heptatonic harp) appeared on the scene. It has a much more harmonic timbre than the balafon. It was tuned not to the 7-edo scale, but to a major JI scale. The same balafon songs were sung to the kora, but in the new scale. Even though vocal harmonies were now possible, the vocals remained solo or unison. This implies that the Mande people wanted to keep their music unchanged as much as possible. But when the timbre changed, the JI pull exerted itself, and the scale <u>had</u> to change.

Middle eastern music breaks sharply from low-limit JI with its quarter-tones and its neutral intervals. And indeed, there are no vocal harmonies. They do use string and wind instruments with a harmonic timbre. However, they avoid the JI pull by playing mostly monophonically over a drone.

The implication is clear: The price you pay for straying too far from just intonation is 1) no vocal harmonies and 2) either only instruments with inharmonic timbres, or no harmonies played except octaves.

However, the converse doesn't hold. The Wagogo use inharmonic timbres (kalimbas, marimbas) even thought they don't have to. And the music of India, which uses ya JI, doesn't use vocal harmonies, even though it could.

It's hard to generalize about world music because whatever you postulate, there will always be a counterexample. In this case, the music from the country of Georgia provides one. Their traditional music is primarily vocal, with 3-part harmonies. So they experience a strong JI pull. But they divide the just fifth into four equal steps!

Table 5.1.1 – The traditional Georgian scale

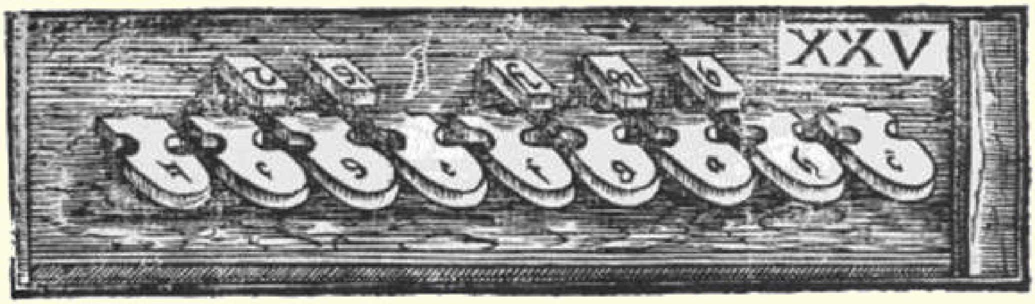
tonic	0¢
lowered 2nd	175¢
neutral 3rd	351¢
raised 4th	526¢
5th	702¢
lowered 6th	877¢
neutral 7th	1053¢
raised 8ve	1228¢

This is an example of a non-octave scale, the EDONOI discussed in chapters 4.1 and 4.8. Some Georgian music:

> Hamlet Gonashvili Gogov Shavtvala + Rustavi Ensemble+Lyrics
> www.youtube.com/watch?v=6045_n_XdiI
>
> Georgian Folk Song Tu Ase Turpa Ikhavi by Hamlet Gonashvili
> www.youtube.com/watch?v=-2QPNRY39aY
>
> Georgian singers imitate Duduki instrument in an incredible polyphony
> youtu.be/9HXvhQ4Sr2k?t=1m15s
>
> Khasanbegura - Ensemble Rustavi
> www.youtube.com/watch?v=EduaIfdYw2s

Chapter 5.2 – The 12-tone Framework

We've seen how in the West, we use a dual-framework system consisting of a 7-note naming framework and a 12-tone sizing framework, or a 12 + 7 system for short. The Western conception of 12 notes to an octave dates back to 1361, with the appearance of the first keyboard with the now-familiar 7 white keys and 5 black keys, the Halberstadt organ.



This chart summarizes the assumptions behind Western music notation ever since then:

Table 5.2.1 – The Western system of 12 tones and 7 notes (12 + 7)

prime	ratio	cents	keyspan	degree	stepspan
2	2/1 = w8	1200¢	12	8ve	7
3	3/2 = w5	702¢	7	5th	4
5	5/4 = y3	386¢	4	3rd	2

The first three columns are objective acoustical facts, but the next three columns reflect certain musical choices the West has made. The **stepspan** is always one less than the degree (except for negative intervals, see chapter 3.3). The stepspan of a fifth is 4, the stepspan of a fourth is 3, etc. Both a major third and a minor third have the same stepspan, because they have the same degree.

Everything follows from this simple table. Every interval is the sum or difference of these three rungs, so every interval's keyspan and stepspan can be deduced from this table. For example, 15/8 is the sum of one wa rung and one yo rung, so its keyspan must be 11 and its stepspan must be 6.

Why does this system work so well for ya music? One reason is that the keyspan column approximates the cents values quite well. In other words, a wa fifth is about 7/12 of an octave, and a yo third is about 4/12. The stepspan column matches the cents almost as well. A wa fifth is about 4\7 = 685¢ of an octave, and a yo third is about 2\7 = 343¢.

To implement and notate this system in the West, 7 letters were chosen (because the octave's stepspan is 7) and distributed evenly across the keyboard. These 7 keys are the natural white keys, and the other 5 keys are the black keys. A somewhat arbitrary choice was made as to the keyspan from each letter to the next, being 1 or 2. In other words, the West decided a long time ago that there would be a black key between D and E, but not one between E and F. Symbols were chosen for a sharpening accidental and a flattening accidental, and each black key was named as both a sharpened white key and as a flattened white key. In other words, C♯ is also D♭.

But there's another way to look at accidentals. If we arrange our 7 note symbols in a chain of fifths, we can continue the chain with the same 7 letters, modified by accidentals:

F – C – G – D – A – E – B – F♯ – C♯ – G♯ – D♯ – A♯ – E♯ – B♯

The sharp sign indicates both melodic distance and harmonic distance. Harmonically, sharpening a note means moving rightwards on this chain through all 7 letters, for example from F to F♯. Each step raises the pitch by 7 semitones (the fifth's keyspan). 7 steps times 7 semitones is 49 semitones, which octave-reduces to 1 semitone. Thus in the 12 + 7 system, the sharp sign raises the pitch by one semitone. This is the sharp's melodic distance. In other systems, a sharp might work out to be 2 semitones, or 0 semitones, or -3 semitones. These systems are much harder to notate, requiring a second pair of accidentals (see chapter 5.5, "Ups and Downs").

So the 7-note and 12-tone frameworks are a natural fit. Do any other naming frameworks work this well with the 12-tone framework? If the keyspan of the octave and the fifth are relatively prime (no common factors), then the sizing framework always implies two natural naming frameworks. The two naming frameworks always add up to the sizing framework. Thus 12-tone = 7-note + 5-note, and the other naming framework is the pentatonic one.

Table 5.2.2 – The pentatonic system of 12 + 5

prime	ratio	cents	keyspan	stepspan
2	2/1 = w8	1200¢	12	5
3	3/2 = w5	702¢	7	3
5	5/4 = y3	386¢	4	2

This complimentary framework can be thought of as the result of naming the black keys as naturals and the white keys as accidentals. This table provides a blueprint for naming every ya interval pentatonically. For example, 15/8 is the musical sum of a 3/2 and a 5/4. Adding up the keyspans and stepspans, 15/8 becomes an interval of keyspan 11 and stepspan 5, which works out to be a diminished octave. 6/5 has keyspan 3 and stepspan 1, making an augmented second. The details of the pentatonic framework are spelled out in the next chapter.

The next table shows how well each rung is approximated in each framework, in cents and as a percentage. For example, the yo third 's pentatonic stepspan = round (5 · 386¢ / 1200¢) = round (1.61) = 2. We have to round up a whopping 39% of a step pentatonically, close to the theoretical maximum of 50%.

Table 5.2.3 – The 12-tone, 7-note and 5-note frameworks

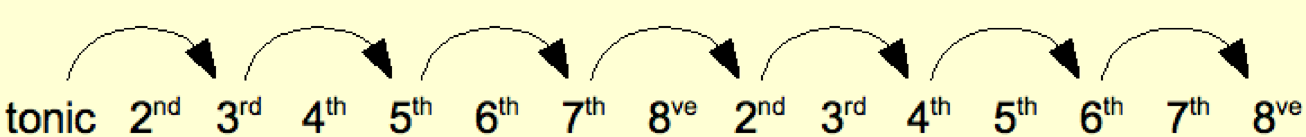
prime	ratio	cents	keyspan	stepspan	
2	2/1 = w8	1200¢	12	7	5
3	3/2 = w5	702¢	7 (700¢ = -2%)	4 (686¢ = -9%)	3 (720¢ = +8%)
5	5/4 = y3	386¢	4 (400¢ = +14%)	2 (343¢ = -25%)	2 (480¢ = +39%)
7	7/4 = z7	969¢	10 (1000¢ = +31%)	6 (1029¢ = +35%)	4 (960¢ = -4%)

Notice that the pentatonic framework represents the zo 7th by far the best. Because the fifth is quite accurately represented by all three frameworks, we can generalize the accuracy of the yo rung to the entire yo row, as well as the entire gu row, and indeed all ya ratios. Likewise the zo rung's accuracy generalizes to all za ratios. Therefore the pentatonic framework is the most accurate for za music. As a result, many of the za commas which are negative heptatonically aren't pentatonically.

As we've seen, our method of naming intervals with both a quality and a degree can be seen as a way of avoiding having to count semitones to compare the size of two intervals. Each interval has several possible qualities. There are two sequences of qualities, "diminished – minor – major – augmented" and "diminished – perfect – augmented". The first one is for imperfect intervals like the second or third, and the second one is for perfect intervals like the fifth or the octave. When exploring new systems, how does one decide which sequence to follow for an interval? In other words, why isn't there a perfect third or a major fifth? There are several ways to answer this question; here's the one I prefer:

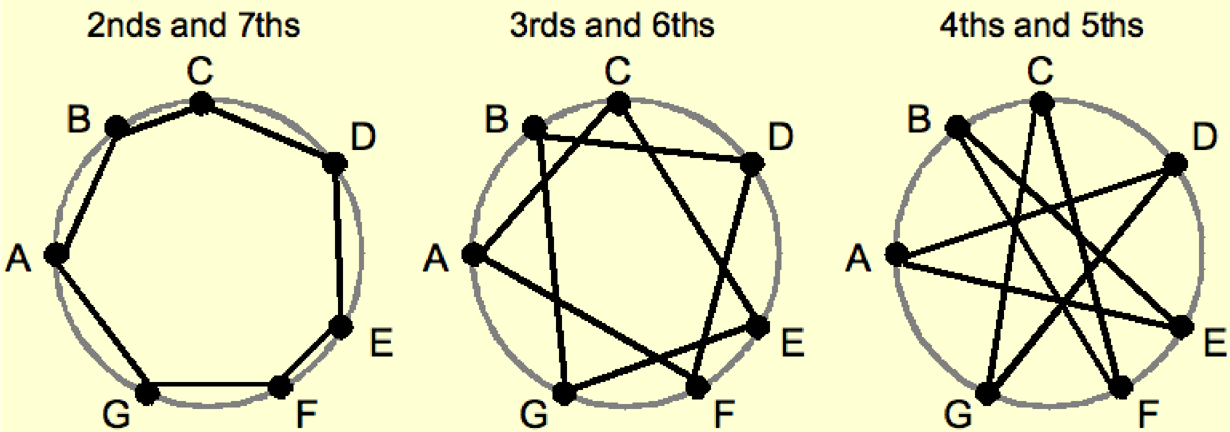
An octave has a stepspan of 7. Take any seven note scale and look at the size of the seven steps it contains. Obviously, they always add up to an octave, and therefore they have an average size of one-seventh of an octave. Now look at the size of the seven thirds the scale contains. They always add up to two octaves. You can visualize this by imagining a two-octave instrument with seven notes to the octave; starting at the lowest note and playing seven ascending thirds takes you to the highest note.

Figure 5.2.1 – Seven thirds always add up to two octaves



Or you can imagine a seven-pointed star inside a circle. Following the star's lines from point to point takes you around the circle two times. Likewise, the seven fourths always add up to three octaves, the seven fifths add up to four octaves, etc.

Figure 5.2.2 – Heptatonic intervals



This means the average size of the seconds is always 1/7 of an octave, which is 171¢. The average size of the thirds is always 2/7 of an octave, which is 343¢. The average size of the fourths is always 3/7 of an octave = 514¢, etc. This is true for every seven-note scale. Harmonic minor, quarter-tone arabic, whatever. Even if it has augmented or neutral intervals, the average size of an interval is always exactly equal to the corresponding 7-edo interval.

An octave has a keyspan of 12. In any twelve-tone tuning, the twelve steps have an average size of 1/12 of an octave = 100¢. The average size of the twelve intervals it contains that span 3 semitones is 300¢. You can confirm this by considering the 3 diminished tetrads contained in the tuning. Each one has 4 intervals that add up to an octave, so the average size is obviously 1/4 of an octave. Likewise, considering the 4 augmented chords it contains, the average size of the twelve 4-semitone intervals is 400¢. This holds for every 12-note tuning, no matter how far off from 12-ET it may be. The average 7-semitone interval is always 700¢. Note that I said average 7-semitone interval, not average fifth. Diminished fifths are only 6 semitones wide, and including them pulls the average down.

This rule holds for intervals wider than an octave too. The average ninth is always 8/7 of an octave = 1371¢. This rule generalizes to any framework. The average pentatonic second is always 1/5 of an octave = 240¢, no matter what pentatonic scale you use. This also holds for non-octave scales. If you divide the wa 5th into four steps like the Georgians do, the average size of the fourths is 3/4 of a fifth = 526¢. In its most general form, our rule is:

Any time you divide an interval of C cents into S steps, the average size of all the N-step intervals is C · N / S

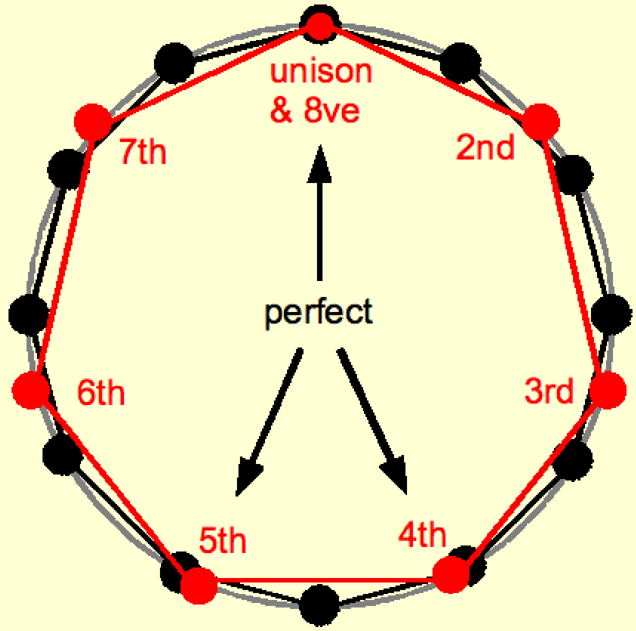
Getting back to perfect and imperfect in the 12 + 7 system: The average third is 2/7 of an octave = 343¢, the average 3-semitone interval is 300¢, and the average 4-semitone interval is 400¢. Now 343¢ is about equally close to both 300¢ and 400¢. Both keyspans, 3 and 4, could reasonably represent a third. A keyspan of 2 or 5 would be a far less reasonable representation. The two reasonable representations are labeled major and minor, and the two less reasonable representations are labeled augmented and diminished. Therefore the third is an imperfect interval.

A fifth has a stepspan of 4, and has an average size of 4/7 of an octave, which is 685¢. A 7-semitone interval is on the average 700¢. 700¢ is a reasonable representation of 685¢, but 600¢ and 800¢ are not. So a keyspan of 7 can reasonably represent a fifth, but a keyspan of 6 or 8 can't. The single reasonable representation is labeled perfect, and the two less reasonable representations are labeled diminished and augmented. Thus the fifth is a perfect interval.

For an interval to be perfect, the corresponding 7-edo interval must be quite close to some 12-edo interval, otherwise it's imperfect. The unison and the octave are always exactly equal, and are always perfect. The next two closest intervals (there will always be two because of symmetry) are defined as perfect, and everything else is imperfect. For the standard 12 tones and 7 names, the expected perfect intervals are produced: 4ths, 5ths and octaves.

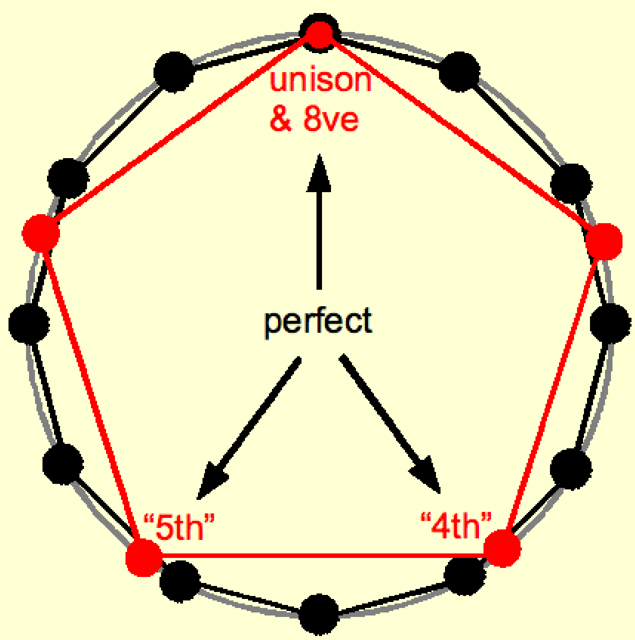
This method depends only on the first rung's keyspan and degree. Neither the size of the rung nor anything about the other rungs matter. Stretching or compressing the octave won't change anything, nor will adding higher rungs. This can be visualized by imagining a pentagon or a heptagon (the naming framework) inside of a 12-sided dodecagon (the sizing framework). The top corner of the heptagon lines up exactly with the top corner of the dodecagon. The two corners of the heptagon that lie closest to any corner of the dodecagon represents perfect intervals.

Figure 5.2.3 – Perfect intervals in the 12 + 7 system



We can make a similar type of diagram for any dual-framework system. For the pentatonic system of 12 + 5, we compare 5-edo to 12-edo. The pentatonic second is on the average 240¢, which misses 12-edo by 40¢. The pentatonic "fifth" (which is actually a fourth, because it's the fourth note of the pentatonic scale) is on the average 720¢, missing by only 20¢. Thus the "fifth", and by symmetry the "fourth", are perfect, and the second and its inverse are imperfect. Therefore the only perfect intervals in the 12 + 5 system are the octave, fourth and fifth, just like in the 12 + 7 system. Here's the diagram:

Figure 5.2.4 – Perfect intervals in the 12 + 5 system



In the next chapter, we'll analyze the pentatonic framework more thoroughly.

Chapter 5.3 – Pentatonicism

The most common naming framework is the diatonic or heptatonic one of 7 steps to the octave. Let's look at the next most common one, the pentatonic framework.

Much of African and East Asian music is pentatonic. It also survives in Western music. There is a trend in modern rock of singing a pentatonic melody over a mildly chromatic chord progression that uses 8 or 9 notes. While a Vivaldi piece with its diatonic runs is unmistakably heptatonic, the blues uses mostly pentatonic or chromatic runs, and seems to imply a 12 + 5 system.

Consider the main yaza chord: w1 – y3 – w5 – z7 – w8 (1/1 – 5/4 – 3/2 – 7/4 – 2/1). Just as the y3 splits the fifth into two similar, complimentary halves, so does the z7 divide the fourth from w5 to w8. However, bisecting a fourth flies in the face of heptatonic terminology. If half a fifth is a third, what's half a fourth? Za intervals don't fit well into the heptatonic framework, but they fit very well into the pentatonic framework. To really understand za and yaza music, I've found it useful to think in both frameworks.

Pentatonic interval names can be confusing. The logical way to name the pentatonic scale degrees would be penta-2nd, penta-3rd, etc. But I'm trained to associate the word "4th" with 4/3, not 3/2. And I don't want to have to say "hexave-equivalent" when I mean octave-equivalent. So I've given the scale degrees heptatonic-friendly names:

> the **subthird** (or penta-2nd), which includes all intervals from 10/9 up to 6/5,
> the **fourthoid** (or penta-3rd), everything from a 5/4 up to 7/5,
> the **fifthoid** (or penta-4th), from 10/7 up to 8/5,
> the **subseventh** (or penta-5th), from 5/3 to 9/5,
> the **octoid** (or hexave), from 15/8 to 15/7.

Keyspans and stepspans add up logically, so a subthird plus a fifthoid always make a sub7th, etc. Since 16/15 is the difference between 4/3 and 5/4, which are both fourthoids, 16/15 is an augmented unison. 25/24 is the difference between 5/4 (fourthoid) and 6/5 (subthird), and is a diminished subthird. A 9/4 is a wide subthird or a **subtenth**, a 5/2 is a wide fourthoid, etc. Octave, fourth and fifth are used for the perfect wa intervals only: there is no diminished fifth, but there is a diminished fifthoid. 2/1 is an octave, but 15/8 is an octoid.

We saw in the last chapter that the octave, 4thoid and 5thoid are perfect, and the sub3rd and sub7th are imperfect. This allows us to assign qualities and degrees to each keyspan. Sub3rd is abbreviated s3, 4thoid is 4d, 5thoid is 5d, sub7th is s7 (not to be confused with the sub-seven chord), and octoid is 8d.

Table 5.3.1 – The quality and degree of each keyspan heptatonically and pentatonically

keyspan	0	1	2	3	4	5	6	7	8	9	10	11	12
diatonic quality & degree	P1	m2	M2	m3	M3	P4	A4 d5	P5	m6	M6	m7	M7	P8
pentatonic quality & degree	P1	A1 ds3	ms3	Ms3	As3 d4d	P4d	A4d d5d	P5d	A5d ds7	ms7	Ms7	As7 d8d	P8d

In the pentatonic framework, the rainbow runs yellow-red-blue-green. The next table shows the main consonant ratios, in both heptatonic and pentatonic terminology.

Table 5.3.2 – Pentatonic terminology for yaza intervals

Ratio	Keyspan in semitones	Diatonic or Heptatonic			Pentatonic		
1/1	0	*perf 1*	wa unison	w1	*perf 1*	wa unison	w1
21/20	1	*min 2*	zogu 2nd	zg2	*aug 1*	zogu aug unison	zg1
16/15	1	*min 2*	gu 2nd	g2	*aug 1*	gu aug unison	g1
10/9	2	*maj 2*	yo 2nd	y2	*min s3*	yo sub3rd	ys3
9/8	2	*maj 2*	wa 2nd	w2	*min s3*	wa sub3rd	ws3
8/7	2	*maj 2*	ru 2nd	r2	*min s3*	ru sub3rd	rs3
7/6	3	*min 3*	zo 3rd	z3	*maj s3*	zo sub3rd	zs3
32/27	3	*min 3*	wa 3rd	w3	*maj s3*	large wa sub3rd	Lws3
6/5	3	*min 3*	gu 3rd	g3	*maj s3*	gu sub3rd	gs3
5/4	4	*maj 3*	yo 3rd	y3	*dim 4d*	yo 4thoid	y4d
9/7	4	*maj 3*	ru 3rd	r3	*dim 4d*	ru 4thoid	r4d
21/16	5	*perf 4*	zo 4th	z4	*perf 4d*	zo 4thoid	z4d
4/3	5	*perf 4*	wa 4th	w4	*perf 4d*	wa 4thoid (4th)	w4
27/20	5	*perf 4*	gu 4th	g4	*perf 4d*	gu 4thoid	g4d
7/5	6	*dim 5*	zogu 5th	zg5	*aug 4d*	zogu 4thoid	zg4d
45/32	6	*aug 4*	yo 4th	y4	*dim 5d*	small yo 5thoid	sy5d
64/45	6	*dim 5*	gu 5th	g5	*aug 4d*	large gu 4thoid	Lg4d
10/7	6	*aug 4*	ruyo 4th	ry4	*dim 5d*	ruyo 5thoid	ry5d
40/27	7	*perf 5*	yo 5th	y5	*perf 5d*	yo 5thoid	y5d
3/2	7	*perf 5*	wa 5th	w5	*perf 5d*	wa 5thoid (5th)	w5
32/21	7	*perf 5*	ru 5th	r5	*perf 5d*	ru 5thoid	r5d
14/9	8	*min 6*	zo 6th	z6	*aug 5d*	zo 5thoid	z5d
8/5	8	*min 6*	gu 6th	g6	*aug 5d*	gu 5thoid	g5d
5/3	9	*maj 6*	yo 6th	y6	*min s7*	yo sub7th	ys7
27/16	9	*maj 6*	wa 6th	w6	*min s7*	small wa sub7th	sws7
12/7	9	*maj 6*	ru 6th	r6	*min s7*	ru sub7th	rs7
7/4	10	*min 7*	zo 7th	z7	*maj s7*	zo sub7th	zs7
16/9	10	*min 7*	wa 7th	w7	*maj s7*	wa sub7th	ws7
9/5	10	*min 7*	gu 7th	g7	*maj s7*	gu sub7th	gs7
15/8	11	*maj 7*	yo 7th	y7	*dim 8d*	yo octoid	y8d
40/21	11	*maj 7*	ruyo 7th	ry7	*dim 8d*	ruyo octoid	ry8d
2/1	12	*perf 8*	wa octave	w8	*perf 8*	wa octoid (8ve)	w8

A big advantage of pentatonicism is that similar, complimentary interval pairs like 7/6 and 8/7, and 7/4 and 12/7, are united into one category. A big disadvantage is that pairs like 6/5 and 5/4 are in separate categories.

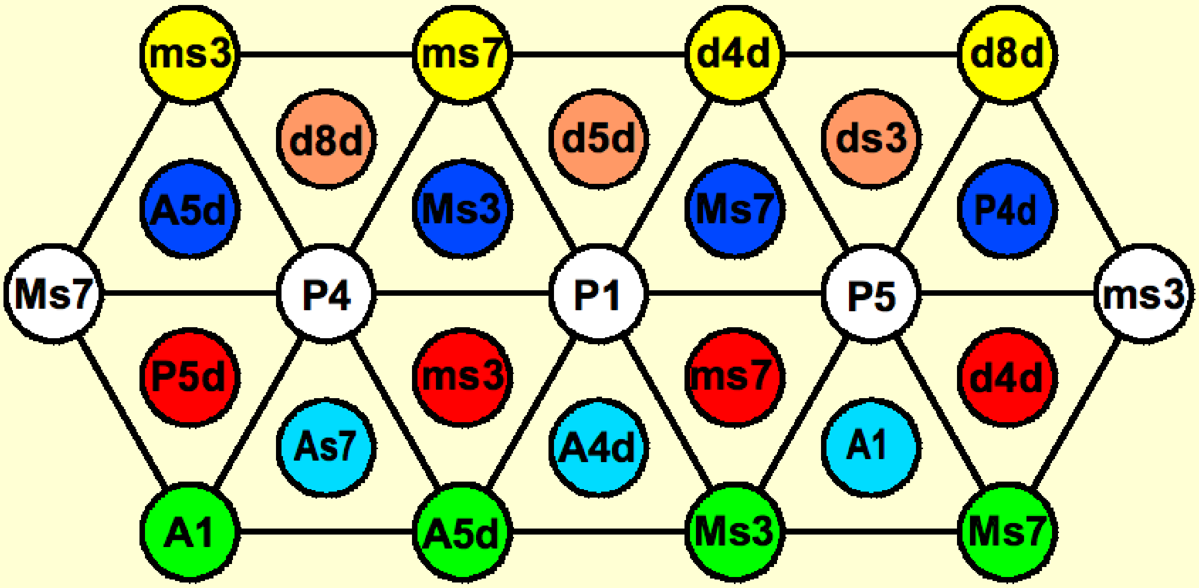
The 12 + 7 system is fifthward, but the 12 + 5 one is fourthward. Therefore large, which means increased by 256/243 = 90¢, is now on the fourthward side. Thus 9/8 is a wa sub3rd and 32/27 is a large wa sub3rd. 16/9 is a wa sub7th and 27/16 is a small wa sub7th. Recall from chapter 3.2 that the magnitude-chain runs 7ss – 7s – 7 central – 7L – 7LL. Pentatonically, that becomes 5LL – 5L – 5 central – 5s – 5ss.

Looking at any one line in table 5.3.2, every major heptatonic interval becomes either minor or diminished, because minor is now fifthward from perfect. The quality-chain runs 5AA – 5A – 2M – 3P – 2m – 5d – 5dd, or

AA – As3 As7 A4d A1 A5d – Ms3 Ms7 – P4d P1 P5d – ms3 ms7 – d4d d8d d5d ds3 ds7 – dd

In the next diagram you can trace this quality-chain running from the wa row to the yo row to the ruyo row:

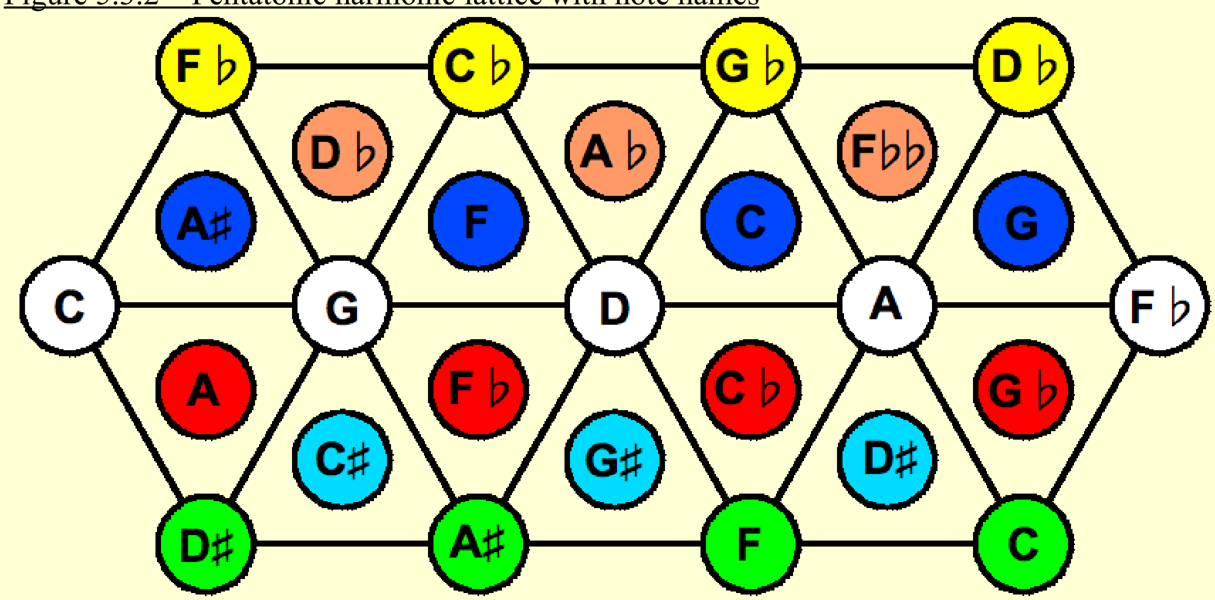
Figure 5.3.1 – Pentatonic harmonic lattice with qualities and degrees



Standard terminology and notation can be used pentatonically, with some modifications. There are only five note names. Of the usual seven letters, omit two. The remaining letters should make a chain of fifths, so there are 3 options: F–C–G–D–A, or C–G–D–A–E, or G–D–A–E–B. Omitting E & B, the chain of 5ths runs F♯ – C♯ – G♯ – D♯ – A♯ – F – C – G – D – A – F♭ – C♭ – G♭ – D♭ – A♭. The sharps are on the fourthward side because this is a fourthward system.

In alt-tuner's keyboard screen, set the number of names to 5, and arrange the keyboard to run A * * C * D * * F * G *. Modulate to D. The harmonic lattice will look like this:

Figure 5.3.2 – Pentatonic harmonic lattice with note names



The staff notation's key signature shows which two notes have been omitted.

Figure 5.3.3 – The D pentatonic chromatic scale

The 5-note framework has fewer negative commas than the 7-note one. As a result, there's fewer naming difficulties. For example in table 5.3.2, the pentatonic column has less overlap between the 4ths and 5ths.

Table 5.3.3 – Nearby yaza commas in both the 5-note and 7-note frameworks

ratio	cents	name	heptatonic		pentatonic	
81/80	22¢	gu comma	g1	perf unison	g1	perf unison
64/63	27¢	ru comma	r1	perf unison	r1	perf unison
$3^{12} / 2^{19}$	23¢	wa comma	LLw-2	desc dim 2nd	ssws3	double-dim sub3rd
49/48	36¢	zozo comma	zz2	min 2nd	zz1	aug unison
50/49	35¢	double ruyo comma	rryy-2	desc dim 2nd	rryys3	double-dim sub3rd
36/35	49¢	rugu comma	rg1	perf unison	rg1	perf unison
225/224	7.7¢	ruyoyo minicomma	ryy-2	desc dim 2nd	ryys3	double-dim sub3rd
$3^8 \cdot 5 / 2^{15}$	1.95¢	yo minicomma	Ly-2	desc dim 2nd	ssys3	double-dim sub3rd
128/125	41¢	triple gu comma	$g^3$2	dim 2nd	Lg^3-s3	desc double-dim sub3rd
$2^{11} / 3^4 \cdot 5^2$	19.5¢	gugu comma	sgg2	dim 2nd	Lgg-s3	desc double-dim sub3rd

The negative commas are shaded. The pentatonic negative commas are all ya. In general, pentatonicism handles za intervals better than heptatonicism, and heptatonicism does better with ya intervals.

As with the 12 + 7 system, purple, la and tha intervals are ambiguous. The ambiguity is rooted in the 12-tone sizing framework, and will arise on a standard keyboard under any naming framework. The purple 3rd can be either Ms3 or d4d, 1o4 = P4d or A4d, and 3o6 = A5d or ms7. Using ambiguous qualities, the ilo 4th is a half-aug fourthoid. For purple and tha, even the degree is ambiguous. The purple 3rd is either a half-aug sub3rd = hAs3, or an extra-dim 4thoid = xd4d. Likewise the tho 6th is either an extra-aug 5thoid = xA5d, or a half-dim sub7th = hds7. The quality-chain for ambiguous pentatonic intervals runs 5xA – 5hA – 2n – 5hd – 5xd, or

xA – hAs3 hAs7 hA4d hA1 hA5d – ns3 ns7 – hd4d hd8d hd5d hds3 hds7 – xd

In general, heptatonicism handles purple and tha intervals better than pentatonicism.

On the standard keyboard, every white key is natural, and every black key has a sharp or a flat.

C	C♯	D	D♯	E	F	F♯	G	G♯	A	A♯	B	C

To picture the keyboard pentatonically, imagine it tuned it down a semitone, so that D♯ becomes D:

D	D♯	F♭	F	G♭	G	A♭	A	A♯	C♭	C	D♭	D

Every black key is natural, every white key has a sharp or a flat. We don't have to actually tune down a semitone, we can just think of the D♯ as a really sharp D, as if we had tuned not to A-440 but to A-466.

Every conventional music term has a pentatonic counterpart. There are 7 conventional modes:

F G A B C D E (lydian)
C D E F G A B (ionian or major)
G A B C D E F (mixolydian)
D E F G A B C (dorian)
A B C D E F G (aeolian or minor)
E F G A B C D (phrygian)
B C D E F G A (locrian, the only mode lacking a perfect fifth)

Pentatonically, there are 5 modes:

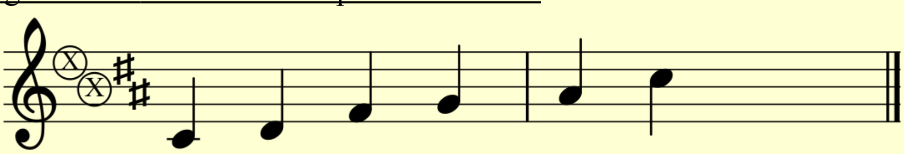
F G A C D (major pentatonic)
C D F G A (mixolydian pentatonic, or thirdless major, so-called because of the major 6th)
G A C D F (dorian pentatonic, or thirdless minor, with a minor 7th)
D F G A C (minor pentatonic)
A C D F G (fifthless pentatonic)

The minor pentatonic could actually be thought of as "penta-major", because it's formed from all the perfect and major intervals in table 5.3.1. The minor pentatonic scale in several keys, written out excluding E and B:

F G♯ A♯ C D♯
C D♯ F G A♯
G A♯ C D F
D F G A C
A C D F♭ G

Note the similarity to conventional charts showing the major scale in various keys, which progress from F major with 1 flat to A major with 3 sharps. Here the progression is from 3 sharps to 1 flat. This suggests a method for key signatures: One sharp = G minor or A♯ major, no sharps or flats = D minor or F major, one flat = A minor or C major, etc. Here's the C minor pentatonic scale written with a key signature. The two (X)'s in the key signature indicate that E and B are never used.

Figure 5.3.4 – The C minor pentatonic scale



Let's revisit table 2.5.3 and notate pentatonic scales pentatonically:

Table 5.3.4 – Pentatonic scales

	heptatonic notation	pentatonic notation
wa minor pentatonic	1, w3, 4, 5, w7	1, Lws3, 4, 5, ws7
wa major pentatonic	1, w2, Lw3, 5, w6	1, ws3, sw4d, 5, sws7
yo pentatonic	1, w/y2, y3, 5, y6	1, w/ys3, y4d, 5, ys7
gu pentatonic	1, g3, w/g4, 5, g7	1, gs3, w/g4d, 5, gs7
zo pentatonic	1, z3, w/z4, 5, z7	1, zs3, w/z4d, 5, zs7
ru pentatonic	1, w2, r3, 5, r6	1, ws3, r4d, 5, rs7
yo zo pentatonic	1, w2, y3, 5, z7	1, ws3, y4d, 5, zs7
zo yo pentatonic	1, z3, 4, 5, y6	1, zs3, 4, 5, ys7
gu ru pentatonic	1, g3, 4, 5, r6	1, gs3, 4, 5, rs7
ru gu pentatonic	1, w2, r3, 5, g7	1, ws3, r4d, 5, gs7

The Wagogo people of Tanzania use the yo zo pentatonic scale exclusively and often sing parallel harmonies in it. In heptatonic notation, the intervals produced are sometimes a 3rd, sometimes a 4th and sometimes a 5th. In pentatonic notation, the intervals are all the same size:

	heptatonic notation						pentatonic notation					
upper voice:	y10	w9	w8	z7	w5	y3	Wy4d	ws10	w8d	zs7	w5	y4d
lower voice:	w8	z7	w5	y3	w2	w1	w8	zs7	w5	y4d	ws3	w1
interval	y3	r3	w4	zg5	w4	y3	y4d	r4d	w4d	zg4d	w4d	y4d

"Without You" in the instrumental chorus becomes entirely pentatonic from about 2:00 to 2:20. The bass and the piano stick strictly to a minor pentatonic scale.

In chapter 5.10, we'll use pentatonic notation for edos 5, 10, 15, etc. In chapter 5.11, we'll see how pentatonic notation can be used for 8-edo, 13-edo and 18-edo. But perhaps the real value of studying pentatonic music theory is realizing how much we take for granted, and how much it shapes our musical thinking!

Chapter 5.4 – The 19-tone Framework

In the last few chapters, we've explored the 12 + 7 system and the 12 + 5 system:

Table 5.4.1 – The 12-tone, 7-note and 5-note frameworks

prime	ratio	cents	keyspan	stepspan	
2	2/1 = w8	1200¢	12	7	5
3	3/2 = w5	702¢	7 (2%)	4 (-9%)	3 (8%)
5	5/4 = y3	386¢	4 (14%)	2 (-25%)	2 (39%)
7	7/4 = z7	969¢	10 (31%)	6 (35%)	4 (-4%)
11	11/8 = 1o4	551¢	6 (49%)	3 (-22%)	2 (-39%)
13	13/8 = 3o6	841¢	8 (-41%)	5 (10%)	4 (50%)

Other naming frameworks are possible but not very useful. For example, consider the hexatonic framework. The wa fifth's stepspan is round (6 * 702¢ / 1200¢) = round (3.51) = either 3 or 4, both very inaccurate. The sharp works out to be six semitones.

Are there other possibilities? What other sizing frameworks are a natural fit with the 7-note framework? It turns out every seventh framework is: 12-tone, 19-tone, 26-tone, 33-tone, etc. And it just so happens that the only other sizing framework ever widely used in the West was 19-tone!

From the 15th to the 17th century, the cembalo cromatico, a "chromatic harpsichord" with 19 keys to the octave, was common in Italy. The five black keys were split in two and two more were added:

photos are from www.ostfriesischelandschaft.de/445.html

Let's use the methods we learned in chapter 5.2. The 19-tone keyspan of each rung is the closest 19-edo step. The other natural naming framework is 19-tone minus 7-note = 12-note. The entire table is easily filled in. Here's the result:

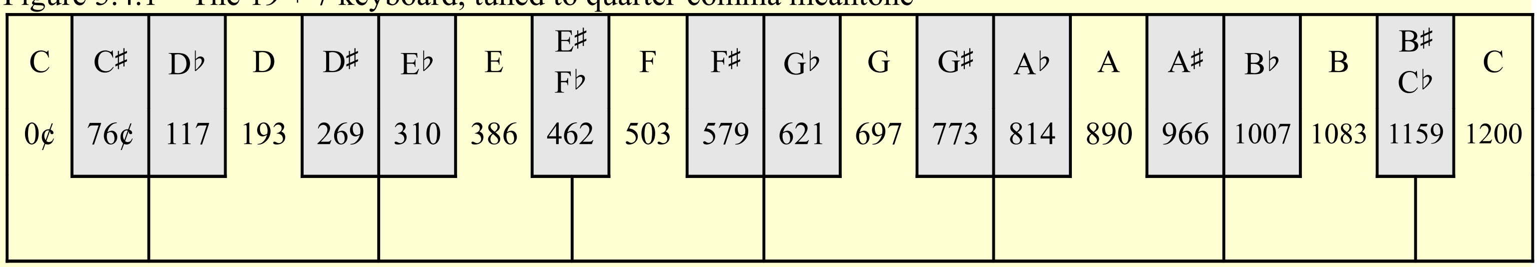
Table 5.4.2 – The 19-tone, 7-note and 12-note frameworks

prime	ratio	cents	keyspan	stepspan	
2	2/1 = w8	1200¢	19	7	12
3	3/2 = w5	702¢	11 (-11%)	4 (-9%)	7 (-2%)
5	5/4 = y3	386¢	6 (-12%)	2 (-25%)	4 (14%)
7	7/4 = z7	969¢	15 (-34%)	6 (35%)	10 (31%)
11	11/8 = 1o4	551¢	9 (27%)	3 (-22%)	6 (49%)
13	13/8 = 3o6	841¢	13 (-31%)	5 (10%)	8 (-41%)

Note the reasonably high accuracy of ya JI in 19-tone. The higher primes are less accurate.

Every dual-framework system implies a keyboard layout with white and black keys. Our familiar seven-white-five-black layout is implied by our 12 + 7 system. The layout of the cembalo cromatico keyboard is implied by the 19 + 7 system. From C we move 11 keys to the right to G. From G we move 19 - 11 = 8 keys to the left to D. From D to A is 11 keys right, from A to E is 8 keys left, etc. This inevitably produces this layout:

Figure 5.4.1 – The 19 + 7 keyboard, tuned to quarter-comma meantone



The D♯ key can also be an E♭♭ key. Double-flats will come in handy later for zo and zogu. The cents are from the quarter-comma meantone tuning common at the time. The B♯ / C♭ key is tuned as B♯ here, C♭ would be 1124¢. The E♯ / F♭ key is tuned as E♯. The complete chain of fifths is:

G♭– D♭– A♭– E♭– B♭– F – C – G – D – A – E – B – F♯ – C♯ – G♯ – D♯ – A♯ – E♯ – B♯

The 19-tone framework misses z7 by 34%. But that only means 19-edo misses z7 by 34%. A 19-tone keyboard isn't necessarily tuned to 19-edo any more than a 12-tone keyboard is tuned to 12-ET. In the meantone tuning, note how the C – A♯ aug 6th is an extremely accurate z7, only 3¢ flat. The F, C, G and D chords, as well as every flat-key chord, have a z7 available. Had this keyboard caught on, the transition to yaza music would be much easier!

The notation for 19 + 7 looks exactly like the notation for 12 + 7. Here are all 19 tones:

Figure 5.4.2 – The 19 + 7 notation

Figure 5.4.3 – The 19-tone guitar fretboard (asterisks indicate frets marked with dots)

E		E♯ / F♭	**F**	F♯	G♭	**G***	G♯	A♭	**A ***
B		B♯ / C♭	**C**	C♯	D♭	**D ***	D♯	E♭	**E ***
G		G♯	A♭	**A**	A♯	B♭*	**B**	B♯ / C♭	**C ***
D		D♯	E♭	**E**	E♯ / F♭	**F ***	F♯	G♭	**G ***
A		A♯	B♭	**B**	B♯ / C♭	**C ***	C♯	D♭	**D ***
E		E♯ / F♭	**F**	F♯	G♭	**G ***	G♯	A♭	**A ***

We saw in chapter 5.2 how using polygons to compare 7-edo to 12-edo gives us perfect 4ths, 5ths and octaves, with 2nds, 3rds, 6ths and 7ths being imperfect and having major and minor forms. If we compare 7-edo to 19-edo, we again get perfect 4ths, 5ths and octaves. Here's the chain of fifths:

Table 5.4.3 – 19-tone keyspans of the chain of fifths

d2	d6	d3	d7	d4	d8	d5	m2	m6	m3	m7	P4	P1	P5	M2	M6	M3	M7	A4	A1	A5	A2	A6	A3	A7
1	12	4	15	7	18	10	2	13	5	16	8	0	11	3	14	6	17	9	1	12	4	15	7	18

We use a chain of fifths, and not 2nds or 3rds, because the fifth is perfect, and is the natural generator for the notation. Rearranging all the intervals by keyspan makes this table:

Table 5.4.4 – 19-tone keyspans as 7-note stepspans

P1	A1 d2	m2	M2	A2 d3	m3	M3	A3 d4	P4	A4 dd5	AA4 d 5	P5	A5 d6	m6	M6	A6 d7	m7	M7	A7 d8	p8
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0¢	63¢	126	189	253	316	379	442	505	568	632	695	758	821	884	947	1011	1074	1137	1200

Aug and dim are gray. The cents are from 19-edo and represent the <u>average</u> size of each interval of that keyspan.

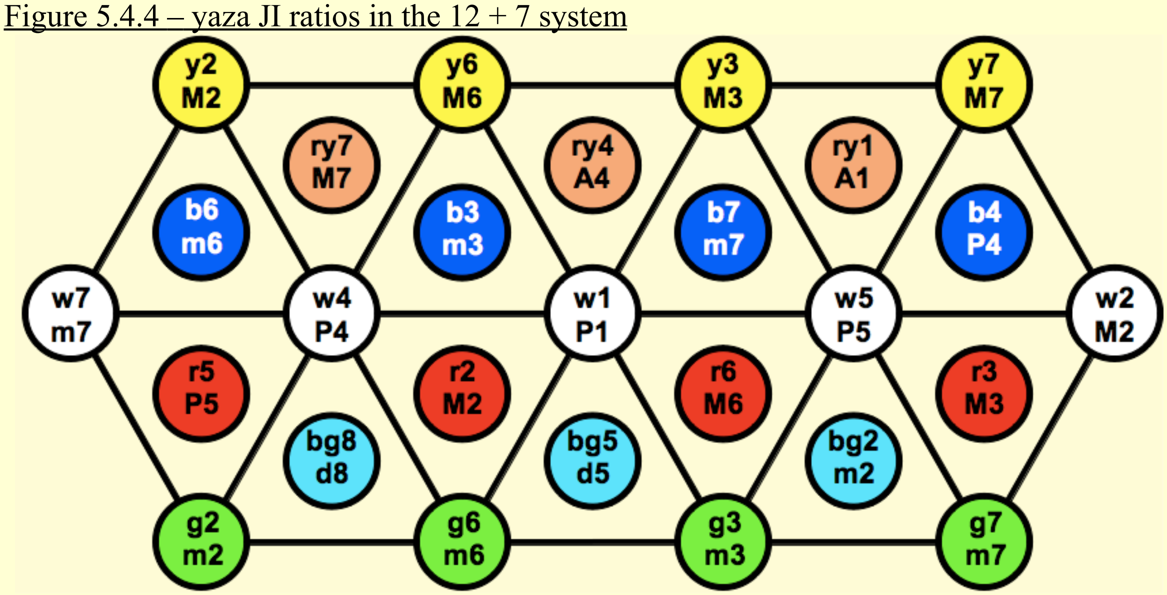
Microtonalists often notate intervals in 19-edo by their keyspan. A perfect fifth is written 11\19. The backwards slash differentiates this from the ratio 11/19. But musicians don't like to count semitones even when there are only 12 of them, let alone 19. They prefer to use qualities and degrees to locate intervals on the keyboard. They would prefer to use the first row of Table 5.4.4 to name 19-edo intervals, not the second row.

Almost all of the familiar interval arithmetic of the 12 + 7 system still applies to 19 + 7. A major 3rd plus a minor 3rd still equals a perfect fifth. The note a major 3rd above C is still E. From D to F is still a minor 3rd. The only difference arises when restating augmented or diminished intervals as other intervals. In 12-tone, M3 + M3 = A5 = m6. But in 19-tone, M3 + M3 = A5 = d6.

12-tone chord names and scale names still apply. The minor chord still has a m3 and a P5. The Dmin7 chord is still D, F, A and C. There are many new chords, such as the dim-three triad C – E♭♭ – G, the aug-three or sharp-three triad C – E♯ – G, and the double-flat-seven tetrad C – E – G – B♭♭. The E minor scale is still E F♯ G A B C D E. There are many new scales such as dim-minor, A B C♭ D E F♭ G♭ A. Chord names and scale names are covered in chapter 5.8.

Every dual-framework system automatically assigns a keyspan and stepspan to every JI ratio by adding up the keyspans and the stepspans of its component rungs. Let's review how the 12 + 7 system assigns qualities and degrees:

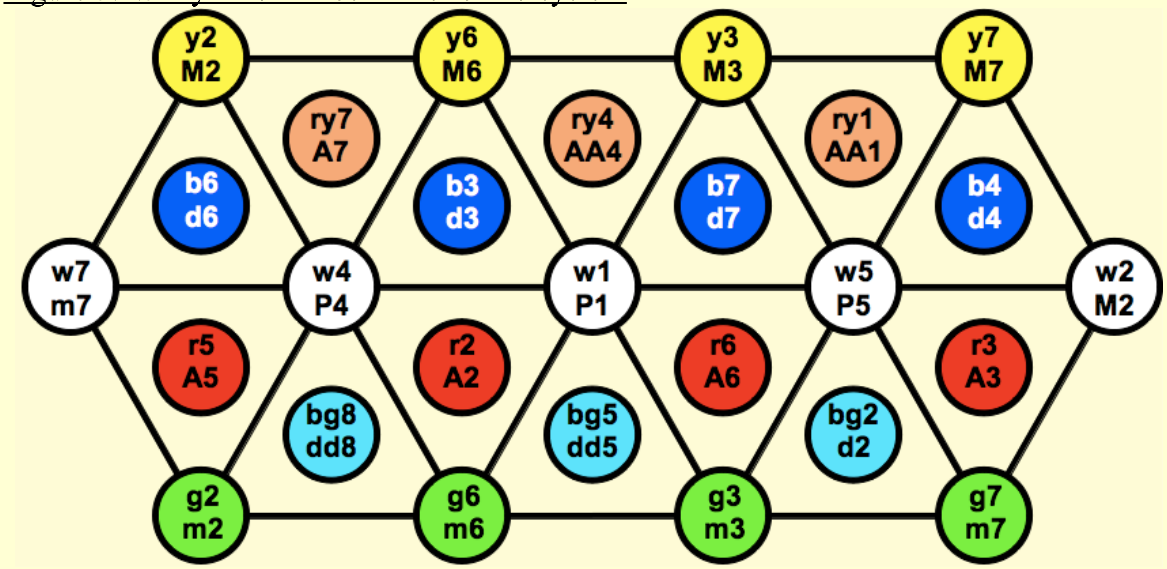
<u>Figure 5.4.4 – yaza JI ratios in the 12 + 7 system</u>



Each row or color tends to have a certain quality. The yo row is all major, the gu one is minor, the zo one is mostly minor, and the ru one is mostly major. This makes it easy to find ratios on the keyboard: the zo 3rd is a minor 3rd, the ru 2nd is a major 2nd, etc. (Of course, if you extend the lattice far enough, each row will run through many qualities, as we saw with the quality-chains in chapter 3.2.)

The 19 + 7 system will have the same degrees as 12 + 7 because the naming framework is the same. We can use table 5.4.2 and table 5.4.4 to calculate the qualities. For example, 7/5 has a keyspan of 9. A 9-edostep fifth is two less than the perfect 11-edostep fifth, so it's a double-diminished fifth.

Figure 5.4.5 – yaza JI ratios in the 19 + 7 system



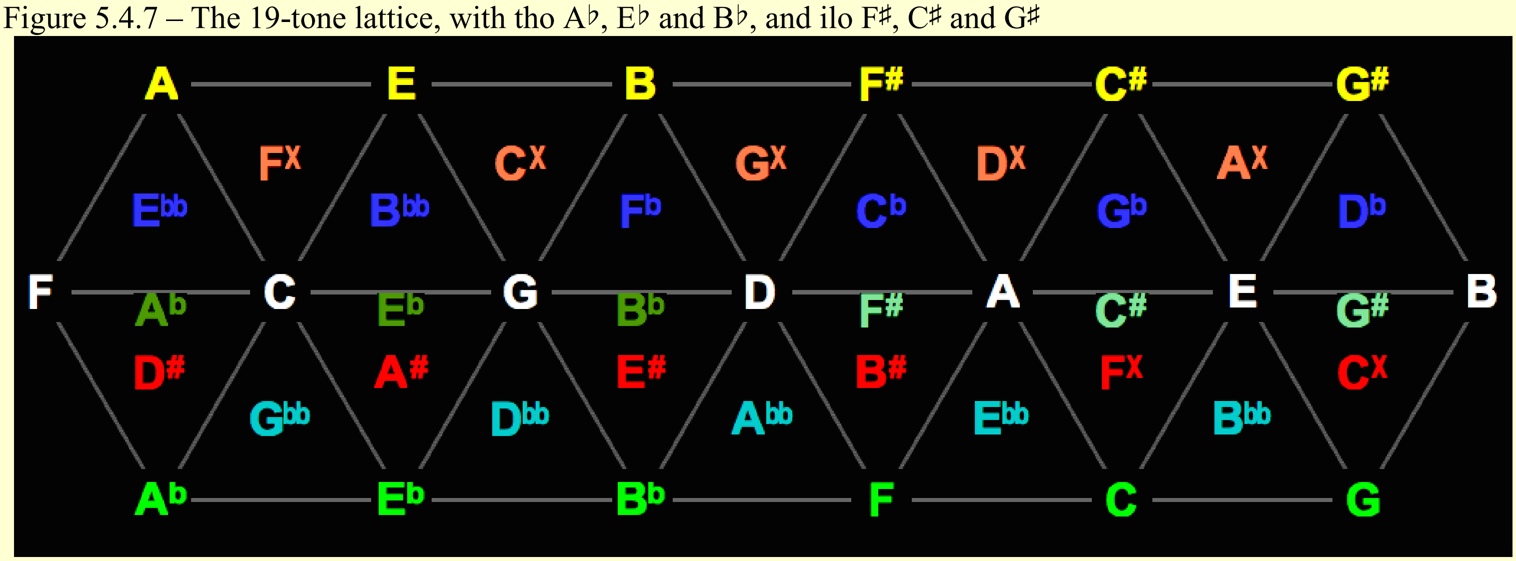
The wa, yo and gu rows are unchanged from 12 + 7. The zo row is diminished, and the ru one is augmented. The net effect of 19-tone is to separate the yaza ratios from the ya ones, allowing simultaneous access. The red-yellow-green-blue rainbow from table 2.1.1 is now spread out over four keys. Because aug and dim share a key, the rainbow overlaps the neighboring rainbows, so you can only access three bands at once. You must choose between ru and zo. Choosing zo suggests this sort of tuning:

Figure 5.4.6 – JI tuning example for the 19 + 7 keyboard

C	D♭♭	D♭	D	E♭♭	E♭	E	F♭	F	G♭♭	G♭	G	A♭♭	A♭	A	B♭♭	B♭	B	C♭	C
w1	z2	g2	w2	z3	g3	y3	z4	w4	zg5	g5	w5	z6	g6	y6	z7	g7	y7	z8	w8

Double flats are used to make the naming consistent. For example, the zogu 5th is the sum of a gu 3rd and a zo 3rd, which are a minor 3rd and a dim 3rd. Since two minor 3rds add up to make a dim 5th, these add up to make a double-dim 5th, written G♭♭ . Here's what the lattice on D looks like:

Figure 5.4.7 – The 19-tone lattice, with tho A♭, E♭ and B♭, and ilo F♯, C♯ and G♯



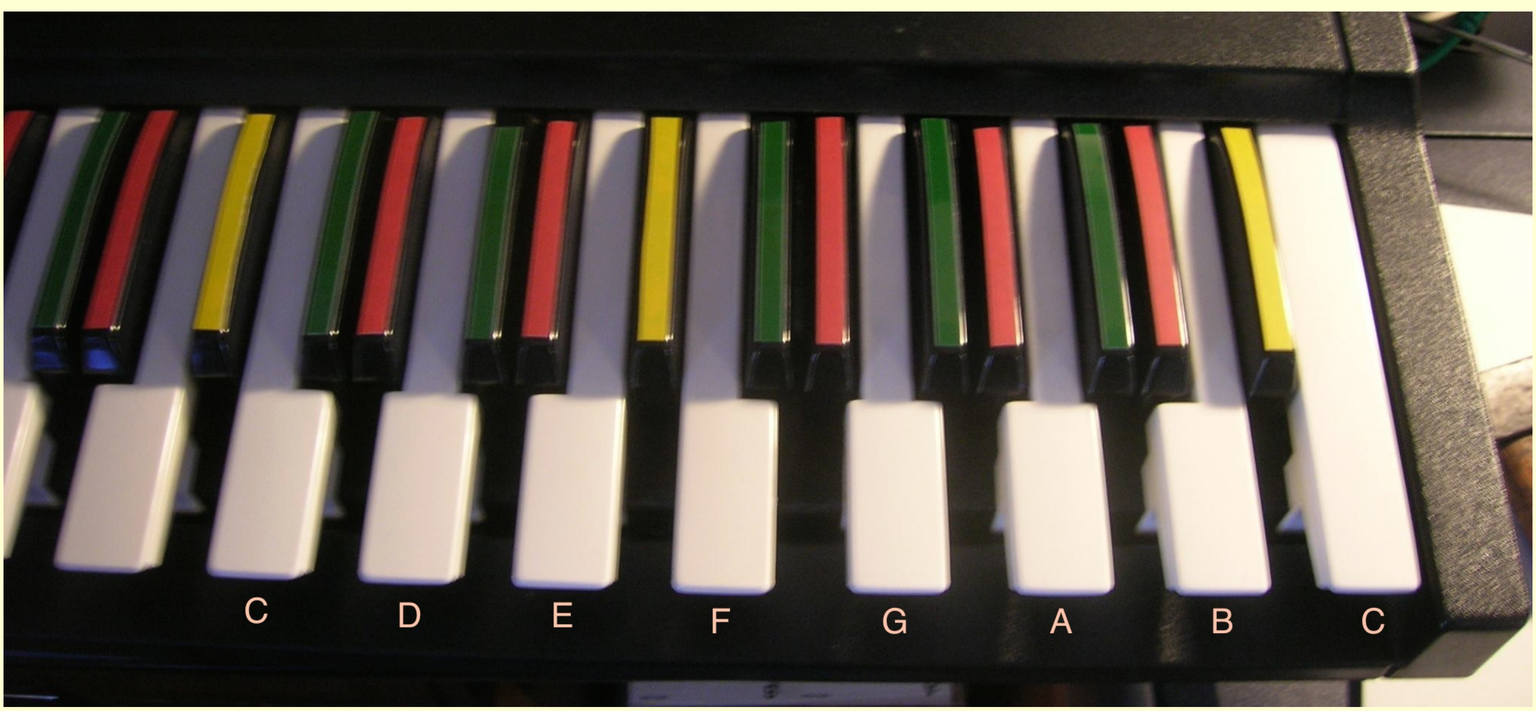
A just Dz chord is still written as wD – zF – wA, even though it uses the F♭ key. The "z" color accidental reduces the keyspan by one, just as the ♭ accidental does. Using zF♭ isn't recommended, because the "z" symbol would no longer be associated with the ru comma 64/63. The commas in table 3.7.2 are the foundation of color notation. While alternate mappings/commas are possible, they shouldn't be tied to the choice of sizing framework.

Why bother with all this? Because nowadays with cheap midi keyboards, making a keyboard with more than 12 tones per octave is quite feasible. It gives you access to more notes, almost always a good thing with microtonal music. Analyzing frameworks like this tells you three things: which frameworks and systems are easy to notate, how to lay out your white and black keys in that system, and how to name your keys.

Fabrizio Fulvio Fausto Fiale is an Italian pianist who has taken apart a midi keyboard and replaced some of the white keys with black ones to make a 19-tone keyboard. Here's what it looks like:

Fabrizio Fulvio Fausto Fiale's 19-tone keyboard:

The colors of the keys are unrelated to color notation. A close-up of one octave:

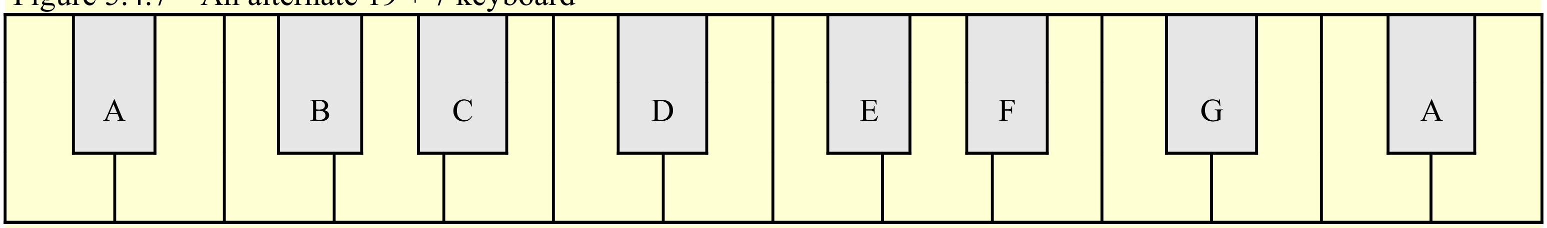


If you look at the backs of the keys in the next picture, you'll see that where the keys are mounted, the white and black keys are of equal width. In 19-tone, the fifth's keyspan is 11, so a fifth is almost the same physical size that an octave is on a conventional keyboard.

A fifth on a 19-tone keyboard:

The large gap between the white keys can be avoided by swapping white and black:

Figure 5.4.7 – An alternate 19 + 7 keyboard



Of course, you don't have to go to these lengths to explore the 19-tone framework. You can set up a standard keyboard to play a 19-tone scale. On a 61 key keyboard running from C2 to C7, one octave runs from C2 to G3, the next from G3 to D5, and the next from D5 to A6. Each octave has a somewhat similar layout of white and black keys:

```
C * D * E F * G * A * B C * D * E F *
G * A * B C * D * E F * G * A * B C *
D * E F * G * A * B C * D * E F * G *
A * B C
```

The effect is one of playing in sharper keys as you play higher.

Another approach is to use edo subsets, described in chapter 4.1. You can use the 12 keys of your keyboard to play a subset of 19-edo. If you use 7 note names to refer to them, then you're using a triple-framework system!

Any of the 19 keys can be the tonic of either a major scale or a minor scale. Just as conventionally each black key produces both a sharp key and a flat key (D♭ major and C♯ minor), each of the 12 black keys of 19-tone produces both, and there are 31 possible keys. However, if avoiding tonics that have double sharps and double flats, most keys have only one name. The only exceptions are the E♯/F♭ key and the B♯/C♭ key. For these two, the flat names are preferred for major keys, and the sharp ones for minor, in order to minimize double sharps and double flats in the key signature. E♯ and B♯ major, and F♭ and C♭ minor, are alternate keys that would only be used in special circumstances, for example if modulating from A♯ major to E♯ major, or from F♭ major to F♭ minor.

Table 5.4.5 – Preferred tonic names for 19-tone

major	C	C♯	D♭	D	D♯	E♭	E	F♭	F	F♯	G♭	G	G♯	A♭	A	A♯	B♭	B	C♭
minor	"	"	"	"	"	"	"	E♯	"	"	"	"	"	"	"	"	"	"	B♯

The next table lists all the key signatures, with the major key and relative minor key that each one indicates. A♯ is preferred over B♭♭, even though its key signature has more double accidentals, in order to avoid a double-flat tonic.

Table 5.4.5 – The 19-tone key signatures, in chain-of-fifths order

key signature	major key	scale	minor key	scale
𝄫 𝄫 𝄫 𝄫 ♭ ♭ ♭			(F♭ minor)	(F♭ G♭ A♭♭ B♭♭ C♭ D♭♭ E♭♭ F♭)
𝄫 𝄫 𝄫 ♭ ♭ ♭ ♭			(C♭ minor)	(C♭ D♭ E♭♭ F♭ G♭ A♭♭ B♭♭ C♭)
𝄫 𝄫 ♭ ♭ ♭ ♭ ♭			G♭ minor	G♭ A♭ B♭♭ C♭ D♭ E♭♭ F♭ G♭
𝄫 ♭ ♭ ♭ ♭ ♭ ♭	F♭ major	F♭ G♭ A♭ B♭♭ C♭ D♭ E♭ F♭	D♭ minor	D♭ E♭ F♭ G♭ A♭ B♭♭ C♭ D♭
♭ ♭ ♭ ♭ ♭ ♭ ♭	C♭ major	C♭ D♭ E♭ F♭ G♭ A♭ B♭ C♭	A♭ minor	A♭ B♭ C♭ D♭ E♭ F♭ G♭ A♭
♭ ♭ ♭ ♭ ♭ ♭	G♭ major	G♭ A♭ B♭ C♭ D♭ E♭ F G♭	E♭ minor	E♭ F G♭ A♭ B♭ C♭ D♭ E♭
♭ ♭ ♭ ♭ ♭	D♭ major	D♭ E♭ F G♭ A♭ B♭ C D♭	B♭ minor	B♭ C D♭ E♭ F G♭ A♭ B♭
♭ ♭ ♭ ♭	A♭ major	A♭ B♭ C D♭ E♭ F G A♭	F minor	F G A♭ B♭ C D♭ E♭ F
♭ ♭ ♭	E♭ major	E♭ F G A♭ B♭ C D E♭	C minor	C D E♭ F G A♭ B♭ C
♭ ♭	B♭ major	B♭ C D E♭ F G A B♭	G minor	G A B♭ C D E♭ F G
♭	F major	F G A B♭ C D E F	D minor	D E F G A B♭ C D
no sharps or flats	C major	C D E F G A B C	A minor	A B C D E F G A
♯	G major	G A B C D E F♯ G	E minor	E F♯ G A B C D E
♯ ♯	D major	D E F♯ G A B C♯ D	B minor	B C♯ D E F♯ G A B
♯ ♯ ♯	A major	A B C♯ D E F♯ G♯ A	F♯ minor	F♯ G♯ A B C♯ D E F♯
♯ ♯ ♯ ♯	E major	E F♯ G♯ A B C♯ D♯ E	C♯ minor	C♯ D♯ E F♯ G♯ A B C♯
♯ ♯ ♯ ♯ ♯	B major	B C♯ D♯ E F♯ G♯ A♯ B	G♯ minor	G♯ A♯ B C♯ D♯ E♯ F G♯
♯ ♯ ♯ ♯ ♯ ♯	F♯ major	F♯ G♯ A♯ B C♯ D♯ E♯ F♯	D♯ minor	D♯ E♯ F♯ G♯ A♯ B C♯ D♯
♯ ♯ ♯ ♯ ♯ ♯ ♯	C♯ major	C♯ D♯ E♯ F♯ G♯ A♯ B♯ C♯	A♯ minor	A♯ B♯ C♯ D♯ E♯ F♯ G♯ A♯
X ♯ ♯ ♯ ♯ ♯ ♯	G♯ major	G♯ A♯ B♯ C♯ D♯ E♯ F𝄪 G♯	E♯ minor	E♯ F𝄪 G♯ A♯ B♯ C♯ D♯ E♯
X X ♯ ♯ ♯ ♯ ♯	D♯ major	D♯ E♯ F𝄪 G♯ A♯ B♯ C𝄪 D♯	B♯ minor	B♯ C𝄪 D♯ E♯ F𝄪 G♯ A♯ B♯
X X X ♯ ♯ ♯ ♯	A♯ major	A♯ B♯ C𝄪 D♯ E♯ F𝄪 G𝄪 A♯		
X X X X ♯ ♯ ♯	(E♯ major)	(E♯ F𝄪 G𝄪 A♯ B♯ C𝄪 D𝄪 E♯)		
X X X X X ♯ ♯	(B♯ major)	(B♯ C𝄪 D𝄪 E♯ F𝄪 G𝄪 A♯ B♯)		

The next table has the same information, with the keys sorted in melodic order. Each major key is on the same row as its relative minor (e.g. C major and A minor). Relative majors and minors have the same key signature, with one exception. A♯ major and G♭ minor have different key signatures, to avoid a tonic with a double sharp (Fx minor) or a double flat (B♭♭ major).

Table 5.4.6 – The 19-tone key signatures, in melodic order

major key	scale	key signature	minor key	scale
C major	C D E F G A B C	no sharps or flats	A minor	A B C D E F G A
C♯ major	C♯ D♯ E♯ F♯ G♯ A♯ B♯ C♯	♯ ♯ ♯ ♯ ♯ ♯ ♯	A♯ minor	A♯ B♯ C♯ D♯ E♯ F♯ G♯ A♯
D♭ major	D♭ E♭ F G♭ A♭ B♭ C D♭	♭ ♭ ♭ ♭ ♭	B♭ minor	B♭ C D♭ E♭ F G♭ A♭ B♭
D major	D E F♯ G A B C♯ D	♯ ♯	B minor	B C♯ D E F♯ G A B
D♯ major	D♯ E♯ Fx G♯ A♯ B♯ Cx D♯	X X ♯ ♯ ♯ ♯ ♯ (𝄫 𝄫 𝄫 ♭ ♭ ♭ ♭)	B♯ minor (C♭ minor)	B♯ Cx D♯ E♯ Fx G♯ A♯ B♯ (C♭ D♭ E♭♭ F♭ G♭ A♭♭ B♭♭ C♭)
E♭ major	E♭ F G A♭ B♭ C D E♭	♭ ♭ ♭	C minor	C D E♭ F G A♭ B♭ C
E major	E F♯ G♯ A B C♯ D♯ E	♯ ♯ ♯ ♯	C♯ minor	C♯ D♯ E F♯ G♯ A B C♯
(E♯ major) F♭ major	(E♯ Fx Gx A♯ B♯ Cx Dx E♯) F♭ G♭ A♭ B♭♭ C♭ D♭ E♭ F♭	(X X X X ♯ ♯ ♯) 𝄫 ♭ ♭ ♭ ♭ ♭ ♭	D♭ minor	D♭ E♭ F♭ G♭ A♭ B♭♭ C♭ D♭
F major	F G A B♭ C D E F	♭	D minor	D E F G A B♭ C D
F♯ major	F♯ G♯ A♯ B C♯ D♯ E♯ F♯	♯ ♯ ♯ ♯ ♯ ♯	D♯ minor	D♯ E♯ F♯ G♯ A♯ B C♯ D♯
G♭ major	G♭ A♭ B♭ C♭ D♭ E♭ F G♭	♭ ♭ ♭ ♭ ♭ ♭	E♭ minor	E♭ F G♭ A♭ B♭ C♭ D♭ E♭
G major	G A B C D E F♯ G	♯	E minor	E F♯ G A B C D E
G♯ major	G♯ A♯ B♯ C♯ D♯ E♯ Fx G♯	X ♯ ♯ ♯ ♯ ♯ ♯ (𝄫 𝄫 𝄫 𝄫 ♭ ♭ ♭)	E♯ minor (F♭ minor)	E♯ Fx G♯ A♯ B♯ C♯ D♯ E♯ (F♭ G♭ A♭♭ B♭♭ C♭ D♭♭ E♭♭ F♭)
A♭ major	A♭ B♭ C D♭ E♭ F G A♭	♭ ♭ ♭ ♭	F minor	F G A♭ B♭ C D♭ E♭ F
A major	A B C♯ D E F♯ G♯ A	♯ ♯ ♯	F♯ minor	F♯ G♯ A B C♯ D E F♯
A♯ major	A♯ B♯ Cx D♯ E♯ Fx Gx A♯	X X X ♯ ♯ ♯ ♯ 𝄫 𝄫 ♭ ♭ ♭ ♭ ♭	G♭ minor	G♭ A♭ B♭♭ C♭ D♭ E♭♭ F♭ G♭
B♭ major	B♭ C D E♭ F G A B♭	♭ ♭	G minor	G A B♭ C D E♭ F G
B major	B C♯ D♯ E F♯ G♯ A♯ B	♯ ♯ ♯ ♯ ♯	G♯ minor	G♯ A♯ B C♯ D♯ E♯ F G♯
(B♯ major) C♭ major	(B♯ Cx Dx E♯ Fx Gx A♯ B♯) C♭ D♭ E♭ F♭ G♭ A♭ B♭ C♭	(X X X X X ♯ ♯) ♭ ♭ ♭ ♭ ♭ ♭ ♭	A♭ minor	A♭ B♭ C♭ D♭ E♭ F♭ G♭ A♭

Triple sharps and triple flats will be needed at times. For example, in E♭, the zg5 is B triple-flat, written B♭^{3}. If the music is atonal, the triple flat can be avoided by simply writing A. But if the music has recognizable chords, perhaps Ih7 – zIIIs6 – IVh7, triple accidentals are unavoidable. Using the chord names of chapter 5.8, this would be I(d7) – ♭♭IIIm(A6) – IV(d7). In E♭, it would be E♭(d7) – G♭♭m(A6) – A♭(d7), and the G♭♭ chord's notes would be G♭♭ – B♭^{3} – D♭♭ – E♭. Spelling B♭^{3} as A would be misleading, because the note is clearly a minor 3rd from the root. The general rule is: ensure that the relative intervals within a chord are correct, even if it creates extreme accidentals. Writing the key as Dx instead of E♭ won't help, because the Ih7 chord becomes Dx – F♯^{3} – Ax – C♯, and we've gone from a triple flat to a triple sharp.

Chapter 5.5 – The 22-tone Framework: Ups and Downs

We've seen that 19-tone is easy to notate heptatonically because 7 fifths reduced by 4 octaves adds up to one key or fret. So C♯ is right next to C, and the sharp symbol retains both its harmonic meaning (7 fifths) and its melodic meaning (one key or fret upwards). The keyboard or fretboard runs C C♯ D♭ D D♯ E♭ E etc. Conventional notation works perfectly with 19-tone as long as you remember that C♯ and D♭ are different notes. Most frameworks are not as easy to notate. For example, here's the 22-tone framework:

Table 5.5.1 – The 22-tone, 7-note and 5-note frameworks

prime	ratio	cents	keyspan	stepspan	
2	2/1 = w8	1200¢	22	7	5
3	3/2 = w5	702¢	13 (13%)	4 (-9%)	3 (8%)
5	5/4 = y3	386¢	7 (-8%)	2 (-25%)	2 (39%)
7	7/4 = z7	969¢	18 (24%)	6 (35%)	4 (-4%)
11	11/8 = 1o4	551¢	10 (-11%)	3 (-22%)	2 (-39%)
13	13/8 = 3o6	841¢	15 (-41%)	5 (10%)	4 (50%)

Using polygons as we did at the end of chapter 5.2 to compare 22-edo with 7-edo, we find that the 4th and 5th are <u>not</u> perfect, instead the 2nd and 7th are. However, we'll treat the 4th and 5th as perfect for now. More on this in chapter 5.x.

The natural fifth-based naming framework for 22-tone is pentatonic (17 note names isn't practical). One sharp = five fourths = one key or fret. The chain of fifths runs ... A♯ – E♯ – C – G – D – A – E – C♭ – G♭...

Table 5.5.2 – 22-edo with pentatonic relative notation

P1	A1	AA1 dds3	ds3	ms3	Ms3	As3	AAs3 dd4d	d4d	P4d	A4d	AA4d dd5d	d5d	P5d	A5d	AA5d dds7	ds7	ms7	Ms7	As7	AAs7 dd8d	d8d	P8d
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
0¢	55¢	109	164	218	273	327	382	436	491	545	600	655	709	764	818	873	927	982	1036	1091	1145	1200

Figure 5.5.1 – Note names in the 22 + 5 system

C	C♯	C𝄪	D♭	D	D♯	D𝄪	E♭	E	E♯	E𝄪	G𝄫	G♭	G	G♯	G𝄪	A♭	A	A♯	A𝄪	C𝄫	C♭	C

But what if we want to use heptatonic notation? Seven fifths adds up to 91 keys or frets, which reduces to 3 keys. The usual chain of fifths E♭ – B♭ – F – C – G – D – A – E – B – F♯ – C♯ etc. creates this scale:

Figure 5.5.2 – Note names in the 22 + 7 system, with numerous negative 2nds and 3rds

C	D♭	B♯	C♯	D	E♭	F♭	D♯	E	F	G♭	E♯	F♯	G	A♭	F𝄪	G♯	A	B♭	C♭	A♯	B	C

This works harmonically but not melodically. C♯ is not next to C, and B♯ – D♭ looks ascending on the page but sounds descending. Also a yo 4:5:6 chord is written C – D♯ – G, and what should be a major 3rd is an aug 2nd.

What if we abandon the chain of fifths, and work melodically, not harmonically? Let's simply spread the natural keys out evenly, and use sharps and flats normally, like this:

Figure 5.5.3 – Note names in the 22 + 7 system, with inconsistent fifths

C	C♯	D♭	D	D♯	E♭	E	E♯	F♭	F	F♯	F𝄪	G♭	G	G♯	A♭	A	A♯	B♭	B	B♯	C♭	C

Unfortunately G – D and A – E are both one key narrower than the other fifths like C – G and D – A, and sound about 50¢ flat. If your piece is in G or A, that's really confusing. A notation system should work in every key!

The problem is that the sharp and flat accidentals can no longer function both melodically and harmonically. The solution is to use a second pair of accidentals, **up** and **down**, written as ^ and v (the caret "^" and the letter "v", preferably in a sans serif font like Arial Narrow), for the melodic meaning of "sharpened/flattened by one key or fret". The sharp and flat accidentals retain their harmonic meaning of "raised/lowered by 7 fifths".

Figure 5.5.4 – Note names in the 22 + 7 system, using flats, ups and downs

C	D♭	D♭^	Dv	D	E♭	E♭^	Ev	E	F	G♭	G♭^	Gv	G	A♭	A♭^	Av	A	B♭	B♭^	Bv	B	C

The notes are spoken as "D-flat-up, D-down", etc. Now the notes run in order, and a yo 4:5:6 chord is written C – Ev – G. Ups and downs can be thought of as a <u>virtual</u> color pair which corresponds to various <u>actual</u> color pairs depending on the context. In the context of 22edo, up = gu and down = yo. In other edos, they correspond to other colors.

Alternatively, we could use sharps instead of flats:

Figure 5.5.5 – Note names in the 22 + 7 system, using sharps, ups and downs

C	C^	C♯v	C♯	D	D^	D♯v	D♯	E	F	F^	F♯v	F♯	G	G^	G♯v	G♯	A	A^	A♯v	A♯	B	C

The names change depending on the key, just like in conventional notation where F♯ in D major becomes G♭ in D♭ major. Thus the B scale would use all sharps and no flats, as above. The names also change according to context, just like conventionally when E♭ in C major becomes D♯ when Gaug is played. Thus in a D♭ yo chord , E becomes Fv.

We can use familiar 12-tone interval arithmetic to locate the note a fourth or fifth above any other note. By stacking, we can find the major 9th, major 2nd and minor 7th too. There are convenient landmarks to find your way around, built into the notation. The notation is a map of unfamiliar territory, and this map should be as easy to read as possible.

Figure 5.5.6 – The 22-tone guitar fretboard (dotted frets have asterisks)

E		**F**	F^ / G♭	F♯v / G♭^	F♯ / Gv	**G ***	G^ / A♭	G♯v / A♭^	G♯ / Av	**A ***
B		**C**	C^ / D♭	C♯v / D♭^	C♯ / Dv	**D ***	D^ / E♭	D♯v / E♭^	D♯ / Ev	**E ***
G		G^ / A♭	G♯v / A♭^	G♯ / Av	**A**	A^ / B♭	A♯v / B♭^	A♯ / Bv	**B**	**C ***
D		D^ / E♭	D♯v / E♭^	D♯ / Ev	**E**	**F ***	F^ / G♭	F♯v / G♭^	F♯ / Gv	**G ***
A		A^ / B♭	A♯v / B♭^	A♯ / Bv	**B**	**C ***	C^ / D♭	C♯v / D♭^	C♯ / Dv	**D ***
E		**F**	F^ / G♭	F♯v / G♭^	F♯ / Gv	**G ***	G^ / A♭	G♯v / A♭^	G♯ / Av	**A ***

As with 12-tone and 19-tone, an interval's quality (major, minor, perfect, aug, or dim) is defined by its position in the chain of fifths, using the quality-chain of chapter 3.2:

Table 5.5.2 – 22-tone keyspans of the chain of fifths

interval	d5	m2	m6	m3	m7	P4	P1	P5	M2	M6	M3	M7	A4
keyspan	10	1	14	5	18	9	0	13	4	17	8	21	12

Rearranging these by keyspan and filling in with ups and downs gives this table:

Table 5.5.3 – Relative notation for 22-tone

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1	m2	^m2	vM2	M2	m3	^m3	vM3	M3	4	^4 d5	vA4 ^d5	A4 v5	5	m6	^m6	vM6	M6	m7	^m7	vM7	M7	8

Ups and downs are trailing in absolute notation, but leading in relative notation. The intervals are pronounced "upminor 2nd", "downmajor 3rd", etc. For conciseness, 4ths, 5ths and 8ves are assumed to be perfect, so there is no upperfect or downperfect. Instead of ^P4, there is ^4 = "upfourth".

These intervals can change according to context. Just as a conventional 12-tone aug chord has an A5 not a m6, a 22-tone aug chord has an A5 not a vM6.

In Chapter 5.1 I described the process of internalizing Western interval arithmetic as:

> "We memorize two sequences ("diminished – minor – major – augmented" and "diminished – perfect – augmented"), learn some relationships ("the minor 6th is next to the perfect 5th"), do a little subconscious math ("perfect to augmented means one semitone wider"), and get our answer (" augmented 5th equals minor 6th")."

19-tone only requires that we learn new relationships ("the minor 6th is next to the augmented 5th"). 22-tone additionally requires us to learn two new sequences:

imperfect: ...dim, updim, downminor, minor, upminor, downmajor, major, upmajor, downaug, aug...
perfect: ...dim, updim, down, perfect, up, downaug, aug...

Interval arithmetic is mostly unchanged, with ups and downs adding up and canceling each other out as expected:

C + M3 = E
C + vM3 = Ev
Cv + M3 = Ev
Cv + vM3 = Evv = E♭^

C to E = M3
C to Ev = vM3
Cv to E = ^M3
C^ to Ev = vvM3 = ^m3

M2 + M2 = M3
M2 + vM2 = vM3
vM2 + ^M2 = M3
vM2 + vM2 = vvM3 = ^m3

Sometimes enharmonic substitutions are needed. Consider this 22-note scale:

C - D♭ - D♭^ - Dv - D - E♭ - E♭^ - Ev - E - F - G♭ - G♭^ - Gv - G - A♭ - A♭^ - Av - A - B♭ - B♭^ - Bv - B - C

Here's our fifths: C – G, D♭ – A♭, D♭^ – A♭^, Dv – Av, D – A, etc. Most fifths look like fifths and are easy to find. Except for three of them, which are spelled as downminor 6ths: B♭^ – G♭, Bv – G♭^, and B – Gv. Here's this scale's chain of 5ths:

G♭^ - D♭^ - A♭^ - E♭^ - B♭^ - G♭ - D♭ - A♭ - E♭ - B♭ - F - C - G - D - A - E - B - Gv - Dv - Av - Ev - Bv

The problem is there are a few places where the sequence of 7 letters breaks, and there are runs of 5 letters. This is the essentially pentatonic-friendly nature of 22-edo asserting itself.

Our maj 2nds are C – D, D♭ – E♭, D♭^ – E♭^, Dv – Ev, D – E, etc. No problem until we reach E♭^ – G♭, which is a major 2nd that's spelled as a downminor 3rd. There are six misspelled major 2nds, nine misspelled major 6ths, etc.

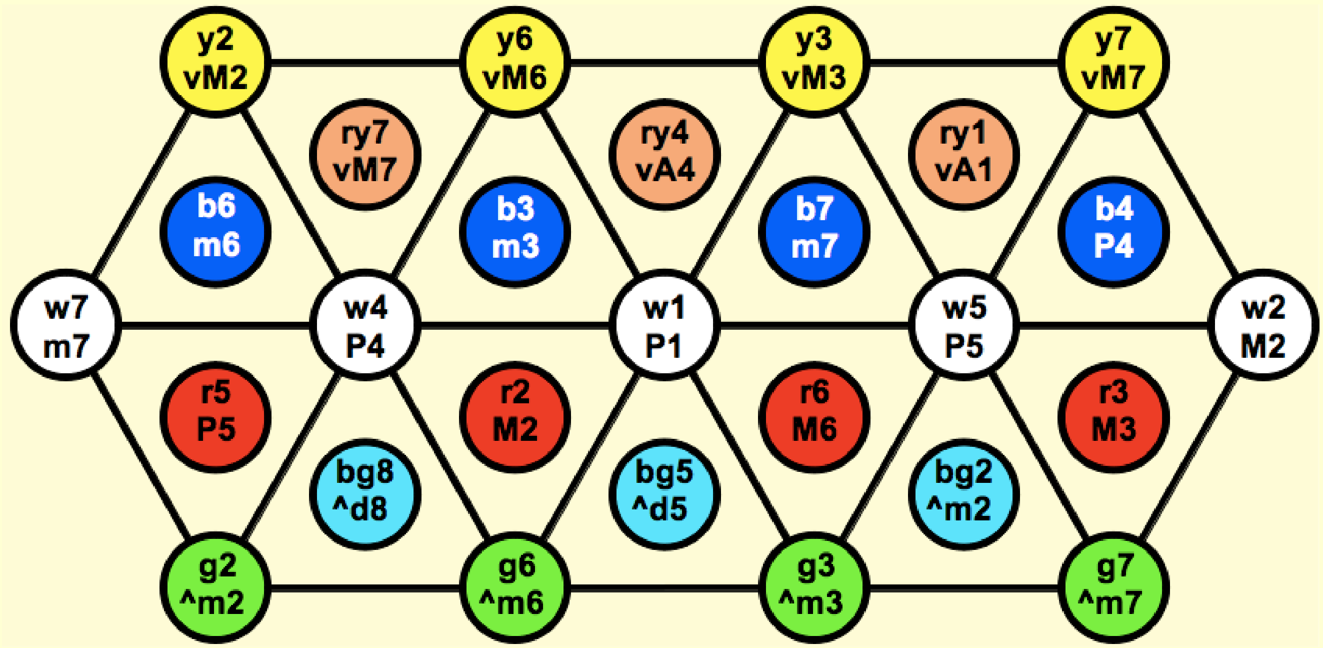
No matter what 22 note names you choose, some misspellings are inevitable. This is analogous to 12-edo, where using only 12 note names will give you a misspelled 5th, e.g. G♯ – E♭, two misspelled major 2nds, e.g. C# – E♭ and G♯ – B♭, three misspelled major 6ths, etc. The solution, as in 12-tone, is to use enharmonic substitutions as needed.

Ups and downs can be loosely related to JI: major = ru or wa, downmajor = yo, upminor = gu, minor = zo or wa. Or simply up = gu, down = yo, and neither = wa, zo or ru. These correlations are for 22-tone only, other frameworks have other correlations.

Table 5.5.2 – 22-edo JI approximations, with prime rungs 2/1, 3/1, 5/1, 7/1 and 11/1 bolded and underlined

0¢	55¢	109	164	218	273	327	382	436	491	545	600	655	709	764	818	873	927	982	1036	1091	1145	1200
P1	m2	^m2	vM2	M2	m3	^m3	vM3	M3	P4	^4	vA4, ^d5	v5	P5	m6	^m6	vM6	M6	m7	^m7	vM7	M7	P8
w1	z2	g2	y2	r2	z3	g3	**y3**	r3	w4	**1o4**	ry4, zg5	1u5	**w5**	z6	g6	y6	r6	**z7**	g7	y7	r7	**w8**

Figure 5.5.7 – yaza ratios in the 22 + 7 system



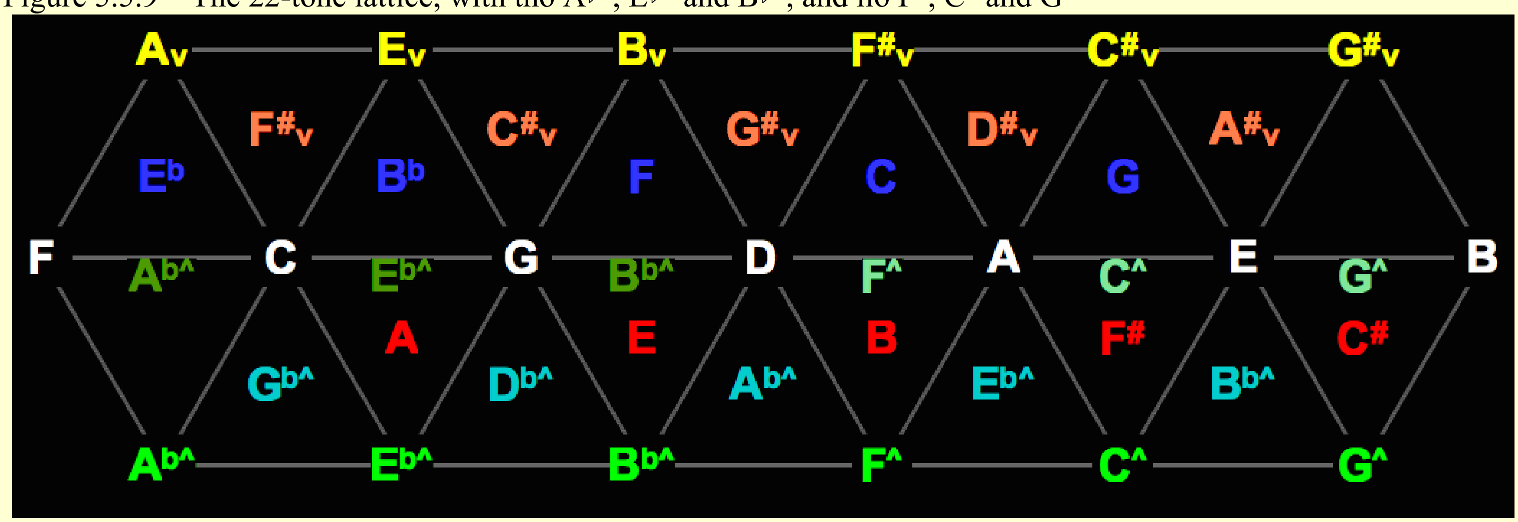
The za plane, which contains the ru, wa and zo rows, has only **plain** notes, those neither up nor down. The yo plane (ruyo and yo rows) has all the down notes. The gu plane (gu and zogu rows) has all the up notes.

A 22-tone keyboard has four 2nds, four 3rds, four 6ths and four 7ths. It's ideal for the red, yellow, green and blue four-band rainbow of Table 2.1.1:

Figure 5.5.8 – An example 22-tone JI tuning

C	D♭	D♭^	Dv	D	E♭	E♭^	Ev	E	F	G♭	G♭^	Gv	G	A♭	A♭^	Av	A	B♭	B♭^	Bv	B	C
w1	z2	g2	y2	w2	z3	g3	y3	r3	w4	g4	zg5	y5	w5	z6	g6	y6	r6	z7	g7	y7	r7	w8

Figure 5.5.9 – The 22-tone lattice, with tho A♭^, E♭^ and B♭^, and ilo F^, C^ and G^



Ups and downs produce many new chords, such as C downmajor C – Ev – G, and C up-seven C – E – G – B♭^. There are new scales too, like E upminor E – F♯ – G^ – A – B – C^ – D^ – E. Chord names and scale names are covered in chapter 5.8.

Chapter 5.6 – 22-tone Staff Notation with Ups and Downs

In staff notation, ups and downs, like sharps and flats, affect all successive notes in the same octave in the same measure. One approach is to always include both the up/down and the sharp/flat/natural for every note:

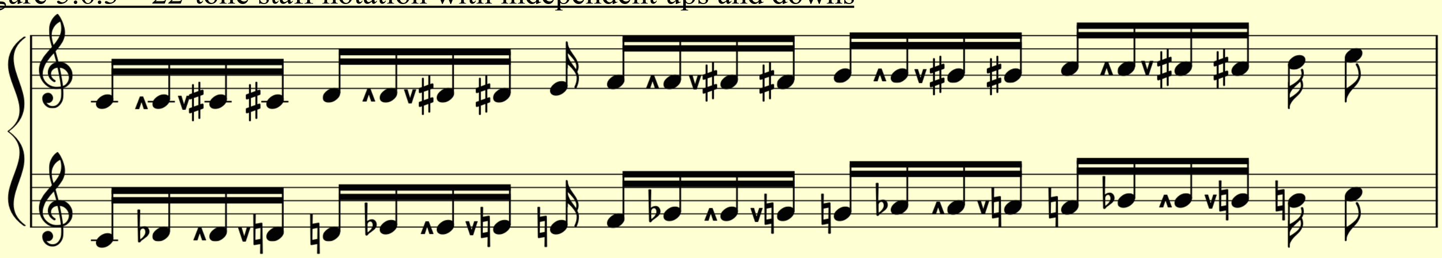
Figure 5.6.1 – 22-tone staff notation with mandatory accidentals

This approach is especially appropriate for atonal or highly chromatic music. A key signature isn't needed and is often omitted. For more tonal music, clutter can be reduced by using a key signature, and omitting accidentals implied by it.

Figure 5.6.2 – 22-tone staff notation with minimal accidentals (assumes a C major key signature)

Clutter can be reduced even more by using ups and downs independently of sharps and flats:

Figure 5.6.3 – 22-tone staff notation with independent ups and downs



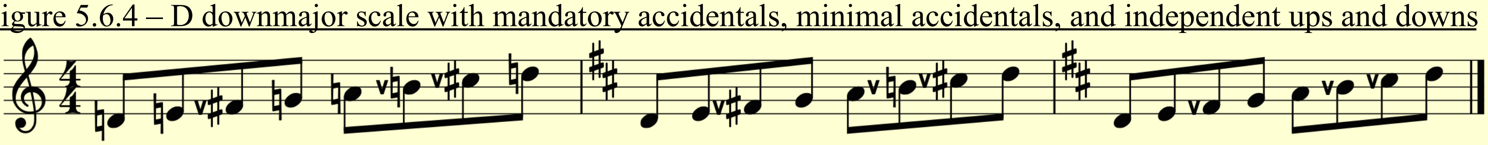
If an up or a down appears without a sharp or a flat, it does not cancel any implied sharp or flat. In the example above, the 3rd note in the lower staff is D♭^. An implied sharp or flat must be explicitly cancelled with a natural sign, as with the 4th note in the lower staff.

However, a sharp, flat or natural without an up or a down does cancel any implied up or down. In fact, this is the only way to cancel an up or a down. Thus the 4th note in the upper staff is C♯. (It would be possible to have an additional accidental, a "plain sign", analogous to the natural sign, that cancels ups and downs without affecting sharps and flats.)

Trills can always be written as a 2nd, e.g. C–D♭ or C^– D♭^ or C♯v–Dv or C♯–D.

One is free to use any of these methods, depending on the music. Here's a D downmajor scale using all three methods:

Figure 5.6.4 – D downmajor scale with mandatory accidentals, minimal accidentals, and independent ups and downs



The mandatory accidental approach is clearly overkill in this example. It's better for more atonal pieces.

Paul Erlich's composition "Tibia" for 22-edo piano (www.TallKite.com/words/Tibia.mp3) is very chromatic, but also very tonal. The chords are written out as 22-edo chords, and also as yaza JI chords, to indicate the general sound. For the JI chords, downmajor is interpreted as yo and upminor as gu. Minor is intrepreted sometimes as wa, sometimes as zo, depending on the context. The edo chord names are covered in Chapter 5.8, "Chord and Scale Names". Independent ups and downs are used. Measure 9 has an example of a down not canceling a sharp.

Figure 5.6.5 – "Tibia" in G with independent ups and downs

Tibia in G in 22edo

Paul Erlich

Tibia is in 22-edo. It uses innate comma chords, so a just intonation rendering is problematic. But when using color notation to write out JI music, a keyspan issue arises, similar to the one noted at the end of chapter 5.4 for zo and ru in 19-tone. A Cy chord could be written either C – Ev – G or as C – yE – G. The "y" accidental reduces the keyspan by one, just as the v accidental does.

In 22-tone music, any of the 22 keys can be the tonic of either a major scale or a minor scale. If the key is a white note, the choice of key signature is easy. But many of the black keys have three or four names. For example, the key midway between G and A is either A♭^, G♯v, Fx or B♭♭. How to choose a name?

One approach is to not allow ups and downs in the tonic or in the key signature. The key names are not in order, so that B♯ is a higher key than C or D♭. Some tonics will have double sharps or flats. If there are two possible names, for example B♯ and E♭♭, choose the one that minimizes double-sharps and double-flats in both the key signature and the tonic. This results in slightly different choices for major vs. minor:

Table 5.6.1 – Preferred tonic names and key signatures for 22-tone, using double sharps and double flats

major	C	D♭	E♭♭	C♯	D	E♭	F♭	D♯	E	F	G♭	E♯	F♯	G	A♭	B♭♭	G♯	A	B♭	C♭	A♯	B
minor	"	"	B♯	"	"	"	"	"	"	"	"	"	"	"	"	Fx	"	"	"	"	"	"

Table 5.6.2 – 22-tone key signatures using double sharps and double flats, in chain-of-fifths order

key signature	major key	major scale	minor key	minor scale
𝄫 𝄫 𝄫 𝄫 ♭ ♭ ♭			F♭ minor	F♭ G♭ A♭♭ B♭♭ C♭ D♭♭ E♭♭ F♭
𝄫 𝄫 𝄫 ♭ ♭ ♭ ♭	E♭♭ major	E♭♭ F♭ G♭ A♭♭ B♭♭ C♭ D♭ E♭♭	C♭ minor	C♭ D♭ E♭♭ F♭ G♭ A♭♭ B♭♭ C♭
𝄫 𝄫 ♭ ♭ ♭ ♭ ♭	B♭♭ major	B♭♭ C♭ D♭ E♭♭ F♭ G♭ A♭ B♭♭	G♭ minor	G♭ A♭ B♭♭ C♭ D♭ E♭♭ F♭ G♭
𝄫 ♭ ♭ ♭ ♭ ♭ ♭	F♭ major	F♭ G♭ A♭ B♭♭ C♭ D♭ E♭ F♭	D♭ minor	D♭ E♭ F♭ G♭ A♭ B♭♭ C♭ D♭
♭ ♭ ♭ ♭ ♭ ♭ ♭	C♭ major	C♭ D♭ E♭ F♭ G♭ A♭ B♭ C♭	A♭ minor	A♭ B♭ C♭ D♭ E♭ F♭ G♭ A♭
♭ ♭ ♭ ♭ ♭ ♭	G♭ major	G♭ A♭ B♭ C♭ D♭ E♭ F G♭	E♭ minor	E♭ F G♭ A♭ B♭ C♭ D♭ E♭
♭ ♭ ♭ ♭ ♭	D♭ major	D♭ E♭ F G♭ A♭ B♭ C D♭	B♭ minor	B♭ C D♭ E♭ F G♭ A♭ B♭
♭ ♭ ♭ ♭	A♭ major	A♭ B♭ C D♭ E♭ F G A♭	F minor	F G A♭ B♭ C D♭ E♭ F
♭ ♭ ♭	E♭ major	E♭ F G A♭ B♭ C D E♭	C minor	C D E♭ F G A♭ B♭ C

♭ ♭	B♭ major	B♭ C D E♭ F G A B♭	G minor	G A B♭ C D E♭ F G
♭	F major	F G A B♭ C D E F	D minor	D E F G A B♭ C D
no sharps or flats	C major	C D E F G A B C	A minor	A B C D E F G A
♯	G major	G A B C D E F♯ G	E minor	E F♯ G A B C D E
♯ ♯	D major	D E F♯ G A B C♯ D	B minor	B C♯ D E F♯ G A B
♯ ♯ ♯	A major	A B C♯ D E F♯ G♯ A	F♯ minor	F♯ G♯ A B C♯ D E F♯
♯ ♯ ♯ ♯	E major	E F♯ G♯ A B C♯ D♯ E	C♯ minor	C♯ D♯ E F♯ G♯ A B C♯
♯ ♯ ♯ ♯ ♯	B major	B C♯ D♯ E F♯ G♯ A♯ B	G♯ minor	G♯ A♯ B C♯ D♯ E♯ F G♯
♯ ♯ ♯ ♯ ♯ ♯	F♯ major	F♯ G♯ A♯ B C♯ D♯ E♯ F♯	D♯ minor	D♯ E♯ F♯ G♯ A♯ B C♯ D♯
♯ ♯ ♯ ♯ ♯ ♯ ♯	C♯ major	C♯ D♯ E♯ F♯ G♯ A♯ B♯ C♯	A♯ minor	A♯ B♯ C♯ D♯ E♯ F♯ G♯ A♯
X ♯ ♯ ♯ ♯ ♯ ♯	G♯ major	G♯ A♯ B♯ C♯ D♯ E♯ F× G♯	E♯ minor	E♯ F× G♯ A♯ B♯ C♯ D♯ E♯
X X ♯ ♯ ♯ ♯ ♯	D♯ major	D♯ E♯ F× G♯ A♯ B♯ C× D♯	B♯ minor	B♯ C× D♯ E♯ F× G♯ A♯ B♯
X X X ♯ ♯ ♯ ♯	A♯ major	A♯ B♯ C× D♯ E♯ F× G× A♯	F× minor	F× G× A♯ B♯ C× D♯ E♯ F×
X X X X ♯ ♯ ♯	E♯ major	E♯ F× G× A♯ B♯ C× D× E♯		

Table 5.6.3 – 22-tone key signatures using double sharps and double flats, in melodic order

major key	major scale	key signature	minor key	minor scale
C major	C D E F G A B C	no sharps or flats	A minor	A B C D E F G A
D♭ major	D♭ E♭ F G♭ A♭ B♭ C D♭	♭ ♭ ♭ ♭ ♭	B♭ minor	B♭ C D♭ E♭ F G♭ A♭ B♭
E♭♭ major	E♭♭ F♭ G♭ A♭♭ B♭♭ C♭ D♭ E♭♭	𝄫 𝄫 𝄫 ♭ ♭ ♭ ♭	C♭ minor	C♭ D♭ E♭♭ F♭ G♭ A♭♭ B♭♭ C♭
C♯ major	C♯ D♯ E♯ Fx G♯ A♯ B♯ C♯	♯ ♯ ♯ ♯ ♯ ♯ ♯	A♯ minor	A♯ B♯ C♯ D♯ E♯ Fx G♯ A♯
D major	D E F♯ G A B C♯ D	♯ ♯	B minor	B C♯ D E F♯ G A B
E♭ major	E♭ F G A♭ B♭ C D E♭	♭ ♭ ♭	C minor	C D E♭ F G A♭ B♭ C
F♭ major	F♭ G♭ A♭ B♭♭ C D♭ E♭ F♭	𝄫 ♭ ♭ ♭ ♭ ♭ ♭	D♭ minor	D♭ E♭ F♭ G♭ A♭ B♭♭ C D♭
D♯ major	D♯ E♯ Fx G♯ A♯ B♯ Cx D♯	X X ♯ ♯ ♯ ♯ ♯	B♯ minor	D♯ E♯ Fx G♯ A♯ B♯ Cx D♯
E major	E F♯ G♯ A B C♯ D♯ E	♯ ♯ ♯ ♯	C♯ minor	C♯ D♯ E F♯ G♯ A B C♯
F major	F G A B♭ C D E F	♭	D minor	D E F G A B♭ C D
G♭ major	G♭ A♭ B♭ C♭ D♭ E♭ F G♭	♭ ♭ ♭ ♭ ♭ ♭	E♭ minor	E♭ F G♭ A♭ B♭ C♭ D♭ E♭
E♯ major	E♯ Fx Gx A♯ B♯ Cx Dx E♯	X X X X ♯ ♯ ♯ 𝄫 𝄫 𝄫 𝄫 ♭ ♭ ♭	 F♭ minor	 F♭ G♭ A♭♭ B♭♭ C♭ D♭♭ E♭♭ F♭
F♯ major	F♯ G♯ A♯ B C♯ D♯ E♯ F♯	♯ ♯ ♯ ♯ ♯ ♯	D♯ minor	D♯ E♯ F♯ G♯ A♯ B C♯ D♯
G major	G A B C D E F♯ G	♯	E minor	E F♯ G A B C D E
A♭ major	A♭ B♭ C D♭ E♭ F G A♭	♭ ♭ ♭ ♭	F minor	F G A♭ B♭ C D♭ E♭ F
B♭♭ major	B♭♭ C♭ D♭ E♭♭ F♭ G♭ A♭ B♭♭	𝄫 𝄫 ♭ ♭ ♭ ♭ ♭	G♭ minor	G♭ A♭ B♭♭ C♭ D♭ E♭♭ F♭ G♭
G♯ major	G♯ A♯ B♯ C♯ D♯ E♯ Fx G♯	X ♯ ♯ ♯ ♯ ♯ ♯	E♯ minor	E♯ Fx G♯ A♯ B♯ C♯ D♯ E♯
A major	A B C♯ D E F♯ G♯ A	♯ ♯ ♯	F♯ minor	F♯ G A B C♯ D E F♯
B♭ major	B♭ C D E♭ F G A B♭	♭ ♭	G minor	G A B♭ C D E♭ F G
C♭ major	C♭ D♭ E♭ F♭ G♭ A♭ B♭ C♭	♭ ♭ ♭ ♭ ♭ ♭ ♭	A♭ minor	A♭ B♭ C♭ D♭ E♭ F♭ G♭ A♭
A♯ major	A♯ B♯ Cx D♯ E♯ Fx Gx A♯	X X X ♯ ♯ ♯ ♯	Fx minor	Fx Gx A♯ B♯ Cx D♯ E♯ Fx
B major	B C♯ D♯ E F♯ G♯ A♯ B	♯ ♯ ♯ ♯ ♯	G♯ minor	G♯ A♯ B C♯ D♯ E F♯ G♯

A disadvantage of these key names is that all the "versions" (flat, sharp, etc.) of any particular note are very far-flung. D♭♭ and Dx are almost a fifth apart, and don't feel like different versions of the same note. The three black keys between A and B, which do feel like different versions of the same note, can be notated as some version of G, A, B, C or D. The connection between a note in all its versions and an approximate pitch range is broken.

Another approach to naming 22-tone keys is to keep this connection by ensuring that all versions of a note fall between the neighboring white keys. For example, all versions of D are contained between the C and the E keys. Each black key's name is some version of one of the two nearest white keys. For example, the black keys between C and D are only notated as some version of C or D, never as some version of B or E.

To achieve this, ups and downs are allowed in tonic names and key signatures. <u>If the tonic has an up or down, all seven notes in the scale do as well</u>. Avoid naming notes as double-sharps or double-flats. Also avoid E♯, F♭, etc.

A major scale starting on the C♯ key wouldn't be C♯ major, because that would contain E♯. Instead it is Dv major. There is much less overlap between major key names and minor key names. Alternate keys are in parentheses:

Table 5.6.4 – Preferred tonic names and key signatures for 22-tone, using ups and downs

major keys	C	D♭ (C^)	D♭^	Dv	D	E♭	E♭^	Ev	E	F	F^	F♯v G♭^	Gv	G	A♭	A♭^	Av	A	B♭	B♭^	Bv	B (Cv)
minor keys	"	C^	C♯v	C♯	"	D^	D♯v E♭^	"	"	"	"	F♯v	F♯	"	G^	G♯v	G♯ (Av)	"	B♭ (A^)	"	"	B

Major keys are mostly natural, down, upflat or flat. Likewise, minor keys are mostly natural, up, downsharp or sharp. The two keys of F♯v / G♭^ major and D♯v / E♭^ minor break the rule for naming black keys because they have either an E♯v or a C♭^. There is unfortunately no way to notate these keys and follow the rule.

An ordinary modulation by a fifth from F^ major to D♭ major would look very odd on paper. In this case, for D♭ major one might use an alternate key, C^ major. Likewise, one might use Cv major, A^ minor, or Av minor.

The key signature contains a "global" up or down that raises or lowers all seven notes, written on the staff as a circled ^ or v, and written in the table below as (^) and (v). The first and last rows are the same notes in 22-edo, G♭^ = F♯v.

Table 5.6.5 – 22-tone key signatures using ups and downs, in chain-of-fifths order

key signature	major key	major scale	minor key	minor scale
♭ ♭ ♭ ♭ ♭ ♭ (^)	G♭^ major	G♭^ A♭^ B♭^ C♭^ D♭^ E♭^ F^ G♭^	E♭^ minor	E♭^ F^ G♭^ A♭^ B♭^ C♭^ D♭^ E♭^
♭ ♭ ♭ ♭ ♭ (^)	D♭^ major	D♭^ E♭^ F^ G♭^ A♭^ B♭^ C^ D♭^	B♭^ minor	B♭^ C^ D♭^ E♭^ F^ G♭^ A♭^ B♭^
♭ ♭ ♭ ♭ (^)	A♭^ major	A♭^ B♭^ C^ D♭^ E♭^ F^ G^ A♭^	F^ minor	F^ G^ A♭^ B♭^ C^ D♭^ E♭^ F^
♭ ♭ ♭ (^)	E♭^ major	E♭^ F^ G^ A♭^ B♭^ C^ D^ E♭^	C^ minor	C^ D^ E♭^ F^ G^ A♭^ B♭^ C^
♭ ♭ (^)	B♭^ major	B♭^ C^ D^ E♭^ F^ G^ A^ B♭^	G^ minor	G^ A^ B♭^ C^ D^ E♭^ F^ G^
♭ (^)	F^ major	F^ G^ A^ B♭^ C^ D^ E^ F^	D^ minor	D^ E^ F^ G^ A^ B♭^ C^ D^
(^) ♭ ♭ ♭ ♭ ♭	(C^ major) D♭ major	(C^ D^ E^ F^ G^ A^ B^ C^) D♭ E♭ F G♭ A♭ B♭ C D♭	(A^ minor) B♭ minor	(A^ B^ C^ D^ E^ F^ G^ A^) B♭ C D♭ E♭ F G♭ A♭ B♭
♭ ♭ ♭ ♭	A♭ major	A♭ B♭ C D♭ E♭ F G A♭	F minor	F G A♭ B♭ C D♭ E♭ F
♭ ♭ ♭	E♭ major	E♭ F G A♭ B♭ C D E♭	C minor	C D E♭ F G A♭ B♭ C
♭ ♭	B♭ major	B♭ C D E♭ F G A B♭	G minor	G A B♭ C D E♭ F G
♭	F major	F G A B♭ C D E F	D minor	D E F G A B♭ C D
no sharps or flats	C major	C D E F G A B C	A minor	A B C D E F G A
♯	G major	G A B C D E F♯ G	E minor	E F♯ G A B C D E
♯ ♯	D major	D E F♯ G A B C♯ D	B minor	B C♯ D E F♯ G A B
♯ ♯ ♯	A major	A B C♯ D E F♯ G♯ A	F♯ minor	F♯ G♯ A B C♯ D E F♯
♯ ♯ ♯ ♯	E major	E F♯ G♯ A B C♯ D♯ E	C♯ minor	C♯ D♯ E F♯ G♯ A B C♯
♯ ♯ ♯ ♯ ♯ (v)	B major (Cv major)	B C♯ D♯ E F♯ G♯ A♯ B (Cv Dv Ev Fv Gv Av Bv Cv)	G♯ minor (Av minor)	G♯ A♯ B C♯ D♯ E♯ F G♯ (Av Bv Cv Dv Ev Fv Gv Av)
♯ (v)	Gv major	Gv Av Bv Cv Dv Ev F♯v Gv	Ev minor	Ev F♯v Gv Av Bv Cv Dv Ev
♯ ♯ (v)	Dv major	Dv Ev F♯v Gv Av Bv C♯v Dv	Bv minor	Bv C♯v Dv Ev F♯v Gv Av Bv
♯ ♯ ♯ (v)	Av major	Av Bv C♯v Dv Ev F♯v G♯v Av	F♯v minor	F♯v G♯v Av Bv C♯v Dv Ev F♯v
♯ ♯ ♯ ♯ (v)	Ev major	Ev F♯v G♯v Av Bv C♯v D♯v Ev	C♯v minor	C♯v D♯v Ev F♯v G♯v Av Bv C♯v
♯ ♯ ♯ ♯ ♯ (v)	Bv major	Bv C♯v D♯v Ev F♯v G♯v A♯v Bv	G♯v minor	G♯v A♯v Bv C♯v D♯v E♯v Fv G♯v
♯ ♯ ♯ ♯ ♯ ♯ (v)	F♯v major	F♯v G♯v A♯v Bv C♯v D♯v E♯v F♯v	D♯v minor	D♯v E♯v F♯v G♯v A♯v Bv C♯v D♯v

Table 5.6.6 – 22-tone key signatures using ups and downs, in melodic order

major key	major scale	key signature	minor key	minor scale
C major	C D E F G A B C	no sharps or flats	A minor	A B C D E F G A
D♭ major (C^ major)	D♭ E♭ F G♭ A♭ B♭ C D♭ (C^ D^ E^ F^ G^ A^ B^ C^)	♭ ♭ ♭ ♭ ♭ (v)	B♭ minor (A^ minor)	B♭ C D♭ E♭ F G♭ A♭ B♭ (A^ B^ C^ D^ E^ F^ G^ A^)
D♭^ major	D♭^ E♭^ F^ G♭^ A♭^ B♭^ C^ D♭^	♭ ♭ ♭ ♭ ♭ (^)	B♭^ minor	B♭^ C^ D♭^ E♭^ F^ G♭^ A♭^ B♭^
Dv major	Dv Ev F♯v Gv Av Bv C♯v Dv	♯ ♯ (v)	Bv minor	Bv C♯v Dv Ev F♯v Gv Av Bv
D major	D E F♯ G A B C♯ D	♯ ♯	B minor	B C♯ D E F♯ G A B
E♭ major	E♭ F G A♭ B♭ C D E♭	♭ ♭ ♭	C minor	C D E♭ F G A♭ B♭ C
E♭^ major	E♭^ F^ G^ A♭^ B♭^ C^ D^ E♭^	♭ ♭ ♭ (^)	C^ minor	C^ D^ E♭^ F^ G^ A♭^ B♭^ C^
Ev major	Ev F♯v G♯v Av Bv C♯v D♯v Ev	♯ ♯ ♯ ♯ (v)	C♯v minor	C♯v D♯v Ev F♯v G♯v Av Bv C♯v
E major	E F♯ G♯ A B C♯ D♯ E	♯ ♯ ♯ ♯	C♯ minor	C♯ D♯ E F♯ G♯ A B C♯
F major	F G A B♭ C D E F	♭	D minor	D E F G A B♭ C D
F^ major	F^ G^ A^ B♭^ C^ D^ E^ F^	♭ (^)	D^ minor	D^ E^ F^ G^ A^ B♭^ C^ D^
F♯v major G♭^ major	F♯v G♯v A♯v Bv C♯v D♯v E♯v F♯v G♭^ A♭^ B♭^ C♭^ D♭^ E♭^ F^ G♭^	♯ ♯ ♯ ♯ ♯ ♯ (v) ♭ ♭ ♭ ♭ ♭ ♭ (^)	D♯v minor E♭^ minor	D♯v E♯v F♯v G♯v A♯v Bv C♯v D♯v E♭^ F^ G♭^ A♭^ B♭^ C♭^ D♭^ E♭^
Gv major	Gv Av Bv Cv Dv Ev F♯v Gv	♯ (v)	Ev minor	Ev F♯v Gv Av Bv Cv Dv Ev
G major	G A B C D E F♯ G	♯	E minor	E F♯ G A B C D E
A♭ major	A♭ B♭ C D♭ E♭ F G A♭	♭ ♭ ♭ ♭	F minor	F G A♭ B♭ C D♭ E♭ F
A♭^ major	A♭^ B♭^ C^ D♭^ E♭^ F^ G^ A♭^	♭ ♭ ♭ ♭ (^)	F^ minor	F^ G^ A♭^ B♭^ C^ D♭^ E♭^ F^
Av major	Av Bv C♯v Dv Ev F♯v G♯v Av	♯ ♯ ♯ (v)	F♯v minor	F♯v G♯v Av Bv C♯v Dv Ev F♯v
A major	A B C♯ D E F♯ G♯ A	♯ ♯ ♯	F♯ minor	F♯ G♯ A B C♯ D E F♯
B♭ major	B♭ C D E♭ F G A B♭	♭ ♭	G minor	G A B♭ C D E♭ F G
B♭^ major	B♭^ C^ D^ E♭^ F^ G^ A^ B♭^	♭ ♭ (^)	G^ minor	G^ A^ B♭^ C^ D^ E♭^ F^ G^
Bv major	Bv C♯v D♯v Ev F♯v G♯v A♯v Bv	♯ ♯ ♯ ♯ ♯ (v)	G♯v minor	G♯v A♯v Bv C♯v D♯v E♯v Fv G♯v
B major (Cv major)	B C♯ D♯ E F♯ G♯ A♯ B (Cv Dv Ev Fv Gv Av Bv Cv)	♯ ♯ ♯ ♯ ♯ (v)	G♯ minor (Av minor)	G♯ A♯ B C♯ D♯ E♯ F G♯ (Av Bv Cv Dv Ev Fv Gv Av)

Both methods of assigning key signatures have their advantages. A quick comparison:

Table 5.6.7 – Preferred tonic names and key signatures for 22-tone

steps	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
cents	0¢	55¢	109	164	218	273	327	382	436	491	545	600	655	709	764	818	873	927	982	1036	1091	1145
JI ratios	w1	z2	g2	y2	r2	z3	g3	**y3**	r3	w4	**1o4**	ry4, zg5	1u5	**w5**	z6	g6	y6	r6	**z7**	g7	y7	r7
using ^/v	P1	m2	^m2	vM2	M2	m3	^m3	vM3	M3	P4	^4 d5	vA4 ^d5	A4 v5	P5	m6	^m6	vM6	M6	m7	^m7	vM7	M7
major keys	C	D♭ (C^)	D♭^	Dv	D	E♭	E♭^	Ev	E	F	F^	F♯v G♭^	Gv	G	A♭	A♭^	Av	A	B♭	B♭^	Bv	B (Cv)
minor keys	"	C^	C♯v	C♯	"	D^	D♯v E♭^	"	"	"	"	F♯v	F♯	"	G^	G♯v	G♯ (Av)	"	B♭ (A^)	"	"	B
no ^/v	P1	m2	d3	A1	M2	m3	d4	A2	M3	P4	d5	A3	A4	P5	m6	d7	A5	M6	m7	d8	A6	M7
major	C	D♭	E♭♭	C♯	D	E♭	F♭	D♯	E	F	G♭	E♯	F♯	G	A♭	B♭♭	G♯	A	B♭	C♭	A♯	B
minor	"	"	B♯	"	"	"	"	"	"	"	"	"	"	"	"	Fx	"	"	"	"	"	"

Since 22-edo's best representation of 5/4 is 7\22, and the major scale contains 8\22 instead, the major scale sounds better lowering the third, sixth and seventh. One might be tempted to use key signatures such as ♯ v♯ v♯ v♯ for E downmajor = E F♯ G♯v A B C♯v D♯v E. I don't recommend this, for two reasons. One reason is the wolf fifth this scale has from F♯ to C♯v. Every time the unaltered F and C appear together on the score, it looks like a perfect fifth but doesn't sound like one.

The other reason is the same reason people don't use key signatures to create dorian or lydian scales. The key signature's main function is to indicate the key, and a general sense of major vs. minor, but not the exact scale. Here's why: the key signature is meant to be not only readable but "speed-readable". The fewer possible key signatures there are, the more instantly recognizable they are. Packing too much information into the key signature inhibits rapid sight reading. We're already asking the musician to cope with new elements in key signatures: for the first method, either double sharps or double flats, and for the second method, either a global up or a global down. That's asking a lot, so I recommend using only the standard key signatures.

"Tibia" is written out below in the key of F♯v. This is a rather extreme example, a very chromatic yet very tonal song in a very remote key. There are many double-downs, abbreviated as "w". The first measure has A♯vv, not A^, because the F♯v chord has a vM3, not a ^^m3. This is analogous to spelling an A♯ major chord with a Cx, not a D. In measures 6 and 8, this is taken further, and double-down double-sharps are used instead of upsharps. As noted at the end of chapter 5.4, the general rule for tonal music is: ensure that the relative intervals within a chord are correct, even if it creates extreme accidentals.

This example shows the bare minimum of accidentals needed. In practice, there would be many courtesy accidentals. Here are all the notes used. The top staff follows the P1 – ^1 – vA1 – A1 – M2 – ^M2 – vA2 – A2 – M3 – P4 pattern of Figure 5.6.1, and the bottom staff follows the P1 – m2 – ^m2 – vM2 – M2 – m3 – ^m3 – vM3 – M3 – P4 pattern.

Figure 5.6.6 – 22-tone staff notation in F♯v with mandatory accidentals

Figure 5.6.7 – "Tibia" in F$^\sharp$v with independent ups and downs

ⓥ makes all seven notes default to down

Tibia in F♯v in 22edo

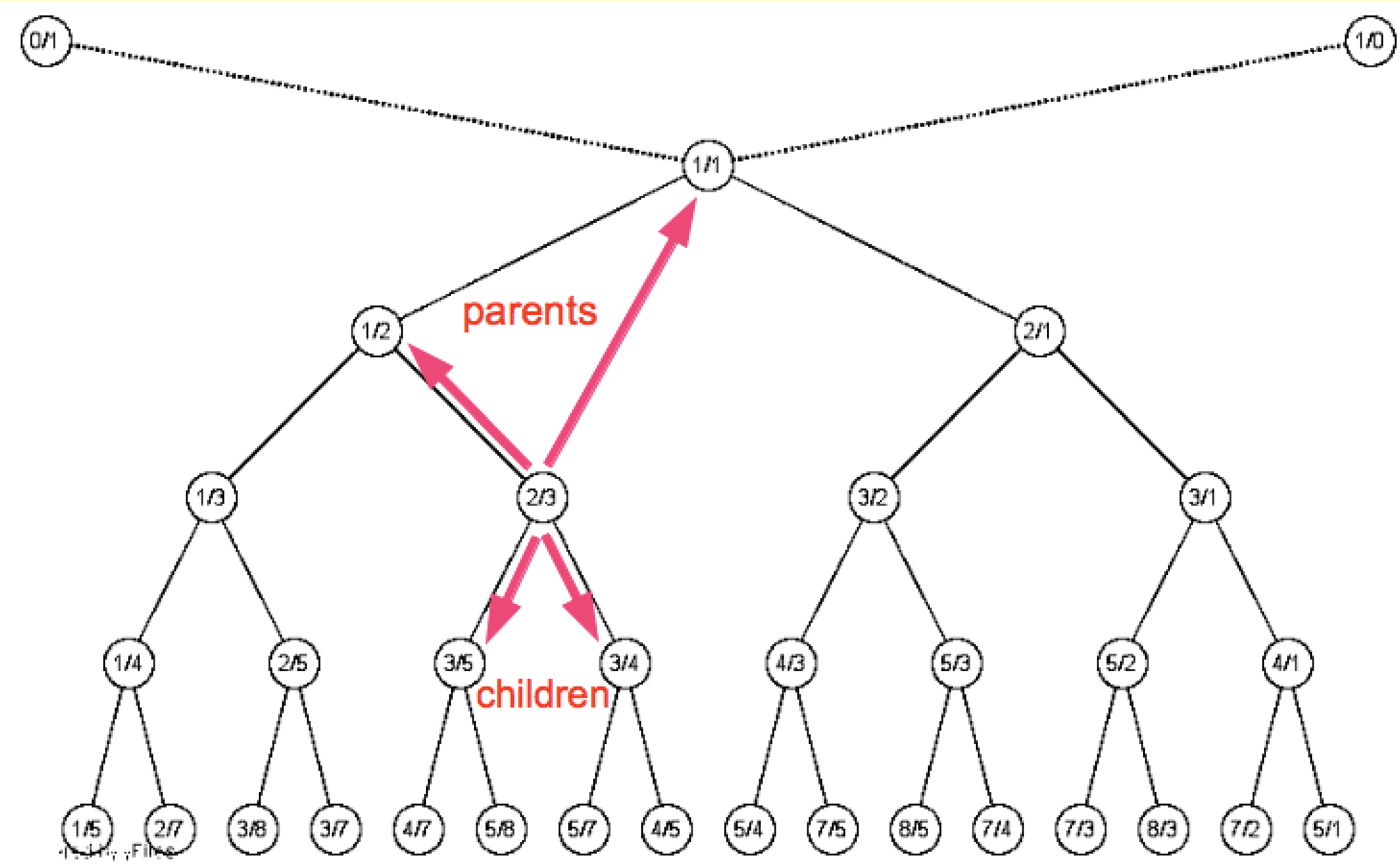
Paul Erlich

The JI chords are written out in the key of F♯. Arguably, they could be written in the key of yF♯, for more compatibility with the 22-edo notation.

Chapter 5.7 – Other Frameworks: The Scale Tree

The Stern-Brocot Tree is a beautiful numerical structure, discovered independently by Stern, a German mathematician, and Brocot, a French clockmaker. It's usually pictured like this:

Figure 5.7.1 – The Stern-Brocot Tree



Each fraction is derived from its two "parents". The parents are found by tracing the tree upwards. The parents of 2/3 are 1/2 and 1/1. The first parent is always immediately above. The other one is always on the other side. If the first one is left of the starting point, the second one will be to the right. In fact, the first upwards move that takes you right leads straight to the righthand parent.

Every fraction is the mediant of its two parents. The mediant of a/b and c/d is (a+b) / (c+d). Each fraction has two "children" directly below it. The children of 2/3 are 3/5 and 3/4. The lefthand child 3/5 is the mediant of its parents, which are 2/3 and 2/3's lefthand parent 1/2. Likewise, the righthand child 3/4 is the mediant of 2/3 and 1/1.

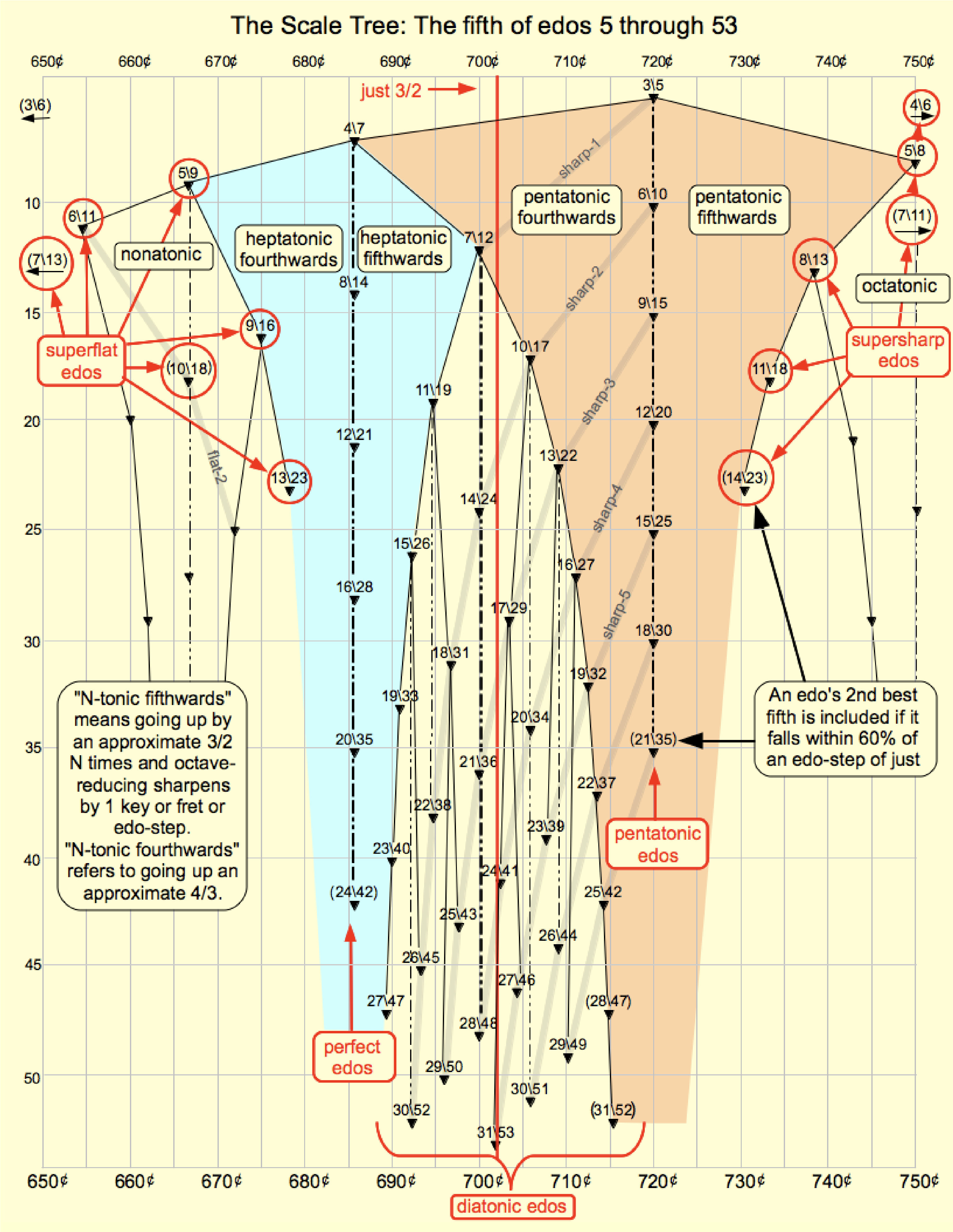
Remarkably, every fraction only appears once, and all fractions are sorted left to right by size. In the chart above, the fractions run 1/5, 1/4, 2/7, 1/3, 3/8, 2/5, etc.

The Stern-Brocot tree can be used to study frameworks. Consider the fractions to be fractions of an octave. Discard the righthand half of the tree, so that the cents range from 0¢ to 1200¢. Thus 3\5 is the 5-edo fifthoid = 720¢, and 4\7 is the 7-edo fifth = 686¢. Include reducible fractions like 2\4 and 3\6, listed directly underneath 1\2. Every non-reducible fraction has a third child, a reducible fraction "clone" child, and every clone has a sole parent and a sole clone child.

Next arrange the tree so that each fraction's horizontal position corresponds exactly to its cents, and each fraction's vertical position corresponds exactly to its denominator, which is the edo that contains that octave fraction.

The Stern-Brocot tree when arranged this way looks very different but has all the same properties. The next figure shows a section of this tree from 650¢ to 750¢. The cents range is chosen to approximate 3/2. The tree has been pruned; for each edo, only the best (and occasionally second best) approximation of 3/2 is shown.

Figure 5.7.2 – The Scale Tree from 650¢ to 750¢



The Scale Tree: The fifth of edos 5 through 53
just 3/2
(3\6)
4\6
3\5
4\7
5\8
5\9
6\10
6\11
(7\11)
7\12
(7\13)
8\13
8\14
9\15
9\16
10\17
(10\18)
11\18
11\19
12\20
12\21
13\22
13\23
(14\23)
14\24
15\25
15\26
16\27
16\28
17\29
18\30
18\31
19\32
19\33
20\34
20\35
(21\35)
21\36
22\37
22\38
23\39
23\40
24\41
(24\42)
25\42
25\43
26\44
26\45
27\46
27\47
(28\47)
28\48
29\49
29\50
30\51
30\52
(31\52)
31\53
sharp-1
sharp-2
sharp-3
sharp-4
sharp-5
flat-2
pentatonic fourthwards
pentatonic fifthwards
heptatonic fourthwards
heptatonic fifthwards
nonatonic
octatonic
superflat edos
supersharp edos
"N-tonic fifthwards" means going up by an approximate 3/2 N times and octave-reducing sharpens by 1 key or fret or edo-step. "N-tonic fourthwards" refers to going up an approximate 4/3.
An edo's 2nd best fifth is included if it falls within 60% of an edo-step of just
pentatonic edos
perfect edos
diatonic edos

This version of the Stern-Brocot tree is called the **scale tree**. It shows all the edos ordered top to bottom, arranged left to right by the size of the 5th. The colored regions of the tree are **kites**. The heptatonic kite is light blue and the pentatonic kite is orange. The nonatonic kite is sketched out on the left, and half of the octatonic kite appears on the far right. There are many more kites; in fact the entire tree is made up of kites.

Every kite has a head (e.g. 4\7 for the heptatonic kite), a central spine (8\14, 12\21, etc.), a fourthward side on the left (5\9, 9\16, etc.), and a fifthward side on the right (7\12, 11\19, etc.). Every fraction on the spine is a reducible fraction. Every non-reducible fraction is part of three kites. It's the head of one kite and it's on the side of two others. These two are the natural naming frameworks for the sizing framework implied by that fraction. For example, 7\12 lies on both the heptatonic and pentatonic kites, and 7-note and 5-note are the two natural naming frameworks for 12-tone.

Every non-reducible fraction implies a notation. 4\7 means that 2/1 spans 7 steps, making it an octave in the sense of "eight notes". 4\7 also means that the notation's generator (which is always perfect) spans 4 steps, making it a fifth. Thus the heptatonic kite represents the standard octave-equivalent fifth-based heptatonic notation. The 3\5 kite implies pentatonic notation, with a period that spans 5 steps and a generator that spans 3 steps. The period is always 2/1 = 1200¢ only because we choose to interpret the fractions as octave fractions. The generator only generates the notation, not necessarily the exact tuning. For example, ya JI isn't generated solely by the fifth, but it's notated as if it were.

An edo is a special case of a sizing framework, but as we saw in chapter 5.2, a framework averages out to an edo, and much of what is true about one is true about the other. For the sake of conciseness, I'm going to use "edo" as a shorthand for "sizing framework" from now on. More on this in chapter 5.x.

The scale tree is a map of the world of edos. Where an edo falls on the tree says a lot about the best way to notate that edo. Every edo on the head or either side of the heptatonic kite (7, 9, 12, 16, 19, 23, etc.) can be notated heptatonically without using ups and downs. All others require ups and downs. Likewise the pentatonic kite, minus the spine, contains the only edos that can be notated pentatonically without ups and downs.

Edos can be placed into six categories. The trivial edos are those that are contained in 12-edo. The other five categories are based on the size of the fifth. From narrowest to widest:

> **superflat** edos (9, 11, 13b, 16, 18b & 23) have a fifth narrower than four-sevenths of an octave = 4\7 = 686¢
> **perfect** edos (7, 14, 21, 28 & 35) have a fifth of 4\7 = 686¢
> **diatonic** edos (12, 17, 19, 22, 24, etc.) have a fifth that hits the "sweet spot" between 686¢ and 720¢
> **pentatonic** edos (5, 10, 15, 20, 25 & 30) have a fifth of three-fifths of an octave = 3\5 = 720¢
> **supersharp** edos (8, 13 & 18) have a fifth wider than 3\5 = 720¢
>
> **trivial** edos (1, 2, 3, 4 and 6) have a fifth about 100¢ from just, and are notated as subsets of 12-edo

Almost all edos, and all edos above 35, are diatonic. There are only a handful of edos in each of the other categories.

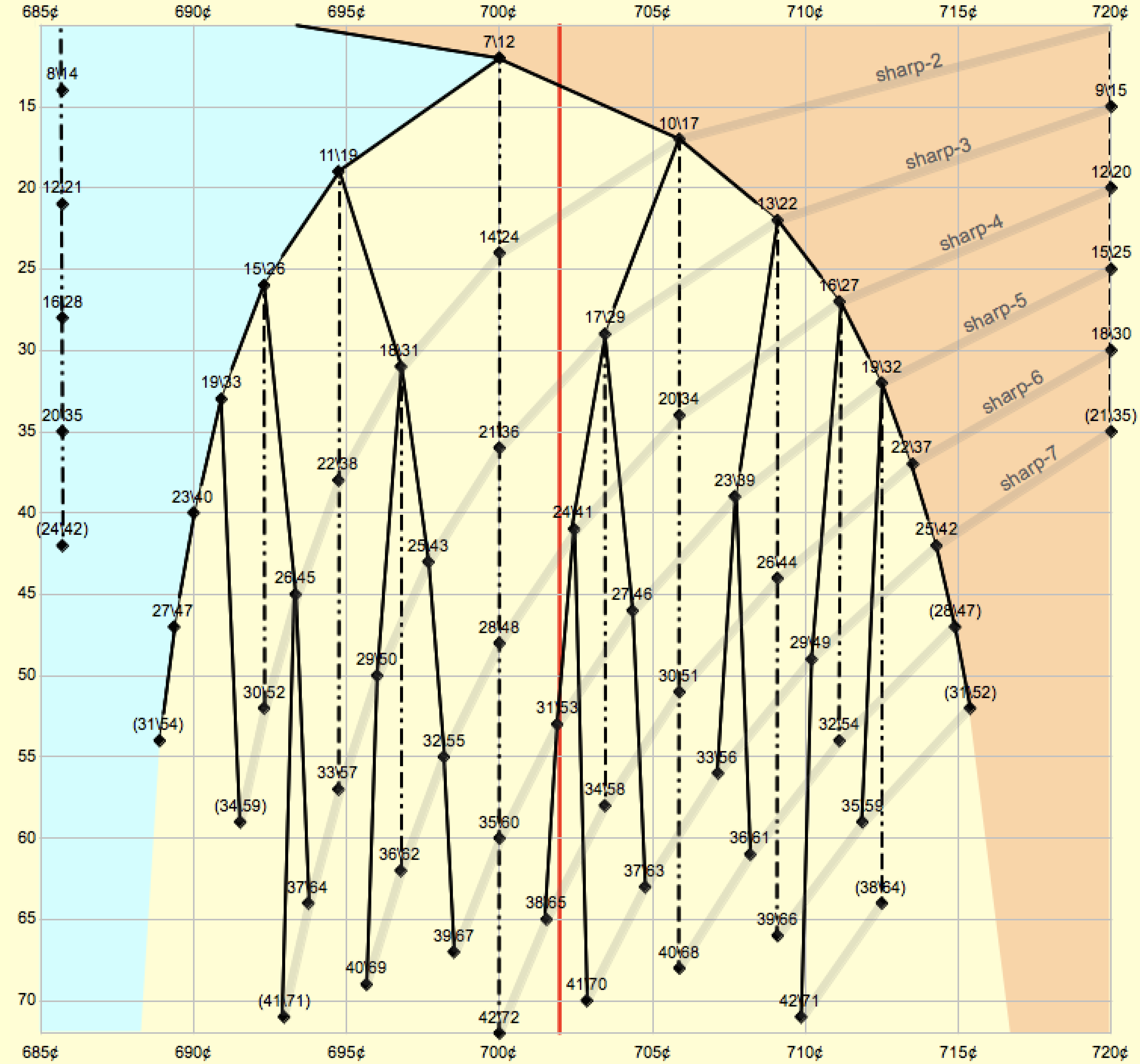
An edo's fifth is defined as the best approximation of 3/2. There is a little leeway to this, because certain edos have an alternate fifth with nearly equal accuracy. For example, 18-edo's best fifth is 11\18. 18b-edo uses the alternate fifth 10\18, which is only 4¢ less accurate. 18-edo is supersharp and 18b-edo is superflat. The maximum discrepancy of the best fifth is 50% of an edostep. An edo's alternate fifth is included in the above chart if its discrepancy is less than 60%, which requires that the best fifth's discrepancy be more than 40%.

All edos except supersharp and trivial ones can be further categorized by the sharp's keyspan. We've seen that in 22-edo, the sharp spans three keys or frets, and the sharp symbol equals 3 ups. Thus 22-edo is in the **sharp**-3 category. The categories are made up of every seventh edo (up to a point, the pattern breaks down with higher edos). Sharp-0 edos (7, 14, 21, 28 and 35) are the perfect edos that lie on the spine of the heptatonic kite. Flat-1 edos lie on the left side of the kite (9, 16 and 23), and sharp-1 edos are on the right (12, 19, 26, etc.). 5-edo is also sharp-1. The light gray lines in Figure 5.7.2 are **sharpness** lines, connecting edos of similar sharpness. They spread out from the heptatonic kite on both sides like ripples in the water.

A composition can often be translated from one edo to a similar one. The scale tree can be used to find similar edos to translate to. For example, "Tibia" in 22-edo might also work in nearby parent/child edos like 17, 27 or 39.

The next figure shows the diatonic frameworks in more detail. There's a 7\12 kite, a 10\17 kite, an 11\19 kite, etc.

Figure 5.7.3 – The scale tree for frameworks 12-72, excluding supersharp and superflat ones

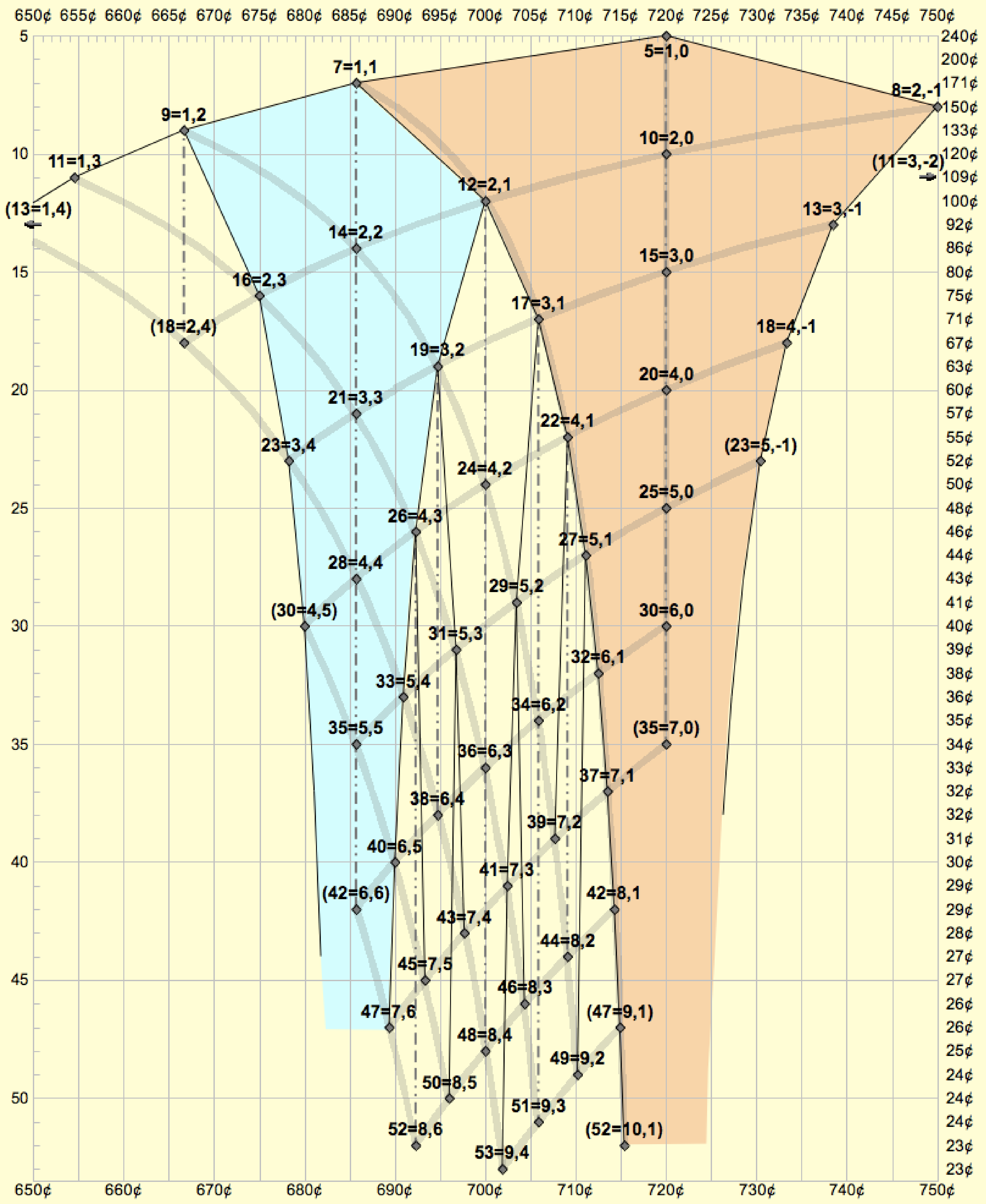


Assuming the seven natural notes (the white keys) form a chain of fifths, a scale is made with five major 2nds and two minor 2nds. For example, 12-edo's 7-semitone fifth makes a scale with 2-semitone major 2nds and 1-semitone minor 2nds. Likewise, 19-edo has 2nds of 3 and 2 edosteps (figure 5.4.1), and 22-edo has 4 and 1 edosteps (figure 5.5.2).

The next figure shows the size of these two seconds in edosteps. The size of the edostep in cents is shown on the right-hand side. Edos that have an alternate fifth, like 11, 13 and 18, have an alternate white key placement, and alternate sizes for the 2nds. The edo's sharpness is simply the difference between these two numbers. The ratio of the two sizes depends directly on the size of the fifth, and increases steadily from right to left (see also Figure 4.3.3).

superflat edos have m2 > M2
perfect edos have m2 = M2
diatonic edos have m2 < M2
pentatonic edos have m2 = 0
supersharp edos have m2 < 0

Figure 5.7.4 – The scale tree, showing the size of the major and minor 2nds in edosteps



This might be a good time to review the graph from chapter 4.1 showing how well each edo represents yaza JI.

Figure 4.1.1 – The discrepancy of the wa, yo and zo rungs in each edo from 5 to 41

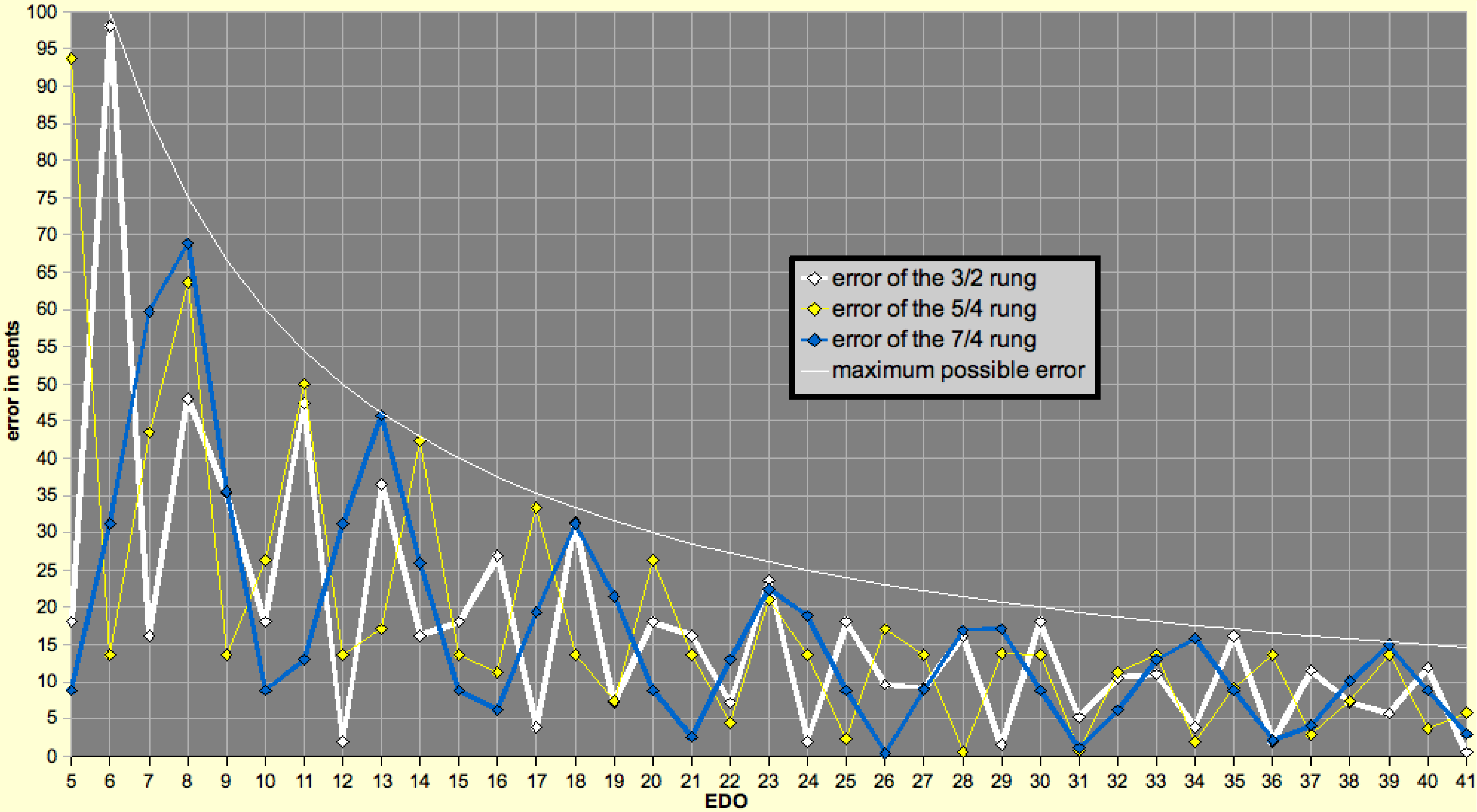
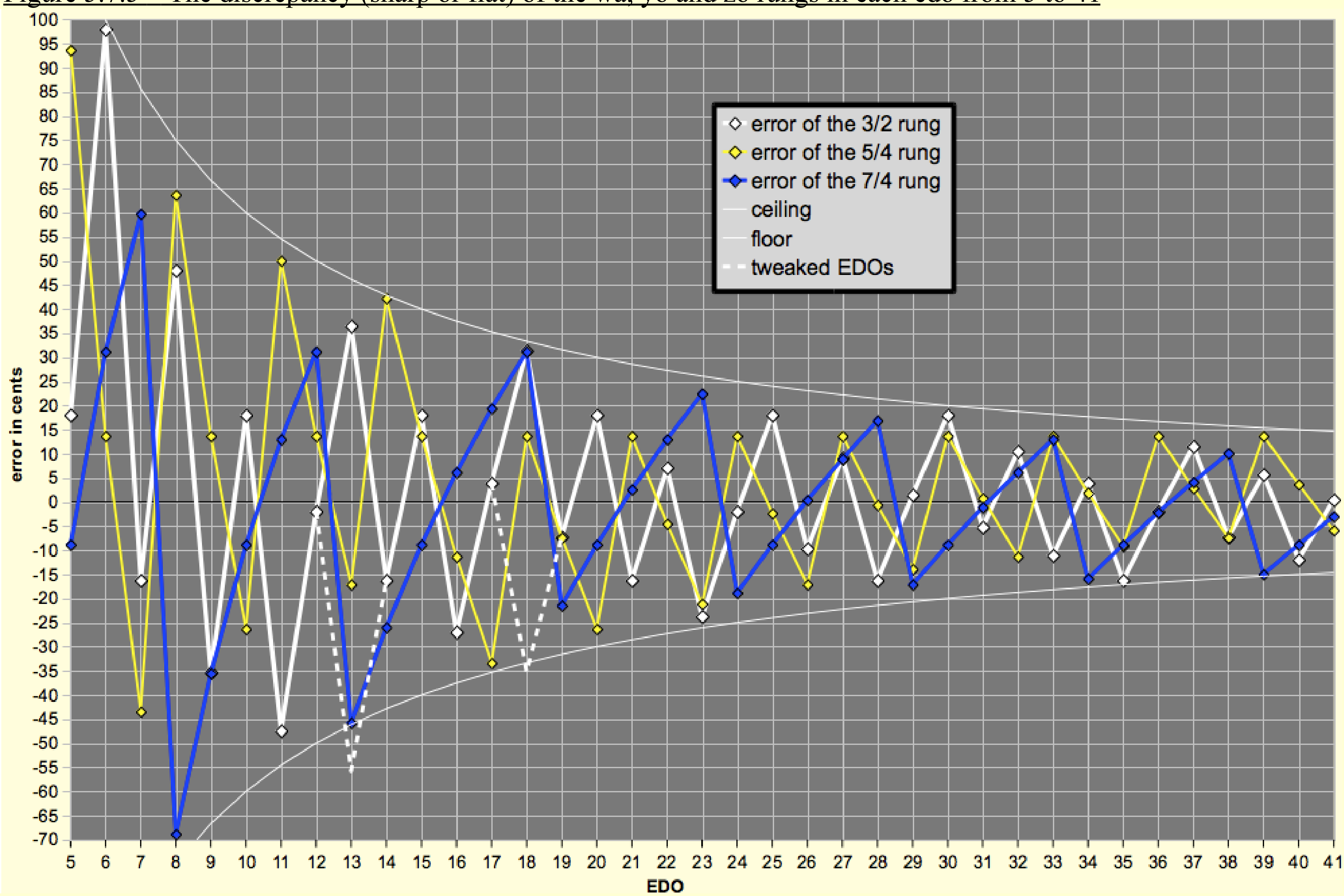


Figure 5.7.5 – The discrepancy (sharp or flat) of the wa, yo and zo rungs in each edo from 5 to 41



In the next figure, red and green lines have been added to the scale tree, showing how well each edo approximates the yo 3rd, 5/4. Green lines represent minimum discrepancy. The edo's discrepancy for y3 equals <u>four</u> times the edo's distance from the nearest green line. For example, 12-edo is 3.5¢ to the right of a green line, and 12-edo's approximation of y3 is 14¢ sharp. This distance is always measured <u>horizontally</u>. 8-edo is a full 16¢ away from the green line, even though it's not far below it.

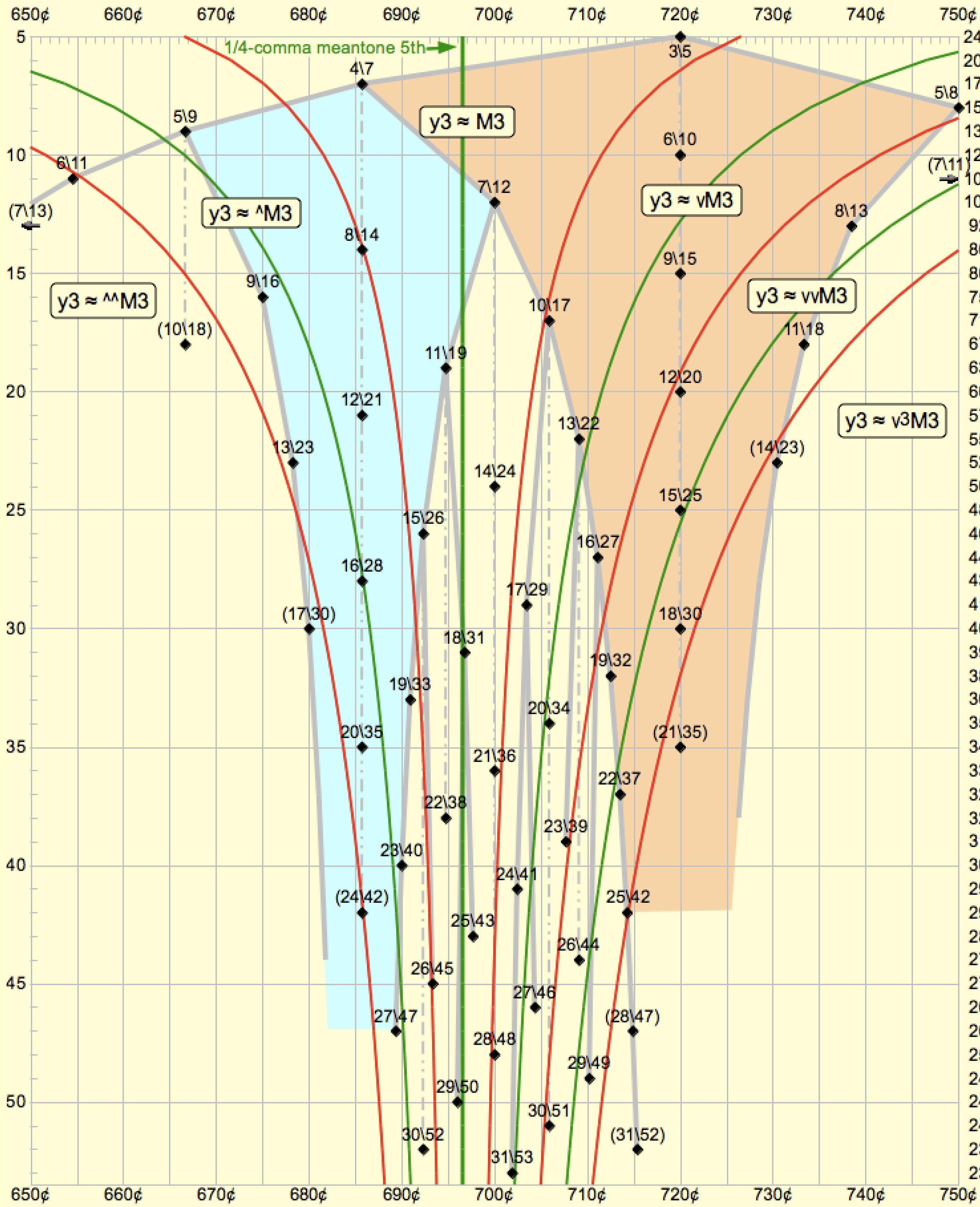
Red lines represent maximum discrepancy. If an edo falls near a red line, like 14-edo and 17-edo, y3 falls almost exactly between two edo notes. These edos are good candidates for tweaking yo.

If an edo tempers out the gu comma g1 = 81/80, it is said to support meantone temperament. In these edos, y3 is best approximated by a major 3rd. The major 3rd is defined as four 5ths minus two octaves, even for superflat edos. More on this later. These edos are all contained in the region of the graph that lies between two red lines, marked "y3 ≈ M3".

The red lines define various regions that correspond to the gu comma's keyspan. In edos like 15 or 22, g1 maps to 1 edostep, and y3 ≈ vM3, a downmajor 3rd. In edos like 16 or 21, g1 is a descending edostep, and y3 ≈ ^M3. Upmajor 3rd is used loosely here to mean the interval one edostep wider than the major 3rd. In many edos in the y3 ≈ ^M3 category, this interval actually has a different name: ^3, A3, or even m3.

The straight green line is the quarter-comma meantone fifth = 696.6¢. The red and green boundary lines are 1/8 of an edostep apart. For example, 15-edo has an 80¢ edostep, and the boundary lines cross the horizontal 15-edo line at points 10¢ apart.

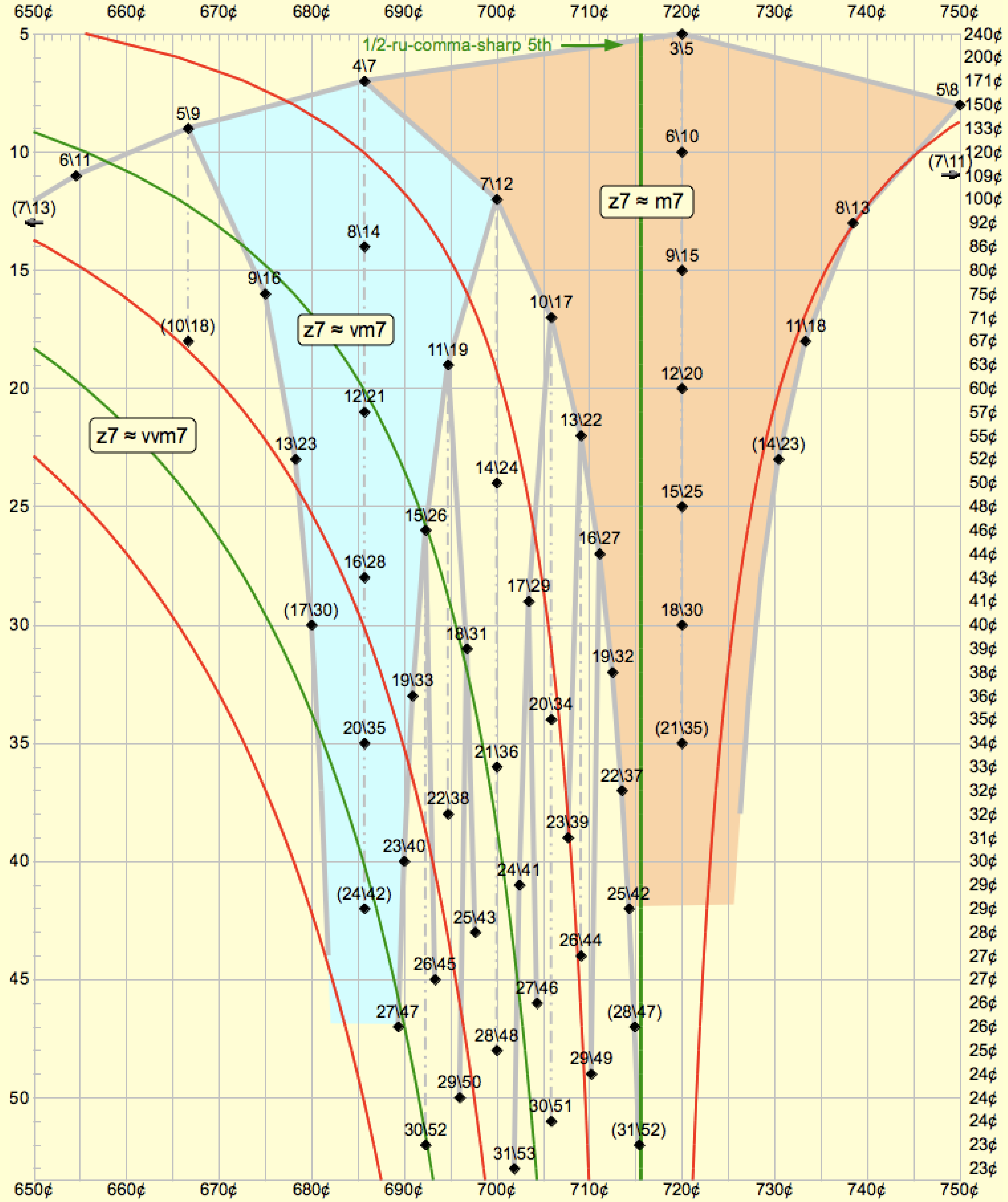
Figure 5.7.6 – The yo 3rd's discrepancy and the keyspan of the gu comma (g1 = 81/80) plotted onto the scale tree



The next figure is a similar scale tree for the zo 7th = 7/4 and the ru comma = 64/63.

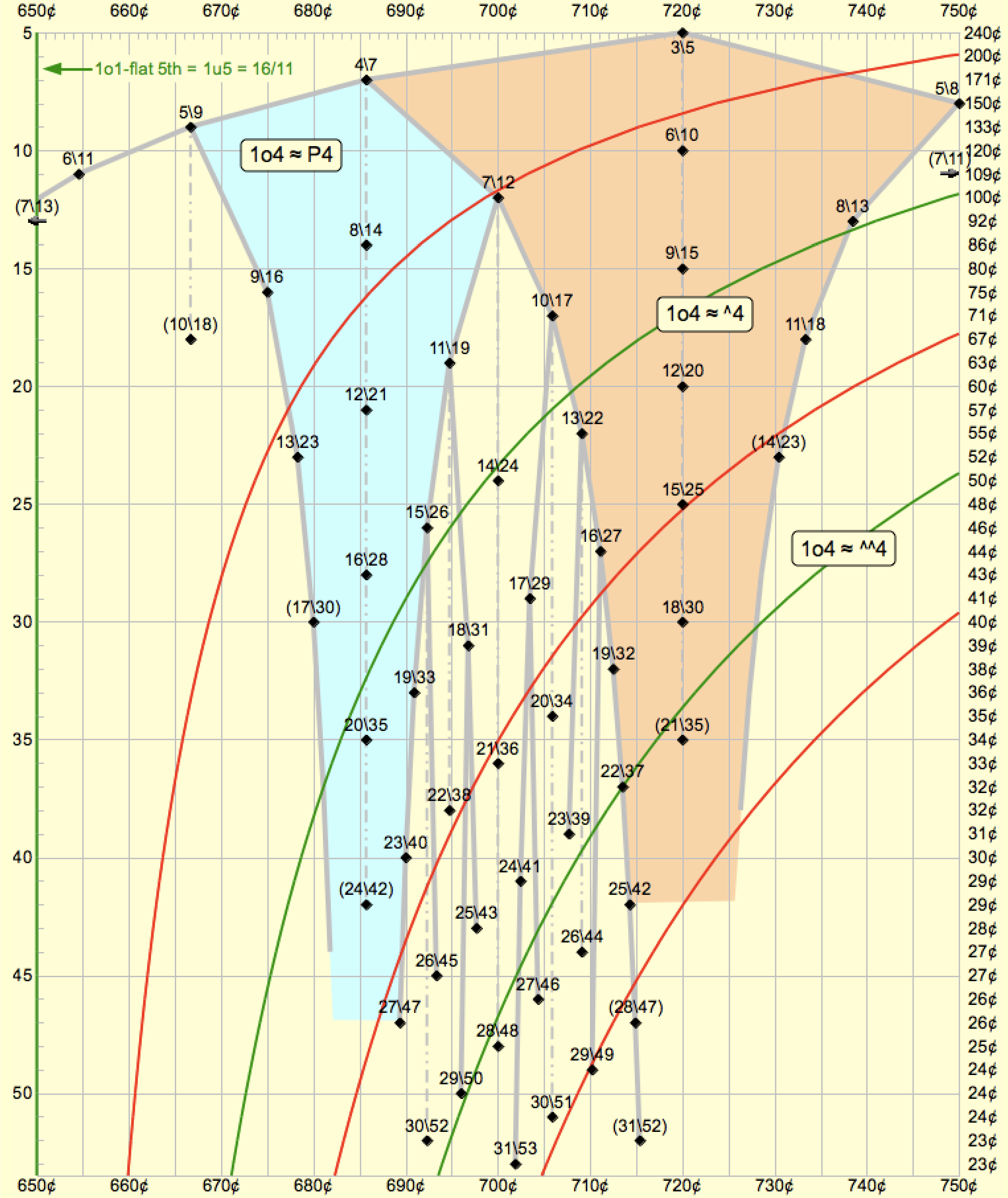
The edo's discrepancy for z7 is <u>twice</u> the distance from the nearest green line. The straight green line is a fifth that's half a comma <u>sharp</u>. Because it's sharp, if the edo falls to the right of a green line, the discrepancy is negative, not positive. The red and green boundary lines are 1/4 of an edostep apart. The vm7 is sometimes called a d7, a v7 or a M7.

<u>Figure 5.7.7 – The zo 7th's discrepancy and the keyspan of the ru comma (r1 = 64/63) plotted onto the scale tree</u>



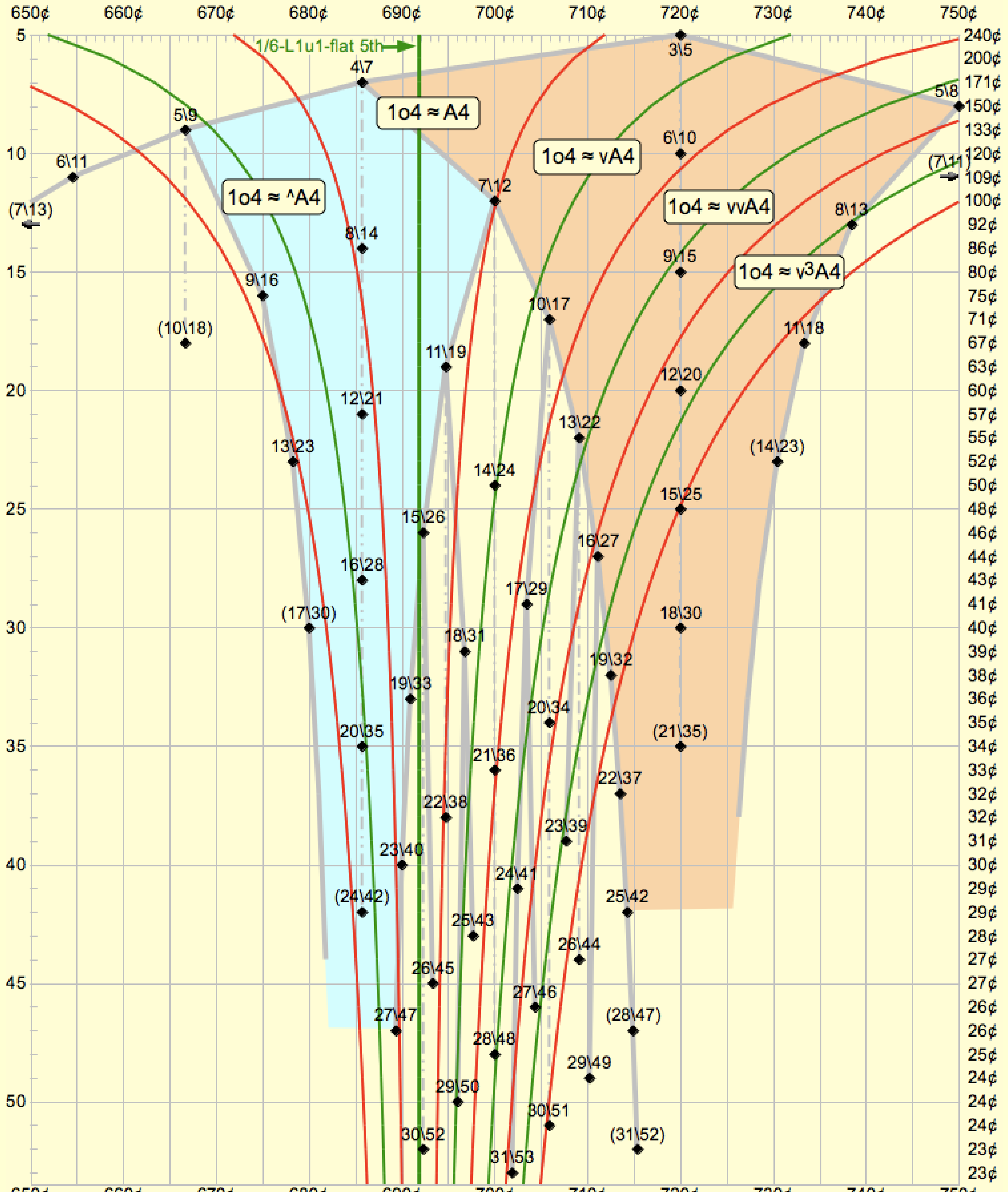
For the ilo 4th 1o4 = 11/8, we have a choice of commas. The ilo comma relates 1o4 to P4, and the large lu comma relates it to A4. The next graph is for the ilo comma. The edo's discrepancy for 1o4 exactly equals the distance from the nearest green line. The red and green boundary lines are half an edostep apart.

Figure 5.7.8 – The ilo 4th's discrepancy and the keyspan of the ilo comma (1o1 = 33/32) plotted onto the scale tree



For the lu comma, the discrepancy of 1o4 is 1/6 the distance to green. The red and green lines are 1/12 edostep apart.

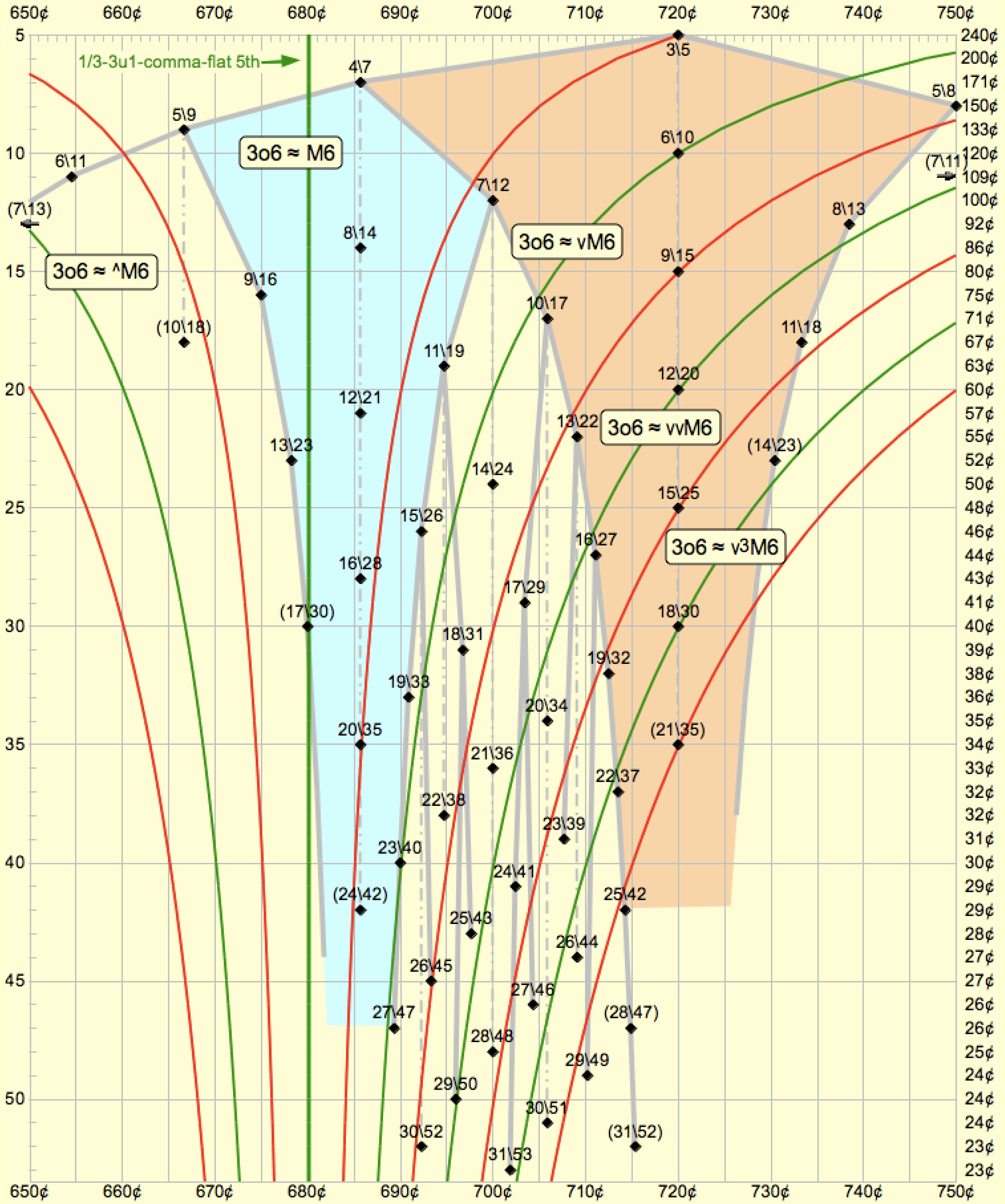
Figure 5.7.9 – The ilo 4th's discrepancy and the keyspan of the large lu comma (L1u1 = 729/704) on the scale tree



The last two scale trees look very different, but they both contain the same information. If an edo falls near a green line or a red line in one scale tree, it also falls near such a line in the other scale tree (e.g. 12edo or 24edo).

For the tho 6th 3o6 = 13/8, we again have a choice of commas, the thu comma for M6 and the large tho comma for m6.

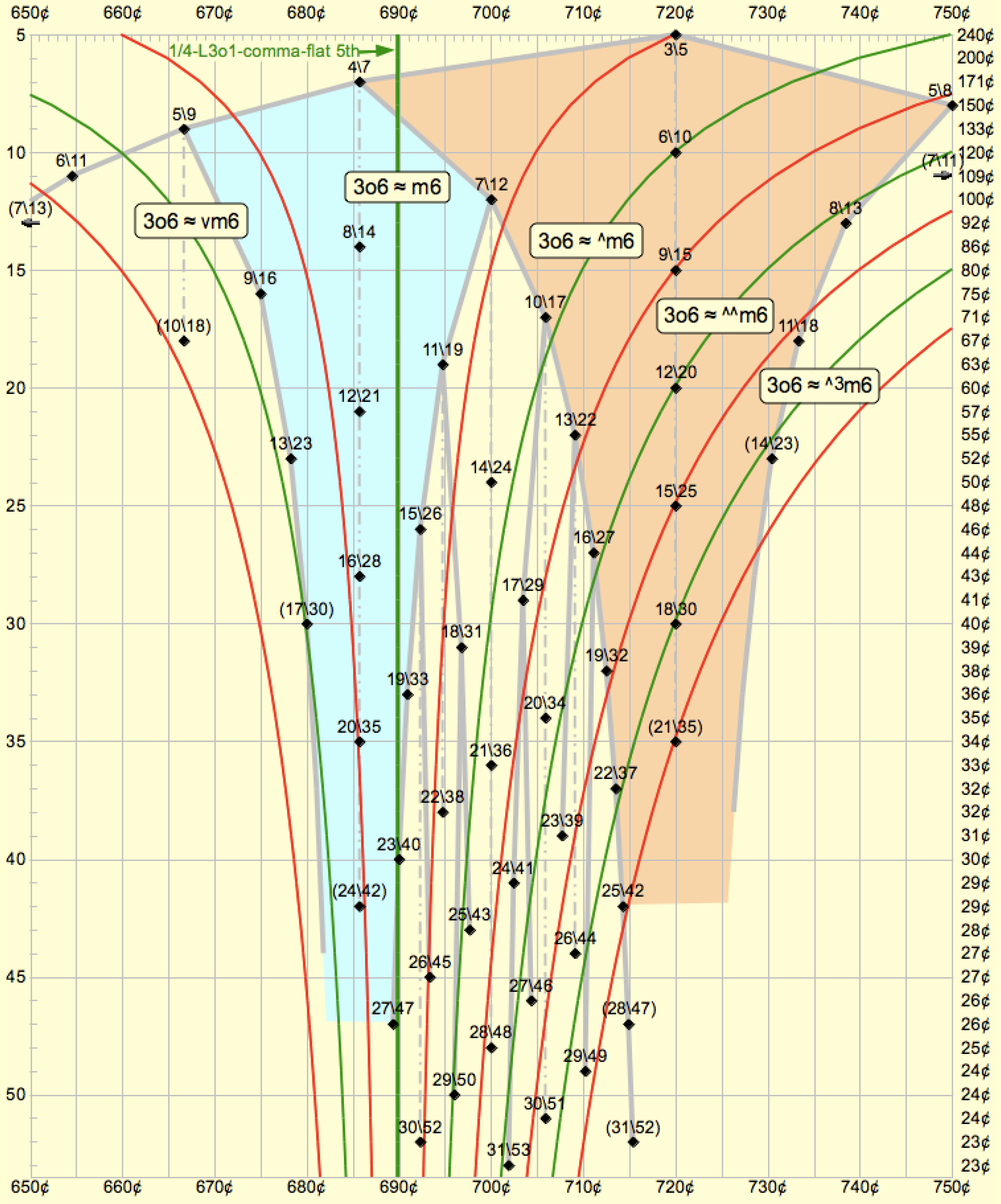
<u>Figure 5.7.10 – The tho 6th's discrepancy and the keyspan of the thu comma (3u1 = 27/26) onto the scale tree</u>



The edo's discrepancy is always some multiple of its distance from the nearest green line. This multiple depends on the comma. The thu comma equates <u>three</u> 5ths to one tho 6th, so the multiplying factor is 3, and the straight line is a <u>third</u>-

comma flat 5th. The red and green lines are 1/6 an edostep apart. For the large tho comma, the multiple is 4.

Figure 5.7.11 – The tho 6th's discrepancy and the keyspan of the large tho comma (L3o1 = 1053/1024)



Mid, written ~, is a quality like major or perfect. It means "exactly midway between major and minor", hence neutral. For example, in sharp-2 edos, upminor equals downmajor, and "mid" replaces both terms. Instead of ^m3 or vM3, we have ~3. In sharp-4 edos, mid replaces both double-upminor and double-downmajor.

This next table shows how to name any quality in any edo up to 72-edo. Diminished intervals can be deduced from augmented ones by symmetry. Sharp-7 and higher edos are rarely used. Upmid and downmid are 2 edosteps apart in sharp-6 edos, but only 1 edostep apart in sharp-5 edos.

Table 5.7.1 – Quality sequences for edos 5-72, excluding 6-edo and 8-edo (**bold** = superflat edos)

category	edos	imperfect and perfect quality sequences (dim is symmetrical to aug)
sharp-0 (perfect)	7, 14, 21, 28, 35	(no imperfect intervals) perfect, up, double-up, triple-up... = P, ^, ^^, ^^{3}...
sharp-1, **flat-1**	5, **9**, 12, **16**, 19, **23**, 26, 33, 40, 47	minor, major, aug, double-aug, triple-aug... = m, M, A, AA, A^3... perfect, aug, double-aug, triple-aug... = P, A, AA, A^3...
sharp-2, **flat-2**	10, **11**, 17, **18b**, 24, 31, 38, 45, 52	minor, mid, major, upmajor, aug, up-aug, double-aug... = m, ~, M, ^M, A, ^A, AA... perfect, up, aug, up-aug, double-aug... = P, ^, A, ^A, AA...
sharp-3, **flat-3**	**13b**, 15, 22, 29, 36, 43, 50, 57, 64	minor, upminor, downmajor, major, upmajor, downaug, aug... = m, ^m, vM, M, ^M, vA, A perfect, up, downaug, aug, up-aug, down-double-aug, double-aug... = P, ^, vA, A, ^A, vAA, AA...
sharp-4	20, 27, 34, 41, 48, 55, 62, 69	m, ^m, ~ (mid), vM, M, ^M, ^^M (double-up major), vA, A... P, ^, ^^ (double-up), vA, A, ^A, ^^A (double-up-aug), vAA (down-double-aug), AA...
sharp-5	25, 32, 39, 46, 53, 60, 67	m, ^m, v~ (downmid), ^~ (upmid), vM, M, ^M, ^^M, vvA, vA, A... P, ^, ^^, vvA, vA, A, ^A, ^^A, vvAA (double-down double-aug), vAA, AA...
sharp-6	30, 37, 44, 51, 58, 65, 72	m, ^m, v~, ~, ^~, vM, M, ^M, ^^M, ^^{3}M (triple-upmajor), vvA, vA, A... P, ^, ^^, ^^{3} (triple-up), vvA, vA, A, ^A, ^^A, ^^{3}A (triple-upaug), vvAA, vAA, AA...
sharp-7 (rare)	42, 49, 56, 63, 70	m, ^m, ^^m, v~, ^~, vvM, vM, M, ^M, ^^M, ^^{3}M, v^{3}A, vvA, vA, A... P, ^, ^^, ^^{3}, v^{3}A, vvA, vA, A, ^A, ^^A, ^^{3}A, v^{3}AA, vvAA, vAA, AA...
sharp-8 (rare)	54, 61, 68	m, ^m, ^^m, v~, ~, ^~, vvM, vM, M, ^M, ^^M, ^^{3}M, ^^{4}M, v^{3}A, vvA, vA, A... P, ^, ^^, ^^{3}, ^^{4}, v^{3}A, vvA, vA, A, ^A, ^^A, ^^{3}A, ^^{4}A, v^{3}AA, vvAA, vAA, AA...
sharp-9 (rare)	59, 66	m, ^m, ^^m, vv~, v~, ^~, ^^~, vvM, vM, M, ^M, ^^M, ^^{3}M, ^^{4}M, v^{4}A, v^{3}A, vvA, vA, A... P, ^, ^^, ^^{3}, ^^{4}, v^{4}A, v^{3}A, vvA, vA, A, ^A, ^^A, ^^{3}A, ^^{4}A, v^{4}AA, v^{3}AA, vvAA, vAA, AA...
sharp-10 (rare)	71	m, ^m, ^^m, vv~, v~, ~, ^~, ^^~, vvM, vM, M, ^M, ^^M, ^^{3}M, ^^{4}M, ^^{5}M, v^{4}A, v^{3}A, vvA, vA P, ^, ^^, ^^{3}, ^^{4}, ^^{5}, v^{4}A, v^{3}A, vvA, vA, A, ^A, ^^A, ^^{3}A, ^^{4}A, ^^{5}A, v^{4}AA, v^{3}AA, vvAA...

The next table is a notation guide for edos 5 through 72. Every note from D to F is represented in at least two ways, and often three ways. Any sharp or flat that is tripled or more is written ♯^{3}, ♭^{4}, etc., whether on the staff or in text. Doubled ups and downs are written ^^ and vv, but if tripled or more, they are written ^^{3}, v^4, etc.

Examples:
- D$^{\wedge\wedge}$ = D double-up = ^^M2, if in the key of C = double-upmajor 2nd
- D^{x} = D double-sharp = AA2 = double-aug 2nd
- D$^{x\wedge\wedge}$ = D double-sharp double-up = ^^AA2 = double-up double-aug 2nd
- D$^{\sharp v3}$ = D sharp triple-down = v^{3}A2 = triple-down aug 2nd
- D$^{\sharp 3v}$ = D triple-sharp down = vA^{3}2 = down triple-aug 2nd

For each edo, the D, E and F naturals are shown as white keys, and the other keys are black. The full layout of natural and sharped/flatted notes can be deduced from these keys. For example, 29-edo has D * * * * E * F, which implies:

C * * * * D * * * * E * F * * * * G * * * * A * * * * B * C

Table 5.7.2 – Notation Guide for Edos 5-72 Using Ups and Downs, Showing White and Black Keys

5-edo	pentatonic sharp-1	**D** E♭ F♭	D♯ **E** **F**					
6-edo	trivial (subset of 12-edo)	**D** E♭♭	DX **E** F♭	EX F♯				
7-edo	perfect sharp-0	**D**	**E**	**F**				
8-edo	supersharp (subset of 24-edo)	**D** E♭♭	D♯^ Ev F♭v	E♯ **F**				
9-edo	superflat flat-1	**D** E♯	D♭ **E** FX	D♭♭ E♭ F♯	E♭♭ **F**			
10-edo	pentatonic sharp-2	**D** E♭ F♭	D^ Ev Fv	D♯ **E** **F**				
11-edo	superflat flat-2	**D** Ev	D^ **E** F♯v	D♭ E^ F♯	D♭^ E♭ Fv	E♭^ **F**		
12-edo	diatonic sharp-1	**D** E♭♭	D♯ E♭ F♭♭	DX **E** F♭	E♯ **F**			
13b-edo (5th = 7\13)	superflat flat-3	**D** Ev	D^ **E** F♯v	D♭v E^ F♯	D♭ E♭v F♯^	D♭^ E♭ Fv	E♭^ **F**	
14-edo	perfect sharp-0	**D** Evv	D^ Ev F^3	D^^ **E** Fvv	D^3 E^ Fv	E^^ **F**		
15-edo	pentatonic sharp-3	**D** E♭ F♭	D^ E♭^ F♭^	D♯v Ev Fv	D♯ **E** **F**			
16-edo	superflat flat-1	**D** EX	D♭ E♯	D♭♭ **E** F♯3	D♭3 E♭ FX	E♭♭ F♯	E♭3 **F**	
17-edo	diatonic sharp-2	**D** E♭v F♭♭	D^ E♭ F♭v	D♯ Ev F♭	D♯^ **E** Fv	DX E^ **F**		
18b-edo (5th = 10\18)	superflat flat-2	**D** E♯	D^ Ev FXv	D♭ **E** FX	D♭^ E^ F♯v	D♭♭ E♭ F♯	D♭♭^ E♭^ Fv	E♭♭ **F**

19-edo	diatonic sharp-1	**D** E♭3	D♯ E♭♭	Dx E♭ F♭3	D♯3 **E** F♭♭	E♯ F♭	Ex **F**				
20-edo	pentatonic sharp-4	**D** E♭ F♭	D^ E♭^ F♭^	D^^ Evv Fvv	D♯v Ev Fv	D♯ **E** **F**					
21-edo	perfect sharp-0	**D** Ev3	D^ Evv	D^^ Ev F^4	D^3 **E** Fv3	D^4 E^ Fvv	E^^ Fv	E^3 **F**			
22-edo	diatonic sharp-3	**D** E♭v F♭♭^	D^ E♭ F♭v	D♯v E♭^ F♭	D♯ Ev F♭^	D♯^ **E** Fv	Dxv E^ **F**				
23-edo	superflat flat-1	**D** E♯3	D♭ Ex	D♭♭ E♯	D♭3 **E** F♯4	F♭4 E♭ F♯3	E♭♭ Fx	E♭3 F♯	E♭4 **F**		
24-edo	diatonic sharp-2	**D** E♭♭	D^ E♭v	D♯ E♭ F♭♭	D♯^ Ev F♭v	Dx **E** F♭	E^ Fv	E♯ **F**			
25-edo	pentatonic sharp-5	**D** E♭ F♭	D^ E♭^ F♭^	D^^ E♭^^ F♭^^	D♯vv Evv Fvv	D♯v Ev Fv	D♯ **E** **F**				
26-edo	diatonic sharp-1	**D** E♭4	D♯ E♭3	Dx E♭♭	D♯3 E♭ F♭4	D♯4 **E** F♭3	E♯ F♭♭	Ex F♭	E♯3 **F**		
27-edo	diatonic sharp-4	**D** E♭v F♭vv	D^ E♭ F♭v	D^^ E♭^ F♭	D♯v Evv F♭^	D♯ Ev Fvv	D♯^ **E** Fv	D♯^^ E^ **F**			
28-edo	perfect sharp-0	**D** Ev4	D^ Ev3	D^^ Evv	D^3 Ev Fv5	D^4 **E** Fv4	D^5 E^ Fv3	E^^ Fvv	E^3 Fv	E^4 **F**	
29-edo	diatonic sharp-3	**D** E♭♭^	D^ E♭v F♭♭	D♯v E♭ F♭♭^	D♯ E♭^ F♭v	D♯^ Ev F♭	Dxv **E** F♭^	Dx E^ Fv	E♯v **F**		
30-edo	pentatonic sharp-6	**D** E♭ F♭	D^ E♭^ F♭^	D^^ E♭^^ F♭^^	D^3 Ev3 Fv3	D♯vv Evv Fvv	D♯v Ev Fv	D♯ **E** **F**			
31-edo	diatonic sharp-2	**D** E♭♭v	D^ E♭♭	D♯ E♭v	D♯^ E♭ F♭♭v	Dx Ev F♭♭	Dx^ **E** F♭v	E^ F♭	E♯ Fv	E♯^ **F**	
32-edo	diatonic sharp-5	**D** E♭v	D^ E♭ F♭v	D^^ E♭^ F♭	D♯vv E♭^^ F♭^	D♯v Evv F♭^^	D♯ Ev Fvv	D♯^ **E** Fv	E^ **F**		
33-edo	diatonic sharp-1	**D** E♭5	D♯ E♭4	Dx E♭3	D♯3 E♭♭ F♭6	D♯4 E♭ F♭5	D♯5 **E** F♭4	D♯6 E♯ F♭3	Ex F♭♭	E♯3 F♭	E♯4 **F**

	0	1	2	3	4	5	6	7	8	9	10			
34-edo diatonic sharp-4	**D** E♭vv F♭♭	D^ E♭v F♭♭^	D^^ E♭ F♭vv	D♯v E♭^ F♭v	D♯ Evv F♭	D♯^ Ev F♭^	D♯^^ **E** Fvv	Dxv E^ Fv	Dx E^^ **F**					
35-edo perfect sharp-0	**D** Ev5	D^ Ev4	D^^ Ev3	D^3 Evv Fv7	D^4 Ev Fv6	D^5 **E** Fv5	D^6 E^ Fv4	D^7 E^^ Fv3	E^3 Fvv	E^4 Fv	E^5 **F**			
36-edo diatonic sharp-3	**D** E♭♭	D^ E♭♭^	D♯v E♭v	D♯ E♭ F♭♭	D♯^ E♭^ F♭♭^	Dxv Ev F♭v	Dx **E** F♭	E^ F♭^	E♯v Fv	E♯ **F**				
37-edo sharp-6	**D** E♭v F♭vv	D^ E♭ F♭v	D^^ E♭^ F♭	D^3 E♭^^ F♭^	D♯vv Ev3 F♭^^	D♯v Evv Fv3	D♯ Ev Fvv	D♯^ **E** Fv	D♯^^ E^ **F**					
38-edo sharp-2	**D** E♭3	D^ E♭♭v	D♯ E♭♭ F♭4	D♯^ E♭v F♭3v	Dx E♭ F♭3	Dx^ Ev F♭♭v	D♯3 **E** F♭♭	D♯3^ E^ F♭v	D♯4 E♯ F♭	E♯^ Fv	Ex **F**			
39-edo sharp-5	**D** E♭vv	D^ E♭v	D^^ E♭ F♭vv	D♯vv E♭^ F♭v	D♯v E♭^^ F♭	D♯ Evv F♭^	D♯^ Ev F♭^^	D♯^^ **E** Fvv	E^ Fv	E^^ **F**				
40-edo sharp-1	**D** E♭6	D♯ E♭5	Dx E♭4	D♯3 E♭3	D♯4 E♭♭ F♭7	D♯5 E♭ F♭6	D♯6 **E** F♭5	D♯7 E♯ F♭4	Ex F♭3	E♯3 F♭♭	E♯4 F♭	E♯5 **F**		
41-edo sharp-4	**D** E♭♭^	D^ E♭vv	D^^ E♭v F♭♭	D♯v E♭ F♭♭^	D♯ E♭^ F♭vv	D♯^ Evv F♭v	D♯^^ Ev F♭	Dxv **E** F♭^	Dx E^ Fvv	E^^ Fv	E♯v **F**			
42-edo sharp-7	**D** E♭v	D^ E♭ F♭v	D^^ E♭^ F♭	D^3 E♭^^ F♭^	D♯v3 E♭^3 F♭^^	D♯vv Ev3 F♭^3	D♯v Evv Fv3	D♯ Ev Fvv	D♯^ **E** Fv	E^ **F**				
43-edo sharp-3	**D** E♭♭v	D^ E♭♭	D♯v E♭♭^	D♯ E♭v F♭3^	D♯^ E♭ F♭♭v	Dxv E♭^ F♭♭	Dx Ev F♭♭^	Dx^ **E** F♭v	D♯3v E^ F♭	E♯v F♭^	E♯ Fv	E♯^ **F**		
44-edo sharp-6	**D** E♭vv	D^ E♭v	D^^ E♭ F♭vv	D^3 E♭^ F♭v	D♯vv E♭^^ F♭	D♯v Ev3 F♭^	D♯ Evv F♭^^	D♯^ Ev Fv3	D♯^^ **E** Fvv	E^ Fv	E^^ **F**			
45-edo sharp-2	**D** E♭3v	D^ E♭3	D♯ E♭♭v	D♯^ E♭♭ F♭4v	Dx E♭v F♭4	Dx^ E♭ F♭3v	D♯3 Ev F♭3	D♯3^ **E** F♭♭v	D♯4 E^ F♭♭	D♯4^ E♯ F♭v	E♯^ F♭	Ex Fv	Ex^ **F**	
46-edo sharp-5	**D** E♭♭^^	D^ E♭vv	D^^ E♭v F♭♭^	D♯vv E♭ F♭♭^^	D♯v E♭^ F♭vv	D♯ E♭^^ F♭v	D♯^ Evv F♭	D♯^^ Ev F♭^	Dxvv **E** F♭^^	Dxv E^ Fvv	E^^ Fv	E♯vv **F**		
47-edo sharp-1	**D** E♭7	D♯ E♭6	Dx E♭5	D♯3 E♭4	D♯4 E♭3 F♭9	D♯5 E♭♭ F♭8	D♯6 E♭ F♭7	D♯7 **E** F♭6	D♯8 E♯ F♭5	D♯9 Ex F♭4	E♯3 F♭3	E♯4 F♭♭	E♯5 F♭	E♯6 **F**
	0	1	2	3	4	5	6	7	8	9	10	11	12	13

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
48-edo ♯ = 4	**D** E♭♭	D^ E♭♭^	D^^ E♭vv	D♯v E♭v F♭♭v	D♯ E♭ F♭♭	D♯^ E♭^ F♭♭^	D♯^^ Evv F♭vv	Dxv Ev F♭v	Dx **E** F♭	Dx^ E^ F♭^	E^^ Fvv	E♯v Fv	E♯ **F**			
49-edo ♯ = 7	**D** E♭vv	D^ E♭v	D^^ E♭ F♭vv	D^3 E♭^ F♭v	D♯v3 E♭^^ F♭	D♯vv E♭^3 F♭^	D♯v Ev3 F♭^^	D♯ Evv F♭^3	D♯^ Ev Fv3	D♯^^ **E** Fvv	E^ Fv	E^^ **F**				
50-edo ♯ = 3	**D** E♭3^	D^ E♭♭v	D♯v E♭♭	D♯ E♭♭^	D♯^ E♭v F♭3	Dxv E♭ F♭3^	Dx E♭^ F♭♭v	Dx^ Ev F♭♭	D♯3v **E** F♭♭^	D♯3 E^ F♭v	E♯v F♭	E♯ F♭^	E♯^ Fv	Exv **F**		
51-edo ♯ = 6	**D** E♭v3	D^ E♭vv	D^^ E♭v	D^3 E♭ F♭v3	D♯vv E♭^ F♭vv	D♯v E♭^^ F♭v	D♯ Ev3 F♭	D♯^ Evv F♭^	D♯^^ Ev F♭^^	D♯^3 **E** Fv3	E^ Fvv	E^^ Fv	E^3 **F**			
52-edo ♯ = 2	**D** E♭4	D^ E♭3v	D♯ E♭3	D♯^ E♭♭v	Dx E♭♭ F♭5	Dx^ E♭v F♭4v	D♯3 E♭ F♭4	D♯3^ Ev F♭3v	D♯4 **E** F♭3	D♯4^ E^ F♭♭v	D♯5 E♯ F♭♭	E♯^ F♭v	Ex F♭	Ex^ Fv	E♯3 **F**	
53-edo ♯ = 5	**D** E♭♭^	D^ E♭♭^^	D^^ E♭vv	D♯vv E♭v F♭♭	D♯v E♭ F♭♭^	D♯ E♭^ F♭♭^^	D♯^ E♭^^ F♭vv	D♯^^ Evv F♭v	Dxvv Ev F♭	Dxv **E** F♭^	Dx E^ F♭^^	E^^ Fvv	E♯vv Fv	E♯v **F**		
54-edo ♯ = 8	**D** E♭vv	D^ E♭v	D^^ E♭ F♭vv	D^3 E♭^ F♭v	D^4 E♭^^ F♭	D♯v3 E♭^3 F♭^	D♯vv Ev4 F♭^^	D♯v Ev3 F♭^3	D♯ Evv Fv4	D♯^ Ev Fv3	D♯^^ **E** Fvv	E^ Fv	E^^ **F**			
55-edo ♯ = 4	**D** E♭♭^	D^ E♭♭	D^^ E♭♭^	D♯v E♭vv	D♯ E♭v F♭♭vv	D♯^ E♭ F♭♭v	D♯^^ E♭^ F♭♭	Dxv Evv F♭♭^	Dx Ev F♭vv	Dx^ **E** F♭v	Dx^^ E^ F♭	E^^ F♭^	E♯v Fvv	E♯ Fv	E♯^ **F**	
56-edo ♯ = 7	**D** E♭v3	D^ E♭vv	D^^ E♭v	D^3 E♭ F♭v3	D♯v3 E♭^ F♭vv	D♯vv E♭^^ F♭v	D♯v E♭^3 F♭	D♯ Ev3 F♭^	D♯^ Evv F♭^^	D♯^^ Ev F♭^3	D♯^3 **E** Fv3	E^ Fvv	E^^ Fv	E^3 **F**		
57-edo ♯ = 3	**D** E♭3	D^ E♭3^	D♯v E♭♭v	D♯ E♭♭ F♭4	D♯^ E♭♭^ F♭4^	Dxv E♭v F♭3v	Dx E♭ F♭3	Dx^ E♭^ F♭3^	D♯3v Ev F♭♭v	D♯3 **E** F♭♭	D♯3^ E^ F♭♭^	D♯4v E♯v F♭v	D♯4 E♯ F♭	E♯^ F♭^	Exv Fv	Ex **F**
58-edo ♯ = 6	**D** E♭♭^^	D^ E♭v3	D^^ E♭vv	D^3 E♭v F♭♭^	D♯vv E♭ F♭♭^^	D♯v E♭^ F♭v3	D♯ E♭^^ F♭vv	D♯^ Ev3 F♭v	D♯^^ Evv F♭	D♯^3 Ev F♭^	Dxvv **E** F♭^^	Dxv E^ Fv3	E^^ Fvv	E^3 Fv	E♯vv **F**	
59-edo ♯ = 9	**D** E♭vv	D^ E♭v	D^^ E♭ F♭vv	D^3 E♭^ F♭v	D^4 E♭^^ F♭	D♯v4 E♭^3 F♭^	D♯v3 E♭^4 F♭^^	D♯vv Ev4 F♭^3	D♯v Ev3 F♭^4	D♯ Evv Fv4	D♯^ Ev Fv3	D♯^^ **E** Fvv	E^ Fv	E^^ **F**		
60-edo ♯ = 5	**D** E♭♭	D^ E♭♭^	D^^ E♭♭^^	D♯vv E♭vv	D♯v E♭v F♭♭v	D♯ E♭ F♭♭	D♯^ E♭^ F♭♭^	D♯^^ E♭^^ F♭♭^^	Dxvv Evv F♭vv	Dxv Ev F♭v	Dx **E** F♭	Dx^ E^ F♭^	E^^ F♭^^	E♯vv Fvv	E♯v Fv	E♯ **F**
61-edo ♯ = 8	**D** E♭v3	D^ E♭vv	D^^ E♭v	D^3 E♭ F♭v3	D^4 E♭^ F♭vv	D♯v3 E♭^^ F♭v	D♯vv E♭^3 F♭	D♯v Ev4 F♭^	D♯ Ev3 F♭^^	D♯^ Evv F♭^3	D♯^^ Ev Fv4	D♯^3 **E** Fv3	E^ Fvv	E^^ Fv	E^3 **F**	
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
62 ♯ = 4	**D** E♭♭vv	D^ E♭♭^	D^^ E♭♭	D♯v E♭♭^	D♯ E♭vv F♭3	D♯^ E♭v F♭3^	D♯^^ E♭ F♭♭vv	DXv E♭^ F♭♭v	DX Evv F♭♭	DX^ Ev F♭♭^	DX^^ **E** F♭vv	D♯3v E^ F♭v	D♯3 E^^ F♭	E♯v F♭^	E♯ Fvv	E♯^ Fv	E♯^ **F**		
63 ♯ = 7	**D** E♭♭^3	D^ E♭v3	D^^ E♭vv	D^3 E♭v	D♯v3 E♭ F♭♭^3	D♯vv E♭^ F♭v3	D♯v E♭^^ F♭vv	D♯ E♭^3 F♭v	D♯^ Ev3 F♭	D♯^^ Evv F♭^	D♯^3 Ev F♭^^	DXv3 **E** F♭^3	E^ Fv3	E^^ Fvv	E^3 Fv	E♯v3 **F**			
64 ♯ = 3	**D** E♭3v	D^ E♭3	D♯v E♭3^	D♯ E♭♭v	D♯^ E♭♭ F♭4v	DXv E♭♭^ F♭4	DX E♭v F♭4^	DX^ E♭ F♭3v	D♯3v E♭^ F♭3	D♯3 Ev F♭3^	D♯3^ **E** F♭♭v	D♯4v E^ F♭♭	D♯4 E♯v F♭♭^	D♯4^ E♯ F♭v	E♯^ F♭	EXv F♭^	EX Fv	EX^ **F**	
65 ♯ = 6	**D** E♭♭^	D^ E♭♭^^	D^^ E♭v3	D^3 E♭vv	D♯vv E♭v F♭♭	D♯v E♭ F♭♭^	D♯ E♭^ F♭♭^^	D♯^ E♭^^ F♭v3	D♯^^ Ev3 F♭vv	D♯^3 Evv F♭v	DXvv Ev F♭	DXv **E** F♭^	DX E^ F♭^^	E^^ Fv3	E^3 Fvv	E♯vv Fv	E♯v **F**		
66 ♯ = 9	**D** E♭v3	D^ E♭vv	D^^ E♭v	D^3 E♭ F♭v3	D^4 E♭^ F♭vv	D♯v4 E♭^^ F♭v	D♯v3 E♭^3 F♭	D♯vv E♭^4 F♭^	D♯v Ev4 F♭^^	D♯ Ev3 F♭^3	D♯^ Evv F♭^4	D♯^^ Ev Fv4	D♯^3 **E** Fv3	E^ Fvv	E^^ Fv	E^3 **F**			
67 ♯ = 5	**D** E♭♭v	D^ E♭♭	D^^ E♭♭^	D♯vv E♭♭^^ F♭3^	D♯v E♭vv F♭3^^	D♯ E♭v F♭♭vv	D♯^ E♭ F♭♭v	D♯^^ E♭^ F♭♭	DXvv E♭^^ F♭♭^	DXv Evv F♭♭^^	DX Ev F♭vv	DX^ **E** F♭v	DX^^ E^ F♭	D♯3vv E^^ F♭^	D♯3v E♯vv F♭^^	E♯v Fvv	E♯ Fv	E♯^ **F**	
68 ♯ = 8	**D** E♭v4	D^ E♭v3	D^^ E♭vv	D^3 E♭v F♭♭^3	D^4 E♭ F♭v4	D♯v3 E♭^ F♭v3	D♯vv E♭^^ F♭vv	D♯v E♭^3 F♭v	D♯ Ev4 F♭	D♯^ Ev3 F♭^	D♯^^ Evv F♭^^	D♯^3 Ev F♭^3	D♯^4 **E** Fv4	DXv3 E^ Fv3	E^^ Fvv	E^3 Fv	E^4 **F**		
69 ♯ = 4	**D** E♭3^	D^ E♭♭vv	D^^ E♭♭v	D♯v E♭♭	D♯ E♭♭^ F♭3vv	D♯^ E♭vv F♭3v	D♯^^ E♭v F♭3	DXv E♭ F♭3^	DX E♭^ F♭♭vv	DX^ Evv F♭♭v	DX^^ Ev F♭♭	D♯3v **E** F♭♭^	D♯3 E^ F♭vv	D♯3^ E^^ F♭v	D♯3^^ E♯v F♭	E♯ F♭^	E♯^ Fvv	E♯^^ Fv	EXv **F**
70 ♯ = 7	**D** E♭♭^^	D^ E♭♭^3	D^^ E♭v3	D^3 E♭vv F♭♭	D♯v3 E♭v F♭♭^	D♯vv E♭ F♭♭^^	D♯v E♭^ F♭♭^3	D♯ E♭^^ F♭v3	D♯^ E♭^3 F♭vv	D♯^^ Ev3 F♭v	D♯^3 Evv F♭	DXv3 Ev F♭^	DXvv **E** F♭^^	DXv E^ F♭^3	DX E^^ Fv3	E^3 Fvv	E♯v3 Fv	E♯vv **F**	
71 ♯=10	**D** E♭v3	D^ E♭vv	D^^ E♭v	D^3 E♭ F♭v3	D^4 E♭^ F♭vv	D^5 E♭^^ F♭v	D♯v4 E♭^3 F♭	D♯v3 E♭^4 F♭^	D♯vv Ev5 F♭^^	D♯v Ev4 F♭^3	D♯ Ev3 F♭^4	D♯^ Evv Fv5	D♯^^ Ev Fv4	D♯^3 **E** Fv3	E^ Fvv	E^^ Fv	E^3 **F**		
72 ♯ = 6	**D** E♭♭	D^ E♭♭^	D^^ E♭♭^^	D^3 E♭v3	D♯vv E♭vv F♭♭vv	D♯v E♭v F♭♭v	D♯ E♭ F♭♭	D♯^ E♭^ F♭♭^	D♯^^ E♭^^ F♭♭^^	D♯^3 Ev3 F♭v3	DXvv Evv F♭vv	DXv Ev F♭v	DX **E** F♭	DX^ E^ F♭^	DX^^ E^^ F♭^^	E^3 Fv3	E♯vv Fvv	E♯v Fv	E♯ **F**
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18

As discussed at the end of chapter 5.6, note names may be affected by the chord they are part of. In 24-edo, a C^ minor chord may well be written C^ E♭^ G^, not C^ Ev G^. This chart shows all the options for 24-edo in full:

Table 5.7.3 – Notation Guide for 24-edo Using Ups and Downs, In Full

24-edo	diatonic sharp-2	**D** D♯vv E♭♭ E♭vv	D^ D♯v E♭♭^ E♭v	D^^ D♯ Dxvv E♭♭^^ E♭ Evv F♭♭ F♭vv	D♯^ Dxv E♭^ Ev F♭♭^ F♭v	D♯^^ Dx E♭^^ **E** E♯vv F♭♭^^ F♭ Fvv	E^ E♯v F♭^ Fv	E^^ E♯ F♭^^ **F**

Of all the diatonic edos, 42-edo has the sharpest fifth, and 47-edo has the flattest. 42-edo requires the use of triple-ups and triple-downs, and 47-edo requires triple sharps and flats. In certain keys, even more. For example, in 47-edo, a h7 or 4:5:6:7 chord three edosteps above D requires at least quintuple sharps or flats to spell correctly.

Extremely large edos require even more ups and downs. For example, in 1200-edo, a sharp-100 edo, an up equals exactly one cent, and the notation is the same as simply writing the cents offset from 12-edo next to each note, with an up or down replacing the plus or minus sign.

The Tonal Plexus by H-Pi is a 205-edo keyboard, with the keys visually grouped into 41 blocks of five. 205-edo is a sharp-20 edo. To notate the Tonal Plexus, it would make sense to borrow lifts and drops (/ and \) from rank-2 pergen notation, and use an extended 41edo notation, such that /^{5}1 = ^1 = 1\41, or ^^{5}1 = /1 = 1\41.

When working with many different edos, it can be hard to remember the keyspan of various intervals. This next table shows a method for easily finding many keyspans. It's based on the fact that the white line in the previous graph tends to go up-up-down-up-down every five edos, especially when 13-edo and 18-edo are tweaked.

Table 5.7.4 – Keyspan of the major 2nd in edos 5-54 (* asterisk indicates tweaked edos 13b and 18b)

5-edo	1	–	1	–	1	9-edo
10-edo	2	1	2	1*	2	14-edo
15-edo	3	2	3	2*	3	19-edo
20-edo	4	3	4	3	4	24-edo
25-edo	5	4	5	4	5	29-edo
30-edo	6	5	6	5	6	34-edo
35-edo	**5**	6	7	6	7	39-edo
40-edo	**6**	7	8	7	8	44-edo
45-edo	**7**	8	**7**	8	9	49-edo
50-edo	**8**	9	**8**	9	10	54-edo

This table shows the size in edosteps of the major 2nd in various edos. The major 2nd is defined as the interval between the best approximations of 4/3 and 3/2. Thus for 16-edo, the major 2nd is 2\16 = 150¢, not 3\16 = 225¢, even though the latter is closer to 9/8.

The top row is for edos 5 through 9, the next row is for edos 10 through 14, etc. Supersharp and trivial edos are either tweaked to be superflat or omitted. The table follows a surprisingly regular pattern. For N-edo, the keyspan of M2 is always even when N is even, and odd when N is odd. It's usually equal to N divided by 5, rounded down to either an even or an odd number, as needed. Exceptions are shown in red.

The M2 keyspan can be used to find other keyspans. For example, for 22-edo:

> M2 = 22 / 5 rounded down = 4.4 rounded down = 4 edosteps
> P5 = half of M9 = half of (P8 + M2) = (22 + 4) / 2 = 13 edosteps
> P4 = P5 - M2 = 13 - 4 = 9 edosteps
> m2 = P4 - 2 M2 = 9 - 2·4 = 1 edostep
> A1 = M2 - m2 = 4 - 1 = 3 edosteps, hence 22-edo is sharp-3

For 31-edo, since 31 is odd, M2's keyspan must be odd as well.

> M2 = 31 / 5 rounded down to an odd number = 5 edosteps
> P5 = (31 + 5) / 2 = 18 edosteps
> P4 = 18 - 5 = 13 edosteps
> m2 = 13 - 2·5 = 3 edosteps
> A1 = 5 - 3 = 2 edosteps, hence 31-edo is sharp-2

More intervals: m7 = P8 - M2 = 31 - 5 = 26 edosteps, and m3 = P4 - M2 = 13 - 5 = 8 edosteps.

In the scale tree, the sharpness lines are ripples spreading out from the heptatonic kite. Every kite has ripple lines. The pentatonic kite has ripple lines that represent the keyspan of the minor 2nd.

The first column in the table is for edos ending in 0 or 5, the 2nd column's edos end in 1 or 6, etc. Each column corresponds to pentatonic ripple lines. The first column is the pentatonic spine, where m2 = 0 keys. The third column is the lefthand edge of the pentatonic kite, where m2 = 1 key. The fifth column is the next ripple line, where m2 = 2 keys.

The formula M2 = roundDown (N / 5) gives the wrong answer for some edos. The first column errs for those edos which are multiples of 5, but not pentatonic, 35-edo and up. On the scale tree, this is where the spine on the pentatonic kite ends. The third column errs where the lefthand edge of the pentatonic kite ends, 47-edo and up. The fifth column errs from 64-edo on. Above 60-edo, this formula is wrong more often than right, and isn't recommended. But it works for most midsized edos.

Chapter 5.8 – Chord Names and Scale Names with Ups and Downs

Imagine if instead of saying "D major chord", musicians had to count semitones and say "0-4-7 in D". For A7 they would say "2-5-7-11". That's what microtonalists often resort to. But ups and downs make naming chords much easier. Chord names are based on jazz chord names (see "A Player's Guide to Chords & Harmony" by Jim Aiken), with ups and downs added in. A few special cases:

In perfect frameworks (7, 14, 21, 28 and 35), every interval is perfect. When naming chords, never use major, minor, dim or aug. Substitute up for upmajor and upminor, and down for downmajor and downminor. See chapter 5.10.

Supersharp frameworks: 13-edo and 18-edo use their second-best fifth, to convert them to superflat. 6-edo is notated as a subset of 12-edo, and 8-edo as a subset of 24-edo. See chapter 5.12.

In superflat frameworks (9, 11, 13b, 16, 18b and 23), one way of notating them is to have major be narrower than minor, and to have sharpening or augmenting lower the pitch. See chapter 5.13.

All lower degrees are assumed to be present: A 7th chord has a 3rd and a 5th, a 9th chord has both these plus a 7th, an 11th chord has all these plus a 9th, and a 13th chord has all these plus an 11th.

Alterations are always enclosed in parentheses, additions never are. In the written name, commas are used as needed to separate added notes: A,v7 = A–C♯–E–Gv. "Add" is never written but must sometimes be spoken. "Sus" is never written or spoken. Unlike JI chords, enharmonic substitutions are allowed.

Conventionally, B♭5 means B♭–F, and B(♭5) means B–D♯–F. A similar issue arises with ups and downs. A period in a chord name indicates whether an up or down applies to the root or to the chord type. For example, Av.m = Av–Cv–Ev, A.vm = A–Cv–E, and Av.vm = Av–Cvv–Ev. No need to put a period before parentheses: A^(v4) not A^.(v4).

Conventionally, B♭5 is spoken as "B-flat five", and B(♭5) is "B flat-five". Likewise, Av.m is "A-down minor", and A.vm is "A downminor". Sometimes the period must be pronounced as "dot". For example, Cv–Ev–Gv is a Cv chord, "C down", and C–Ev–G is a C.v chord, "C dot down". Also, A.v7 is "A dot down-seven", because "A down-seven" would be A,v7 = A–C♯–E–Gv. Even if the period doesn't need to be pronounced, it's always acceptable to do so.

Applying "dot up" or "dot down" (or "dot double-up", etc.) to a chord raises or lowers the 3rd, and also the 6th, 7th or 11th, if present. Thus "G dot down nine" is the usual G9 chord with the 3rd and 7th lowered: G.v9 = G–Bv–D–Fv–A. Likewise, a "dot mid" chord has a neutral 3rd and a neutral 6th/7th, and possibly a half-aug 11th. The rationale for this rule is that a chord often has a note a perfect fourth or fifth above the 3rd. Furthermore, in many edos, the upfifth, downfifth, upfourth and downfourth will all be quite dissonant and rarely used in chords. Thus if the 3rd is upped or downed, the 6th or 7th likely would be too. If the 7th is, the 11th would be too. However, the 9th likely wouldn't, since that would create an upfifth or a downfifth with the 5th. Nor would the 13th, in order to make a good fifth with the 9th.

An alteration such as "up-three" makes the 3rd either upmajor or upminor, depending on the context. For example, C7(^3) has an up<u>major</u> 3rd, but Cm7(^3) has an up<u>minor</u> 3rd. Without context, as with added notes, the usual assumptions are made: M2, M3, P4, P5, M6, m7, M9, P11, M13. Thus C,^7 has an upminor 7th.

Chord progressions use ups/downs notation to name the roots. Here's the first four chords of Paul Erlich's 22-edo composition "Tibia" from chapter 5.6:

> G.vM7no5 = G dot downmajor seven, no five
> E♭^.v,9 = E-flat-up dot down, add nine
> C7(4) = C-seven four
> A7(v3) = A-seven down-three

Relative notation applies ups and downs to the usual roman numerals. Periods are used as before for consistency. Thus 1–M3–5–vm7 = I,v7 = one down-seven, 1–vM3–5–vm7 = I.v7 = one dot down-seven, and v1–vM3–v5–vm7 = vI7 = down-one seven. We must write VIm not vi, because "vi" could mean "down-one minor". Thus **roman numerals are always upper-case**. The "Tibia" chords:

> I.vM7no5 = one dot down major seven, no five

^♭VI.v,9 = upflat six dot down, add nine
IV7(4) = four-seven sus-four
II7(v3) = two-seven down-three

In 19-edo, the 4:5:6:7 chord is C–E–G–B♭♭. This is C,♭♭7 = C double-flat-seven. Sharp and flat are relative to M7, the default 7th in a scale. One might wonder, why M7 and not m7, since m7 is the default 7th in a chord? C–E–G–B♭♭ would then be C,♭7 = C flat-seven. The problem is that in the key of D, C–E–G–B♭♭ would be ♭VII,♭7 = flat-seven flat-seven. The root of the chord would be ♭VII = min 7th, but the 7th of the chord would be ♭7 = dim 7th, and "flat-seven" would have two different meanings, very confusing. Alternate names for C double-flat-seven are C major dim-seven and C add dim-seven, both written as C,d7.

To find a chord's name, determine its component intervals, then use the following tables. These tables aren't exhaustive, but they do provide enough examples to extrapolate from. As noted at the end of chapter 2.5, slash chords in relative notation can be notated relative to either the chord root or the scale's tonic.

Chord names should be constructed whenever possible as conventional chords modified unconventionally, rather than as unconventional chords modified conventionally. For C–E♭^–G–A, C minor six up-three is preferred over C upminor add six, because C minor six is conventional, and C upminor isn't. However, "dot up" and "dot down" chords are an exception to this rule: C.v9 is preferred over C9(v3,v7).

If ups and downs are removed from the name, the result should be the closest conventional 12-edo chord. Therefore, avoid double-ups and double-downs if possible: in sharp-3 edos, "upflat-five" is preferred to "double-down five".

Chord names are mostly independent of the edo: A–C^–E is usually A.^m = A upminor. But in perfect edos, major, minor, aug and dim aren't used, and A–C^–E is A.^ = A dot up. In edos with a sharpness of 2, 4, or higher, mid, up-mid, etc. are used. In sharp-2 edos, A–C^–E is A~ = A mid. There are extra columns in the tables below that cover perfect edos and edos with large sharpness. If there is no entry in this column, use the 2nd and 3rd columns instead.

Table 5.8.1 – Various triads

Chord	Written name	Spoken name	In perfect edos	In certain edos
C E G	C	C or C major	C or C perfect	
C E^ G	C.^	C upmajor or C dot up	C dot up	
C E^^ G	C.^^	C double-upmajor or C dot double-up	C dot double-up	
C Ev G	C.v	C downmajor or C dot down	C dot down	C~ = C mid[1]
C Evv G	C.vv	C double-downmajor or C dot double-down	C dot double-down	C~ = C mid[2] C.^~ = C up-mid[3]
C E♭ G	Cm	C minor	C or C perfect	
C E♭^ G	C.^m	C upminor	C.^ = C dot up	C~ = C mid[1]
C E♭^^ G	C.^^m	C double-upminor	C.^^ = C dot double-up	C~ = C mid[2] C.v~ = C down-mid[3]
C E♭v G	C.vm	C downminor	C.v = C dot down	
C E♭vv G	C.vvm	C double-downminor	C.vv = C dot double-down	

[1] In sharp-2 edos
[2] In sharp-4 edos
[3] In sharp-5 edos and sharp-6 edos

In some edos, it's common to have chords containing dim 4ths, dim 3rds, aug 7ths, double-dim 5ths, etc. Examples are given in each table of such extreme chords.

Table 5.8.2 – More triads

Chord	Written name	Spoken name	In perfect edos
C D G	C2 or C(2)	C two	
C D^ G	C(^2) (never C.^2)	C up-two (never "C dot up-two")	
C D♯ G	C(♯2) or C(A2)	C sharp-two or C aug-two	C2

C E♭♭ G	C(d3) or C(♭♭3)	C dim-three or C double-flat-three	C
C E♯ G	C(♯3) or C(A3)	C sharp-three or C aug-three	C

C F G	C4 or C(4)	C four	
C Fv G	C(v4) (never C.v4)	C down-four (never "C dot down-four")	
C F♭ G	C(♭4) or C(d4)	C flat-four or C dim-four not "C-flat four" = C♭(4)	C4

C E♭ F G	Cm,4	C minor, add four	C,4 = C add four
C E♭ F^ G	Cm,^4	C minor, add up-four	C,^4 = C add up-four
C D E G	C,2	C add two	

C G	C5	C five	
C E	Cno5	C no-five	
C E♭	Cm,no5	C minor, no-five	Cno5
C Ev	C.v,no5	C downmajor no-five or C dot down no-five	C dot down no-five
C E♭^	C.^m,no5	C upminor, no-five	

As noted in chapter 3.8, "JI Chord Names", Cdim is a triad, not a tetrad. A diminished tetrad is a dim7 chord.

In names like C.^dim = C up-dim, since "dim" refers to the 5th, one might expect the 5th to be upped. But it's the 3rd that's upped, because "dot up" always applies to only the 3rd, 6th, 7th and 11th. Likewise C.vaug = C down-aug downs the 3rd.

Table 5.8.3 – Triads with altered fifths

Chord notes	Written name	Spoken name	In perfect edos	In sharp-2 edos
C E G^	C(^5)	C up-five		
C G^	C(^5)no3	C up-five, no-three		
C E Gv	C(v5)	C down-five		
C Ev Gv	C.v(v5)	C dot down, down-five or C downmajor, down-five		C~(v5) = C mid down-five
C E♭ Gv	Cm(v5)	C minor, down-five	C(v5)	
C E♭v Gv	C.vm(v5)	C downminor, down-five	C.v(v5)	
C E♭ G♭	Cdim	C dim	C	
C E♭^ G♭	C.^dim	C up-dim	C.^	C~dim = C mid-dim
C E♭ G♭^	Cdim(^5)	C dim, up-five	C(^5)	Cm(v5)
C E♭^ G♭^	C.^dim(^5)	C up-dim, up-five	C.^(^5)	C~(v5) = C mid down-five
C E G♯	Caug	C aug	C	
C Ev G♯	C.vaug	C down-aug	C.v	C~aug = C mid-aug
C E G♯v	Caug(v5)	C aug, down-five	C(v5)	
C Ev G♯v	C.vaug(v5)	C down-aug, down-five	C.v(v5)	
C E G♭	C(♭5) or C(d5)	C major, flat-five or C major, dim-five	C	
C E♭ G♭♭	Cm(♭♭5) or Cm(dd5)	C minor, double-flat-five or C minor, double-dim five	C	
C E♭♭ G♭	Cdim(♭♭3) or Cdim(d3)	C dim, double-flat three or C dim, dim-three	C	
C E♭♭ G♭♭	C(♭♭3,♭♭5) or C(d3,dd5)	C double-flat three, double-flat five or C dim-three, double-dim five	C	
C E♭ G♯	Cm(♯5) or Cm(A5)	C minor, sharp-five or C minor, aug-five	C	
C E♯ G♯	Caug(♯3) or Caug(A3)	C aug, sharp-three or C aug, aug-three	C	
C E G♯♯	C(♯♯5) or C(AA5)	C double-sharp-five or C double-aug-five	C	

Table 5.8.4 – Seventh chords

Chord notes	Written name	Spoken name	In perfect edos	In sharp-2 edos
C E G B♭	C7	C seven		
C E B♭	C7no5	C seven, no-five		
C Ev G B♭	C7(v3)	C seven, down-three		C7(~3)
C E G B♭v	C,v7	C down-seven		
C Ev G B♭v	C.v7	C dot down seven		C~,v7
C Evv G B♭vv	C.vv7	C dot double-down seven		
C E Gv B♭	C7(v5)	C seven, down-five		
C Ev Gv B♭	C7(v3,v5)	C seven, down-three, down-five		C7(~3,v5)
C Ev Gv B♭v	C.v7(v5)	C dot down seven, down-five		same, or C~(v5)v7
C Ev G B♭vv	C.v,vv7	C dot down, double-down seven		
C E G B♭^	C,^7	C up-seven		C,~7 C mid-seven
C Ev G B♭^	C.v,^7	C dot down, up-seven		C.~7 C dot mid seven
C E♭ G B♭	Cm7	C minor seven	C7	
C E♭^ G B♭	Cm7(^3)	C minor seven, up-three	C7(^3)	C7(~3)
C E♭ G B♭^	Cm,^7	C minor, up-seven	C,^7	Cm,~7
C E♭^ G B♭^	C.^m7	C dot up minor-seven	C.^7	C.~7
C E G B	CM7	C major seven	C7	
C Ev G B	CM7(v3)	C major seven, down-three	C7(v3)	CM7(~3)
C E G Bv	C,vM7	C downmajor-seven	C,v7	C,~7
C Ev G Bv	C.vM7	C dot down major-seven	C.v7	C.~7
C F G B♭	C7(4) or C4,7	C seven, four or C four, seven		
C F^ G B♭	C7(^4) or C(^4),7	C seven, up-four or C up-four, seven		
C F G B♭^	C4^7 or C.^7(4)	C four, up-seven or C dot up-seven, four		C4,~7 or C.~7(4)
C F^ G B♭^	C(^4)^7 or C.^7(^4)	C up-four, up-seven or C dot up-seven, up-four		C(^4)~7 or C.~7(^4)
C E♭ G♭ B♭♭	Cdim7	C dim seven (Cdim would be a triad)	C7	
C E♭^ G♭ B♭♭	Cdim7(^3)	C dim seven, up-three	C7(^3)	Cdim7(~3)

C E♭ G♭^ B♭♭	Cdim7(^5)	C dim seven, up-five	C7(^5)	
C E♭ G♭ B♭♭^	Cdim,^d7 or Cdim, ^♭♭7	C dim, updim-seven or possibly Cdim, up-double-flat-seven	C,^7	Cdim,v7
C E♭^ G♭ B♭♭^	C.^dim7	C dot up dim-seven	C.^7	same, or C.v7(♭5)
C E♭^ G♭^ B♭♭^	C.^dim7(^5)	C dot up dim-seven, up-five	C.^7(^5)	C.v7(v5)
C E♭ G♭ B♭	Cm7(♭5) or Cm7(d5)	C minor-seven, flat-five or C half-dim or C minor-seven, dim-five	C7	
C E♭^ G♭ B♭	Cm7(♭5,^3) or Cm7(d5, ^3)	C minor-seven, flat/dim-five, up-three or C half-dim, up-three	C7(^3)	C7(♭5,~3) or C7 (d5,~3)
C E♭ G♭^ B♭	Cm7(^♭5) or Cm7(^d5)	C minor-seven, upflat-five or C minor-seven updim-five or C half-dim, up-five	C7(^5)	Cm7(v5)
C E♭ G♭ B♭^	Cdim,^7	C dim, up-seven or C half-dim, up-seven	C,^7	Cdim,~7
C E♭^ G♭^ B♭	Cm7(^3,^♭5) Cm7(^3,^d5)	C minor seven, up-three, upflat-five or C half-dim, up-three, up-five	C7(^3,^5)	C7(~3,v5)
C E♭^ G♭ B♭^	C.^m7(♭5) or C.^m7(d5)	C dot up minor-seven, flat-five or C dot up half-dim	C.^7	C.~7(♭5) or C.~7(d5)
C E♭ G♭^ B♭^	Cdim(^5)^7	C dim, up-five, up-seven or C half-dim, up-five, up-seven	C(^5)^7	Cdim(^5)~7
C E♭^ G♭^ B♭^	C.^m7(^♭5) or C.^m7(^d5)	C dot up minor-seven, upflat-five or C dot up half-dim, up-five	C.^7(^5)	C.~7(v5)

C E G♯ B♭	Caug7	C aug seven	C7	
C E^ G♯ B♭	Caug7(^3)	C aug seven, up-three	C7(^3)	
C E G♯ B♭^	Caug,^7	C aug, up-seven	C,^7	
C E^ G♯ B♭^	C.^aug7	C dot up aug seven	C.^7	
C E G♯v B♭	Caug7(v5)	C aug seven, down-five	C7(v5)	
C E^ G♯v B♭^	C.^aug7(v5)	C dot up aug-seven, down-five	C7	

C E♯ G B♭	C7(♯3) or C7(A3)	C seven, sharp-three or C seven, aug-three	C7	
C E G B♭♭	C,♭♭7 or C,d7	C double-flat-seven or C major dim-seven or C add dim-seven (not "C dim-seven" = Cdim7)	C7	
C E♭♭ G B♭♭	C(♭♭3)♭♭7 or C(d3)d7	C double-flat-three, double-flat-seven or C dim-three, dim-seven	C7	
C E G B♯	C,♯7 or C,A7	C sharp-seven or C major aug-seven or C add aug-seven (not "C aug-seven" = Caug7 = C E G♯ B♭)	C7	
C E G C♭	C,♭8 or C,d8	C flat-eight or C dim-eight	C	

Table 5.8.5 – Ninth chords

Chord notes	Written name	Spoken name	In perfect edos	In sharp-2 edos
C E G D	C,9	C add nine		
C Ev G D	C.v,9	C dot down, add nine or C downmajor, add nine		C~,9 C mid add nine
C E G D^	C,^9	C add up-nine		
C Ev G D^	C.v,^9	C dot down, add up-nine or C downmajor, add up-nine		C~,^9

C E G B♭ D	C9	C nine		
C Ev G B♭ D	C9(v3)	C nine, down-three		C9(~3)
C E G B♭^ D	C9(^7)	C nine, up-seven		C9(~7)
C E G B♭ Dv	C7,v9	C seven, down-nine		
C Ev G B♭v D	C.v9	C dot down nine		C.~9
C Ev G B♭ Dv	C7(v3)v9	C seven, down-three, down-nine		C7(~3)v9
C E G B♭v Dv	C,v7,v9	C, down-seven, down-nine		
C Ev G B♭v Dv	C.v7,v9	C dot down seven, down-nine		C.~7,v9

C E G B D	CM9	C major nine	C9	
C Ev G B D	CM9(v3)	C major nine, down-three	C9(v3)	CM9(~3)
C E G Bv D	CM9(v7)	C major nine, down-seven	C9(v7)	C9(~7)
C Ev G Bv D	C.vM9	C dot down major-nine	C.v9	C.~9

C E♭ G B♭ D	Cm9	C minor nine	C9	
C E♭^ G B♭ D	Cm9(^3)	C minor nine, up-three	C9(^3)	C9(~3)
C E♭ G B♭^ D	Cm9(^7)	C minor nine, up-seven	C9(^7)	Cm9(~7)
C E♭^ G B♭^ D	C.^m9	C dot up minor-nine	C.^9	C.~9

C E G B♭ D♭	C7,♭9 or C7,m9	C seven, flat-nine (or minor-nine)	C9	
C Ev G B♭ D♭	C7,♭9(v3)	C seven, flat-nine, down-three	C9(v3)	C7,♭9(~3)
C E G B♭v D♭	C,v7,♭9	C, down-seven, flat-nine	C9(v7)	
C E G B♭ D♭v	C7,v♭9	C seven, downflat-nine	C7,v9	
C Ev G B♭v D♭	C.v7,♭9	C dot down seven, flat-nine	C.v9	
C Ev G B♭ D♭v	C7(v3)v♭9	C seven, down-three, downflat-nine	C7(v3)v9	C7(~3)v♭9
C E G B♭v D♭v	C,v7,v♭9	C down-seven, downflat-nine	C,v7,v9	
C Ev G B♭v D♭v	C.v7,v♭9	C dot down seven, downflat-nine	C.v7,v9	

Table 5.8.6 – Sixth and sixth/ninth chords

Chord notes	Written name	Spoken name	In perfect edos	In sharp-2 edos
C E G A	C6	C six		
C Ev G A	C6(v3)	C six, down-three		C6(~3)
C E G Av	C,v6	C down-six		C,~6 = C mid-six
C Ev G Av	C.v6	C dot down six		C.~6 = C dot mid six
C Ev G A^	C.v,^6	C dot down up-six		C~,^6
C E G A♭	C,♭6 or C,m6	C flat-six or C add minor six (not "C minor-six" = Cm6 = C E♭ G A)	C6	
C E♭ G A♭	Cm,♭6 or Cm,m6	C minor, flat-six (or minor-six)	C6	
C E♭ G A♯	Cm,♯6 or Cm,A6	C minor, sharp-six or C minor, aug-six	C6	
C E♭♭ G A	C6(d3) or C6(♭♭3)	C six, dim-three or C six, double-flat three	C6	
C E G A D	C6,9	C six, nine		
C Ev G A D	C6,9(v3)	C six, nine, down-three		C6,9(~3)
C E G Av D	C,v6,9	C down-six, nine		C,~6,9
C E G A Dv	C6,v9	C six, down-nine		
C Ev G Av D	C.v6,9	C dot down six, nine		C.~6,9
C Ev G A Dv	C6(v3)v9	C six, down-three, down-nine		C6(~3)v9
C E G Av Dv	C,v6,v9	C down-six, down-nine		C,~6,v9
C Ev G Av Dv	C.v6,v9	C dot down six, down-nine		C.~6,v9
C Ev G A^ D	C.v,^6,9	C dot down, up-six, nine or C downmajor, up-six, nine		C.~,^6,9
C E♭ G A D	Cm6,9	C minor six, nine	C6,9	
C E♭^ G A D	Cm6,9(^3)	C minor six, nine, up-three	C6,9(^3)	C6,9(~3)
C E♭ G Av D	Cm,v6,9	C minor, down-six, nine	C,^6,9	Cm,~6,9
C E♭^ G A^ D	C.^m6,9	C dot up minor-six, nine	C.^6,9	same, or C~,^6,9
C E G A D♭	C6,♭9 or C6,m9	C six, flat-nine (or minor nine)	C6,9	
C E G A♭ D	C,♭6,9	C flat-six, nine	C6,9	
C E G A♭ D♭	C,♭6,♭9	C flat-six flat-nine	C6,9	
C E♭ G A♭ D♭	Cm,♭6,♭9	C minor flat-six flat-nine	C6,9	
C E♭♭ G A♭ D	C(♭♭3)♭6,9	C dim-three flat-six nine	C6,9	

In a "dot mid" eleven or thirteen chord, the 11th is a perfect 5th above the mid-7th, and is therefore half-augmented.

Table 5.8.7 – Eleventh chords

Chord notes	Written name	Spoken name	In perfect edos	In sharp-2 edos
C E G B♭ D F	C11	C eleven		
C G B♭ D F	C11no3 or C9(4)	C eleven, no three or C nine, four		
C Ev G B♭ D F	C11(v3)	C eleven, down three		
C E^ G B♭^ D F	C11(^3,^7)	C eleven, up-three, up-seven		
C E^ G B♭^ D F^	C.^11	C dot up eleven		
C E G B♭ D F^	C9,^11	C nine, up-eleven		
C Ev G B♭^ D F^	C.^11(v3)	C dot up eleven, down-three		C.~11
C E G B♭^ D F^	C9(^7)^11	C nine, up-seven, up-eleven		C9(~7)^11
C E^ G B♭^ D F♯	C.^9,♯11 or C.^9,A11	C dot up nine, sharp-eleven or C dot up nine, aug-eleven	C.^9,11	
C E^ G B^ D F^	C.^M11	C dot up major eleven	C.^11	
C E^ G B^ D F♯^	C.^M9,^♯11	C dot up major nine, upsharp-eleven	C.^11	
C E♭^ G B♭^ D F^	C.^m11	C dot up minor eleven	C.^11	C.~11
C E♯ G B♯ D F♯♯ (Cy11 in 33-edo)	C9(♯3♯7)♯♯11 or C9(A3A7)AA11	C nine, sharp-three, sharp-seven, double-sharp eleven or C nine, aug-three, aug-seven, double-aug eleven	C11	
C E♭♭ G B♭♭ D F♭ (Cg11 in 33-edo)	C9(♭♭3♭♭7)♭11 or C9(d3d7)d11	C nine, double-flat three, double-flat seven, flat-eleven or C nine, dim-three, dim-seven, dim-eleven	C11	

Table 5.8.8 – Thirteenth chords

Chord notes	Written name	Spoken name	In perfect edos	In sharp-2 edos
C E G B♭ D F A	C13	C thirteen		
C E G B♭ D A	C9,13	C nine, thirteen		
C E^ G B♭^ D F^ A	C.^13	C dot up thirteen		
C E^ G B♭^ D F A	C13(^3,^7)	C thirteen, up-three, up-seven		C13(^3,~7)
C E♭^ G B♭^ D F^ A	C.^m13	C dot up minor thirteen	C.^13	C.~13
C Ev G Bv D Fv A	C.vM13	C dot down major thirteen	C.v13	C.~13(v11)
C Ev G Bv D F♯v A	C.vM13(v♯11)	C dot down major 13, downsharp-11	C.v13	
C Ev G Bv D F♯ A	C.vM13(♯11)	C dot down major 13, sharp-11	C.v13	

If two edos map the gu and ru commas the same, yaza chord names will be similar. Especially if the sharpness is the same. For example, 15-edo and 22-edo both map g1 to 1 edostep and r1 to zero edosteps. 19-edo and 26-edo both map g1 to 0 and r1 to 1. The following table groups edos with the same mappings into the same column:

Table 5.8.9 – Examples of various yaza JI chords in various edos

JI chord	12-edo (sharp-1)	15 & 22-edo (sharp-3)	19 & 26-edo (sharp-1)	21 & 28-edo (sharp-0)	24, 31 & 38 (sharp-2)	34, 41 & 48 (sharp-4) 46 & 53 (sharp-5)	72-edo (sharp-6)
Cy6	CM6	C.vM6	CM6	C.^6	CM6	C.vM6	C.vM6
Cy7	CM7	C.vM7	CM7	C.^7	CM7	C.vM7	C.vM7
Cg7	Cm7	C.^m7	Cm7	C.v7	Cm7	C.^m7	C.^m7
Cz7	Cm7	Cm7	C(d3)d7	C.v7	C.vm7	C.vm7	C.vvm7
Cr6	C6	C6	C(♯3)♯6	C.^6	C.^6	C.^6	C.^^6
Ch9	C9	C9(v3)	C9(d7)	C.^,v7,^9	C9(v7)	C.v9	C9(v3,vv7)
Cs6,11	Cm6,11	Cm6,11(^3)	Cm,♯6,11	C.v,^6,11	Cm,^6,11	C.^m6,11	C.vm,^^6,11
Cz,y6	Cm6	Cm,^6	C6(d3)	C.v,^6	Cm6(v3)	C.v6	C.vvm,v6
Cs9	C9	C9(^7)	C9(♯3)	C.^,v7,^9	C9(^3)	C.^9	C9(^^3,^7)
Cg7(zg5)	Cm7(♭5)	C.^m7(^♭5)	Cm7(♭♭5)	C.v7(vv5)	Cm7(v♭5)	C.^m7(♭5)	C.^m7(v♭5)
Cs7	Cm7(♭5)	Cm7(^♭5)	C(d3,dd5,d7)	C.v7(vv5)	C.vm7(v♭5)	C.vm7(♭5)	C.vvm7(v♭5)

Translating a JI chord to an edo can be tricky. In 21-edo and 28-edo, w9 = 9/4 is best approximated by an upmajor 9th. Approximating each note individually, Ch9 becomes C.^,v7,^9. But this chord would sound better with a major 9th, avoiding a dissonant upfifth between the 5th and the 9th. However, Cy,9no5 would be better with an upmajor 9th.

In 21-edo and 28-edo, Tw5 + Tw5 ≠ Tw9. Inequalities like this are inevitable. Every edo approximates a JI rung with a certain cents discrepancy. As JI intervals become more remote, the discrepancies accumulate, until they eventually total more than half an edostep. At this point, the best approximation of a JI interval is not equal to the sum of the best approximations of each component rung. (See also paradoxical intervals in appendix 6.)

Heptatonic scales are named as if they were chords with seven notes. While chords need a highly condensed name that can be fit several times into a measure of sheet music, scale names needn't be as concise. The ^, v and ~ symbols aren't used in scale names, instead the words are spelled out. Analogous to "dot down" chords, downmajor is a major scale with the 3rd, 6th and 7th lowered. Likewise, the aug scale is a major scale with the 3rd, 6th and 7th augmented.

Periods aren't needed because "C downmajor" is distinct from "C-down major" = Cv Dv Ev Fv Gv Av Bv Cv. As usual, major is assumed, a C scale is a C major scale.

In perfect edos, scale names never use the words "major" or "minor". One says C scale not C major scale, C downperfect not C downmajor, etc. Names usually never use "perfect" either, but one says C downperfect not C down, because "C down" could be either C downperfect or C-down perfect.

Table 5.8.11 – Scales

Scale notes	Written/spoken name	In perfect edos	In sharp-2 edos
C D E F G A B C	C or C major	C or C perfect	
C D Ev F G Av Bv C	C downmajor	C downperfect	C mid
C D Ev F G A B C	C down-3		C mid-3
C D Ev F G A Bv C	C down-3 down-7		C mid-3 mid-7
C D E♭ F G A♭ B♭ C	C minor	C	
C D E♭v F G A♭v B♭v C	C downminor	C downperfect	
C D E♭^ F G A♭^ B♭^ C	C upminor	C upperfect	C mid
C D E♭^ F G A♭ B♭ C	C minor up-3	C up-3	C minor mid-3
C D E♭^ F G A♭^ B♭ C	C minor up-3 up-6	C up-3 up-6	C minor mid-3 mid-6
C D E♭ F G A B♭ C	C dorian	C	
C D E♭^ F G A^ B♭^ C	C updorian	C upperfect	C mid upmajor-6
C D E♭^ F G A B♭^ C	C dorian up-3 up-7	C up-3 up-7	C mid-3 mid-7
C D E F G A B♭ C	C mixolydian	C	
C D E^ F G A^ B♭^ C	C upmixolydian	C upperfect	C upmajor mid-7
C D E♯ F G A♯ B♯ C	C aug	C	
C D E♯ F G A♯ B C	C aug-3 aug-6	C	
C D Ex F G Ax Bx C	C double-aug	C	
C D E♯3 F G A♯3 B♯3 C	C triple-aug	C	
C D E♭♭ F G A♭♭ B♭♭ C	C dim	C	
C D E♭3 F G A♭3 B♭3 C	C double-dim	C	
C D E♭♭ F G A B♭♭ C	C dorian dim-3 dim-7	C	

Chapter 5.9 – MOS Scales *

All frameworks contain a number of MOS scales. While lots of music uses non-MOS scales, MOS scales are a good place to start one's exploration of a framework.

The generator is listed in several forms, excluding those larger than half an octave, except that the perfect 5th is substituted for the perfect 4th, for consistency with historical practice.

The list excludes trivial scales: those with fewer than five notes, or fewer that three unused notes. Also excluded are scales generated by a single EDOstep, which are a single chromatic run. But if there is more than one period per 8ve, the generator is also a period plus an EDOstep, and such scales are not excluded.

The example scale is in D, and its genchain is centered on D. Each scale implies all its modes.

Table 5.9.1 – 12-tone MOS scales

periods per 8ve	generator	implied rank-2 temperaments	notes per 8ve	L and s steps		example scale
1 (P8)	P5	gT (gen: w5) rT (gen: w5)	5	m3	M2	D E G A C D
			7	M2	m2	D E F G A B C D
2 (A4, d5)	P5, m2	sggT (per: y4, gen: w5) LrrT (per: r4, gen: w5)	6	M3	m2	D E♭ G A♭ A D♭ D
			8	m3	m2	D E♭ E G A♭ A B♭ D♭ D
3 (M3)	P5, m3, m2	g^{3}T (per: y3, gen: w5) r^{3}T (per: r3, gen: w5)	6	m3	m2	D F F♯ A B♭ C♯ D
			9	M2	m2	D E♭ F F♯ G A B♭ B C♯ D
4 (m3)	P5, M3, M2, m2	g^{4}T (per: g3, gen: w5) r^{4}T (per: z3, gen: w5)	8	M2	m2	D E♭ F F♯ A♭ A B C D
6 (M2)	–	rryy&g^3-wT (per: r2, no gen)	6	M2	–	D E F♯ G♯ B♭ C D

These scales are derived mathematically solely from the number 12, without any reference to JI ratios. How many are actually used in music? Most of them. The first two (including all their modes) are the two most popular scales of all time. The last four are the augmented scale, the Tcherepnin scale, the diminished scale, and the whole tone scale. The other two have a large step at least three times as big as the small step, a little too lopsided for good melody.

Often MOS scales arise from temperaments, which result from needing to pump a certain comma. The "implied temperaments" column reverses this process. It starts with the scale, and asks "what commas can this scale pump?" The table lists 7-limit temperaments. To find higher limit commas, first decide on the keyspan of the higher primes. Then go comma-hunting!

To find a comma that implies a generator of a fifth, find an interval with keyspan zero and color depth of one (no double or triple colors, and not wa). As long as the framework isn't ringy (see the chapter on diatonic frameworks), every row will have one such interval every N nodes.

To find a comma that splits the octave in half, find any JI interval with a keyspan of half an octave. Your comma is this interval squared, minus an octave. If you get a descending interval, invert the ratio. Sometimes this comma won't work, because it's the square of another comma. To avoid this, and to make the comma easier to pump, avoid double colors. To find a ya comma, look along only the yo and gu rows for a half-octave interval. To find a za comma, find a ru or zo half-octave. To find a comma that splits the octave into thirds, find a yo or gu or zo or ru interval with a third-octave keyspan. And so forth.

Other temperaments can be derived from those listed. sggT and LrrT can be combined into sgg&LrrT. The new period is both y4 and r4. Or a non-splitting comma like g1 or r1 can be added in to make sgg&rT and g&LrrT. The period becomes the old period plus the new comma, which in both cases is zg5. The generator remains w5.

Because 19 is prime, 19-edo has no scales with multiple periods per octave. But there are still plenty of scales, because every scale degree of a prime edo can serve as a generator. We will also exclude scales with a L:s ratio of 3 or higher.

Table 5.9.2 – 19-tone MOS scales

periods per 8ve	generator	implied rank-2 temperaments	notes per 8ve	L and s steps		example scale centered on D
1 (P8)	P5	gT (gen: w5) LrT (gen: w5)	5	m3	M2	D E G A C D
			7	M2	m2	D E F G A B C D
			12	m2	A1	D E♭ E F F♯ G G♯ A B♭ B C C♯ D
1 (P8)	A4, d5					either too many notes, or L:s is too large
1 (P8)	A3, d4		5			
			8			
			11			
1 (P8)	M3					
1 (P8)	m3					
1 (P8)	A2, d3	zzT (r2, z3)				
1 (P8)	M2					
1 (P8)	m2					

Chapter 5.10 – Diatonic Frameworks: 12, 17, 19, 22, 24, 26 and 27 *

All diatonic frameworks have an average fifth within 12¢ of just. They all represent ya and zo at least as accurately as 12-edo does, except that 17-tone completely misses ya, with an almost 50% discrepancy. 19-tone and 22-tone have been covered earlier, and 12-tone needs no explanation. A quick review:

19-tone: One sharp = one key. Scale fragment: C – C♯ – D♭ – D. JI color associations: perfect = white, aug = ru, major = yo, minor = gu, dim = zo. Aug and dim overlap.

Table 5.10.1 – 19-tone notation, with preferred tonic names and key signatures, and the layout of black and white keys

steps	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
edo cents	0¢	63¢	126	189	253	316	379	442	505	568	632	695	758	821	884	947	1011	1074	1137	1200
JI ratios	w1	z2	g2	y2 w2	r2 z3	g3	**y3**	r3 z4	w4	zg5	ry4	**w5**	r5 z6	g6	y6	r6 **z7**	w7 g7	y7	r7	**w8**
intervals	P1	A1 d2	m2	M2	A2 d3	m3	M3	A3 d4	P4	A4 dd5	AA4 d 5	P5	A5 d6	m6	M6	A6 d7	m7	M7	A7 d8	p8
note names	**D**	D♯ E♭♭	D^X E♭	**E**	E♯ F♭	**F**	F♯ G♭♭	F^X G♭	**G**	G♯ A♭♭	G^X A♭	**A**	A♯ B♭♭	A^X B♭	**B**	B♯ C♭	**C**	C♯ D♭♭	C^X D♭	**D**
major keys	**D**	D♯	E♭	**E**	F♭	**F**	F♯	G♭	**G**	G♯	A♭	**A**	A♯	B♭	**B**	C♭	**C**	C♯	D♭	**D**
minor keys	"	"	"	"	E♯	"	"	"	"	"	"	"	"	"	"	B♯	"	"	"	"

22-tone: One sharp = three keys. Scale fragment: C – D♭ – [_] – C♯ – D. JI color associations: perfect = wa, major = ru, downmajor = yo, upminor = gu, minor = zo.

Table 5.10.2 – 22-tone notation, with preferred tonic names and key signatures, and the layout of black and white keys

steps	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21

cents	0¢	55¢	109	164	218	273	327	382	436	491	545	600	655	709	764	818	873	927	982	1036	1091	1145
JI ratios	w1	z2	g2	y2	r2	z3	g3	**y3**	r3	w4	**1o4**	ry4 zg5	1u5	**w5**	z6	g6	y6	r6	**z7**	g7	y7	**r7**
using ^/v	P1	m2	^m2	vM2	M2	m3	^m3	vM3	M3	P4	^4 d5	vA4 ^d5	A4 v5	P5	m6	^m6	vM6	M6	m7	^m7	vM7	M7
major keys	**D**	E♭	E♭^	Ev	**E**	**F**	F^	F♯v G♭^	Gv	**G**	A♭	A♭^	Av	**A**	B♭	B♭^	Bv	**B** (Cv)	**C**	D♭ (C^)	D♭^	Dv
minor keys	"	D^	D♯v E♭^	"	"	"	"	F♯v	F♯	"	G^	G♯v	G♯ (Av)	"	B♭ (A^)	"	"	**B**	"	C^	C♯v	C♯
no ^/v	P1	m2	d3	A1	M2	m3	d4	A2	M3	P4	d5	A3	A4	P5	m6	d7	A5	M6	m7	d8	A6	M7
major keys	**D**	E♭	F♭	D♯	**E**	**F**	G♭	E♯	F♯	**G**	A♭	B♭♭	G♯	**A**	B♭	C♭	A♯	**B**	**C**	D♭	E♭♭	C♯
minor keys	"	"	"	"	"	"	"	"	"	"	"	Fx	"	"	"	"	"	"	"	"	B♯	"

The next diatonic framework is 17-tone. One sharp = two keys. Scale fragment: C – D♭ – C♯ – D. Color associations: perfect = wa, major = ru, mid = ilo/lu/tho/thu, minor = zo.

Figure 5.10.1 – The 17-tone lattice, with tho Av, Ev and Bv, and ilo F^, C^ and G^

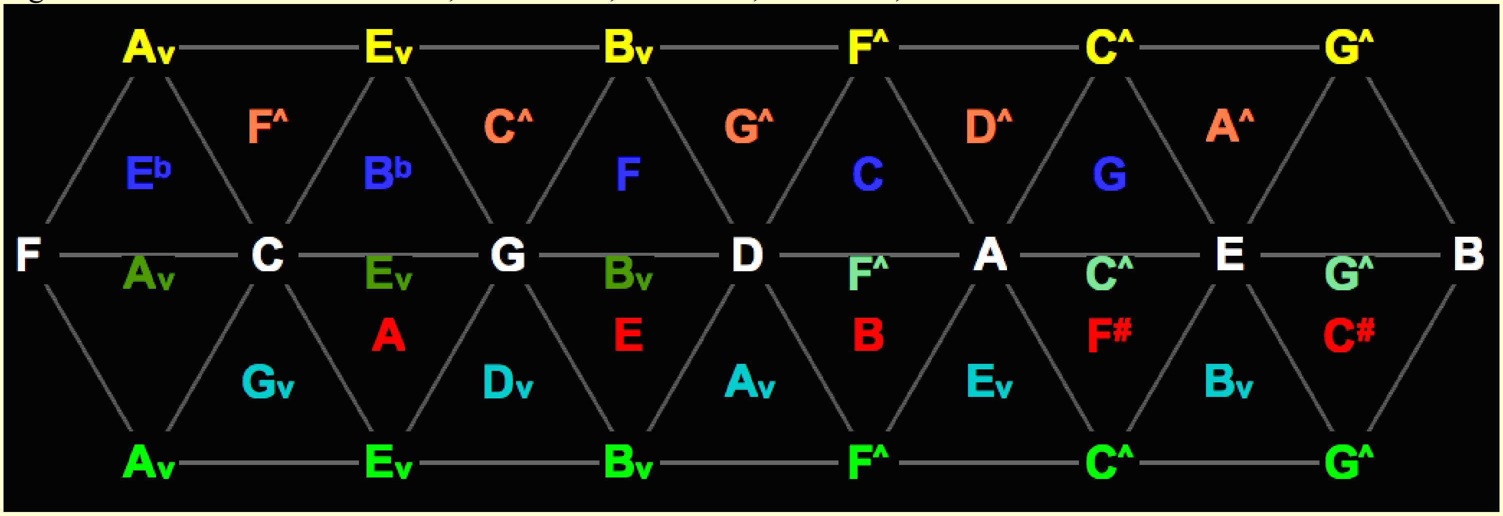


Table 5.10.3 – 17-tone notation, with preferred tonic names and key signatures, and the layout of black and white keys

steps	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
edo cents	0¢	71	141	212	282	353	424	494	565	635	706	776	847	918	988	1059	1129	1200
JI ratios	w1	z2	1u2	r2	z3	1o3	r3	w4	**1o4**	1u5	**w5**	z6	**3o6**	r6	**z7**	1o7	r7	**w8**
intervals (with ^/v)	P1	^1 m2	A1 ~2	M2	m3	~3 d4	M3 v4	P4	^4 d5	A4 v5	P5	^5 m6	A5 ~6	M6	m7	~7 d8	M7 v8	P8
major keys	**D**	E♭	E♭^ (Ev)	**E**	**F**	F^	Gv	**G**	A♭	Av (A♭^)	**A**	B♭	Bv	**B** (Cv)	**C**	D♭ (C^)	Dv	**D**
minor keys	"	D^	Ev	"	"	F♯v (F^)	F♯	"	G^	G♯ (Av)	"	B♭ (A^)	"	**B**	"	C^ (C♯v)	C♯	"
intervals (without ^/v)	P1	m2	A1 d3	M2	m3	A2 d4	M3	P4	d5	A4	P5	m6	A5 d7	M6	m7	d8	M7	P8
major keys	**D**	E♭	F♭ (D♯)	**E**	**F**	G♭	F♯	**G**	A♭	G♯	**A**	B♭	C♭	**B**	**C**	D♭	C♯	**D**
minor keys	"	"	D♯	"	"	E♯ (G♭)	"	"	"	"	"	"	A♯	"	"	"	"	"

As noted in chapter 5.6 with 22-tone, there are two approaches to assigning key signatures, using ups and downs vs. using double-sharps and double-flats. Table 5.10.3 lists the preferred tonic names and key signatures for both methods.

Figure 5.10.2 – The 17-tone guitar fretboard (asterisks indicate frets marked with dots)

E	**F**	F^ / G♭	F♯ / Gv	**G ***	G^ / A♭	G♯ / Av	**A ***
B	**C**	C^ / D♭	C♯ / Dv	**D ***	D^ / E♭	D♯ / Ev	**E ***
G	G^ / A♭	G♯ / Av	**A**	A^ / B♭	A♯ / Bv	**B**	**C ***
D	D^ / E♭	D♯ / Ev	**E**	**F ***	F^ / G♭	F♯ / Gv	**G ***
A	A^ / B♭	A♯ / Bv	**B**	**C ***	C^ / D♭	C♯ / Dv	**D ***
E	**F**	F^ / G♭	F♯ / Gv	**G ***	G^ / A♭	G♯ / Av	**A ***

Recall from chapter 5.6 that if the tonic has an up or down, all seven notes in the scale do as well. This is indicated with a "global" up or down, which appears in every key signature enclosed in a circle. See the "Tibia in F♯v" example.

Table 5.10.3 – 17-tone key signatures using ups and downs

key signature	major key	major scale	minor key	minor scale
♭ ♭ ♭ ♭ (^)	(A♭^ major)	(A♭^ B♭^ C^ D♭^ E♭^ F^ G^ A♭^)	(F^ minor)	(F^ G^ A♭^ B♭^ C^ D♭^ E♭^ F^)
♭ ♭ ♭ (^)	E♭^ major	E♭^ F^ G^ A♭^ B♭^ C^ D^ E♭^	C^ minor	C^ D^ E♭^ F^ G^ A♭^ B♭^ C^
♭ ♭ (^)	B♭^ major	B♭^ C^ D^ E♭^ F^ G^ A^ B♭^	G^ minor	G^ A^ B♭^ C^ D^ E♭^ F^ G^
♭ (^)	F^ major	F^ G^ A^ B♭^ C^ D^ E^ F^	D^ minor	D^ E^ F^ G^ A^ B♭^ C^ D^
(^) ♭ ♭ ♭ ♭ ♭	(C^ major) D♭ major	(C^ D^ E^ F^ G^ A^ B^ C^) D♭ E♭ F G♭ A♭ B♭ C D♭	(A^ minor) B♭ minor	(A^ B^ C^ D^ E^ F^ G^ A^) B♭ C D♭ E♭ F G♭ A♭ B♭
♭ ♭ ♭ ♭	A♭ major	A♭ B♭ C D♭ E♭ F G A♭	F minor	F G A♭ B♭ C D♭ E♭ F
♭ ♭ ♭	E♭ major	E♭ F G A♭ B♭ C D E♭	C minor	C D E♭ F G A♭ B♭ C
♭ ♭	B♭ major	B♭ C D E♭ F G A B♭	G minor	G A B♭ C D E♭ F G
♭	F major	F G A B♭ C D E F	D minor	D E F G A B♭ C D
no sharps or flats	C major	C D E F G A B C	A minor	A B C D E F G A
♯	G major	G A B C D E F♯ G	E minor	E F♯ G A B C D E
♯ ♯	D major	D E F♯ G A B C♯ D	B minor	B C♯ D E F♯ G A B
♯ ♯ ♯	A major	A B C♯ D E F♯ G♯ A	F♯ minor	F♯ G♯ A B C♯ D E F♯
♯ ♯ ♯ ♯	E major	E F♯ G♯ A B C♯ D♯ E	C♯ minor	C♯ D♯ E F♯ G♯ A B C♯
♯ ♯ ♯ ♯ ♯ (v)	B major (Cv major)	B C♯ D♯ E F♯ G♯ A♯ B (Cv Dv Ev Fv Gv Av Bv Cv)	G♯ minor (Av minor)	G♯ A♯ B C♯ D♯ E♯ F G♯ (Av Bv Cv Dv Ev Fv Gv Av)
♯ (v)	Gv major	Gv Av Bv Cv Dv Ev F♯v Gv	Ev minor	Ev F♯v Gv Av Bv Cv Dv Ev
♯ ♯ (v)	Dv major	Dv Ev F♯v Gv Av Bv C♯v Dv	Bv minor	Bv C♯v Dv Ev F♯v Gv Av Bv
♯ ♯ ♯ (v)	Av major	Av Bv C♯v Dv Ev F♯v G♯v Av	F♯v minor	F♯v G♯v Av Bv C♯v Dv Ev F♯v
♯ ♯ ♯ ♯ (v)	(Ev major)	(Ev F♯v G♯v Av Bv C♯v D♯v Ev)	(C♯v minor)	(C♯v D♯v Ev F♯v G♯v Av Bv C♯v)

Table 5.10.2 – Various 17-tone chords

JI chord	JI ratios	EDOsteps	notes	name	spoken name
Dz	1/1 – 7/6 – 3/2	0-4-10	D F A	Dm	D minor
D1o	1/1 – 11/9 – 3/2	0-5-10	D F^ A	D~	D mid
Dr	1/1 – 9/7 – 3/2	0-6-10	D F♯ A	D	D or D major
Dz7	1/1 – 7/6 – 3/2 – 7/4	0-4-10-14	D F A C	Dm7	D minor seven
D1o7	1/1 – 11/9 – 3/2 – 11/6	0-5-10-15	D F^ A C^	D.~7	D dot mid seven

The next diatonic framework is 24-tone, the first framework to contain 12-tone within it. It represents za about as poorly as 12-tone does, but represents la very well.

One sharp = two keys. Scale fragment: C – [_] – C♯/D♭ – [_] – D. Color associations: perfect = wa, upmajor = ru, major = yo, mid = ilo/lu/purple, minor = gu, downminor = zo. Upmajor and downminor overlap, so zo and ru share the same key or fret.

Figure 5.10.3 – The 24-tone lattice, with tho Av, Ev and Bv, and ilo F^, C^ and G^

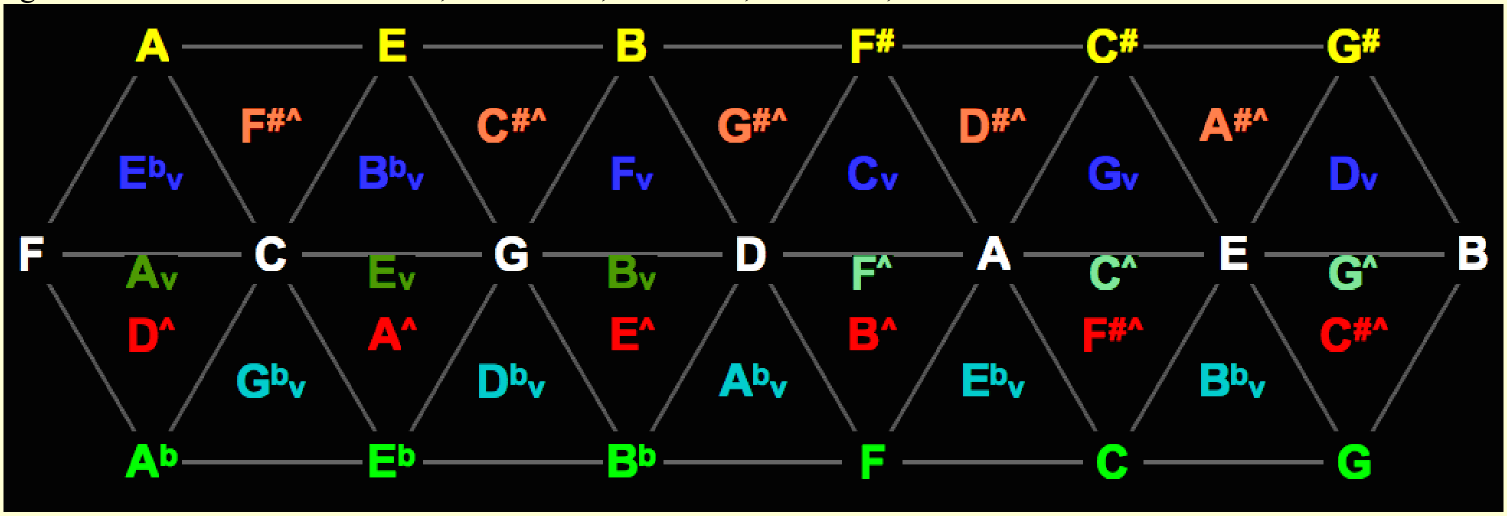


Table 5.10.4 – 24-tone notation and key signatures

steps	edo cents	JI	interval (with ^/v)	note names	major keys	minor keys
0	0¢	w1	P1	**D**	**D**	"
1	50	1o1	^1 / vm2	D^ / E♭v	D^	"
2	100	g2	A1 / m2	D♯ / E♭	E♭	D♯ / E♭
3	150	1u2	^A1 / ~2	D♯^ / Ev	Ev	"
4	200	y2, w2	M2	**E**	**E**	"
5	250	r2, z3	^M2 / vm3	E^ / Fv	E^ / Fv	"
6	300	w3, g3	A2 / m3	**F**	**F**	"
7	350	1o3	~3 / vd4	F^ / G♭v	F^	"
8	400	**y3**	M3 / d4	F♯ / G♭	F♯ / G♭	F♯
9	450	r3, z4	^M3 / v4	F♯^ / Gv	Gv	"
10	500	w4	P4	**G**	**G**	"
11	550	**1o4**	^4 / vd5	G^ / A♭v	G^	"
12	600	y4, g5	A4 / d5	G♯ / A♭	A♭	G♯
13	650	1u5	^A4 / v5	G♯^ / Av	Av	"
14	700	**w5**	P5	**A**	**A**	"
15	750	r5, z6	^5 / vm6	A^ / B♭v	A^	"
16	800	g6	A5 / m6	A♯ / B♭	B♭	"
17	850	**3o6**, 1u6	^A5 / ~6	A♯^ / Bv	Bv	"
18	900	y6, w6	M6 / d7	**B**	**B**	"
19	950	r6, **z7**	^M6 / vm7	B^ / Cv	B^ / Cv	"
20	1000	w7, g7	m7	**C**	**C**	"
21	1050	1o7	~7 / vd8	C^ / D♭v	C^	"
22	1100	y7	M7 / d8	C♯ / D♭	D♭	C♯
23	1150	1u7	^M7 / v8	C♯^ / Dv	Dv	"
24	1200	**w8**	P8	**D**	**D**	"

Figure 5.10.4 – The 24-tone guitar fretboard (*asterisks indicate frets marked with dots)

E	E^ / Fv	**F**	F^	F♯ / G♭	Gv	**G** *	G^	G♯ / A♭	Av	**A** *
B	B^ / Cv	**C**	C^	C♯ / D♭	Dv	**D** *	D^	D♯ / E♭	Ev	**E** *
G	G^	G♯ / A♭	Av	**A**	A^	A♯ / B♭*	Bv	**B**	B^ / Cv	**C** *
D	D^	D♯ / E♭	Ev	**E**	E^ / Fv	**F** *	F^	F♯ / G♭	Gv	**G** *
A	A^	A♯ / B♭	Bv	**B**	B^ / Cv	**C** *	C^	C♯ / D♭	Dv	**D** *
E	E^ / Fv	**F**	F^	F♯ / G♭	Gv	**G** *	G^	G♯ / A♭	Av	**A** *

Table 5.10.5 – 24-tone chords

JI chord	JI ratio	EDOsteps	notes	name	spoken name
Db	1/1 – 7/6 – 3/2	0-5-14	D Fv A	D.vm	D downminor
Dg	1/1 – 6/5 – 3/2	0-6-14	D F A	Dm	D minor
D1o	1/1 – 11/9 – 3/2	0-7-14	D F^ A	D~	D mid
Dy	1/1 – 5/4 – 3/2	0-8-14	D F♯ A	D	D or D major
Dr	1/1 – 9/7 – 3/2	0-9-14	D F♯^ A	D.^	D upmajor
Dh7	1/1 – 5/4 – 3/2 – 7/4	0-8-14-19	D F♯ A Cv	D(v7)	D down-seven
Ds6	1/1 – 6/5 – 3/2 – 12/7	0-6-14-19	D F A B^	Dm(^6)	D minor up-six
D1o7	1/1 – 11/9 – 3/2 – 11/6	0-7-14-21	D F^ A C^	D.~7	D dot mid seven

The 24-edo circle of fifths closes before it reaches all the notes, thus it takes two circles or **rings** to name all the notes. Ups and downs are used to distinguish between the different rings. 24-edo is notated with the plain (neither up nor down) ring and the up ring. Just as G♯ can alternatively be written as A♭, all the up notes can alternatively be written as down notes.

plain ring: E♭ – B♭ – F – C – G – D – A – E – B – F♯ – C♯ – G♯/A♭ – E♭
up ring: E♭^ – B♭^ – F^ – C^ – G^ – D^ – A^ – E^ – B^ – F♯^ – C♯^ – G♯^/A♭^ – E♭^

17, 19 and 22 are **single-ring** edos and 24 is a **multi-ring** edo. There are many other multi-ring edos. On the scale tree, multi-ring edos are the "clone" children on the dotted lines. The further down the dotted line, the more rings there are. For example, all perfect frameworks other than 7-tone are multi-ring. 14-tone is 2-ring, 21-tone is 3-ring, etc.

Multi-ring edos use ups and downs for a different reason than single-ring edos like 22-edo do. In 22-edo, ups and downs are used to avoid negative 2nds. In 24-edo, ups and downs are used to jump from one ring to the next. While 22-edo could be notated without ups and downs, if one tolerates out-of-order notes, 24-edo absolutely requires ups and downs.

In a single-ring edo, we can require that the tonic be a plain note. For example in 22-edo, rather than using C♯v as a tonic, we could use B♯. But multi-ring edos force the use of tonics that are not plain. For example, the key of C^ in 24-edo runs C^ – D♭– D♭^ – D – D^ – E♭– E♭^ – E – E^ – F – F^ – F^^ – G♭^ – G – G^ etc. F^^ is preferred over F♯ because up-4th is preferred to down-aug-4th.

With 17-tone and 22-tone, there are two approaches to assigning key signatures, using ups and downs vs. using double-sharps and double-flats. But in any multi-ring framework, there's no point to the second approach. 24-tone key signatures are like conventional ones, but with the addition of the (^) or (v) symbol that raises or lowers all notes of the scale. A few keys can be written two ways. E-upmajor with ♯ ♯ ♯ ♯ (^) could instead be F-downmajor with ♭ (v).

26-tone is notated much like 19-tone. The lattice looks exactly the same. The JI color associations are not as accurate. Scale fragment: C – C♯ – [_] – D♭ – D. One sharp = one key, no ups and downs. There is only one way to assign key signatures. There can be up to six double-sharps or double-flats in the key signature.

Table 5.10.6 – 26-tone notation, with 26-edo cents

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
0¢	46	92	138	185	231	277	323	369	415	462	508	554	600	646	692	738	785	831	877	923	969	1015	1062	1108	1154	1200
w1	1o1	g2	1u2	y2	r2	z3	g3	**y3**	r3	z4	w4	**1o4**	ry4, zg5	1u5	**w5**	r5	z6	**3o6**	y6	r6	**z7**	g7	1o6	y2	1u7	**w8**
P1	A1	d2	m2	M2	A2	d3	m3	M3	A3	d4	P4	A4	AA4 dd5	d5	P5	A5	d6	m6	M6	A6	d7	m7	M7	A7	d8	P8
D	D♯	E♭♭	E♭	**E**	E♯	F♭	**F**	F♯	F^X	G♭	**G**	G♯	G^X A♭♭	A♭	**A**	A♯	B♭♭	B♭	**B**	B♯	C♭	**C**	C♯	C^X	D♭	**D**

Figure 5.10.7 – Preferred tonic names and key signatures for 26-tone

major keys	**D**	D♯	E♭♭	E♭	**E**	E♯	F♭	**F**	F♯	F^X G♭♭	G♭	**G**	G♯	A♭♭	A♭	**A**	A♯	B♭♭	B♭	**B**	B♯	C♭	**C**	C♯	D♭♭	D♭
minor keys	"	"	D^X E♭♭	"	"	"	"	"	"	F^X	"	"	"	G^X	"	"	"	"	"	"	"	"	"	"	C^X	"

27-tone is notated much like 22-tone. Scale fragment: C – D♭ – [_] – [_] – C♯ – D. One sharp = 4 keys. The lattice looks exactly the same, except that the tho rung is a ~6, not an ^m6. The color associations are not as accurate.

Table 5.10.7 – 27-tone notation

steps	edo cents	JI	interval (with ^/v)	major keys	minor keys	interval (no ^/v)	major keys	minor keys
0	0¢	w1	P1	**D**	"	P1	**D**	"
1	44	1o1, rg1	^1 / m2	E♭	D^	m2	E♭	"
2	89	zg2	^m2	E♭^	"	d3	F♭	"
3	133	3o2	~2	Evv	"	A7 / dd4	G♭♭ (C^X)	"
4	178	y2	vM2	Ev	"	A1	D♯	"
5	222	r2	M2	**E**	"	M2	**E**	"
6	267	z3	m3	**F**	"	m3	**F**	"
7	311	g3	^m3	F^	"	d4	G♭	"
8	356	1o3, 3u3	~3	F^^		dd5 / AA1	A♭♭	D^X (A♭♭)
9	400	**y3**	vM3	F♯v		A2	E♯	"
10	444	r3	M3 / v4	F♯	F♯	M3	F♯	"
11	489	w4	P4	**G**	"	P4	**G**	"
12	533	**1o4**	^4 / d5	A♭	G^	d5	A♭	"
13	578	zg5	^^4 / ^d5	G^^		d6	B♭♭	"
14	622	ry4	vA4 / vv5	Avv		A3 / dd7	F^X (C♭♭)	"
15	667	1u5	A4 / v5	Av		A4	G♯	"
16	711	**w5**	P5	**A**	"	P5	**A**	"

17	756	z6	^5 / m6	B♭		m6	B♭	"
18	800	g3	^m6	B♭^		d7	C♭	"
19	844	**3o6**	~6	Bvv		AA4 / dd8	D♭♭	G^X
20	889	y6	vM6	Bv		A5	A♯	"
21	933	r6	M6	**B**	"	M6	**B**	"
22	978	**z7**	m7	**C**	"	m7	**C**	"
23	1022	g7	^m7	D♭	C^	d8	D♭	"
24	1067	3u7	~7	C^^	C^^	d9 / AA5	E♭♭	E♭♭ (A^X)
25	1111	ry7	vM7	C♯v	C♯v	A6	B♯	"
26	1156	1u8, zy8	M7	Dv	C♯	M7	C♯	"
27	1200	**w8**	P8	**D**	"	P8	**D**	"

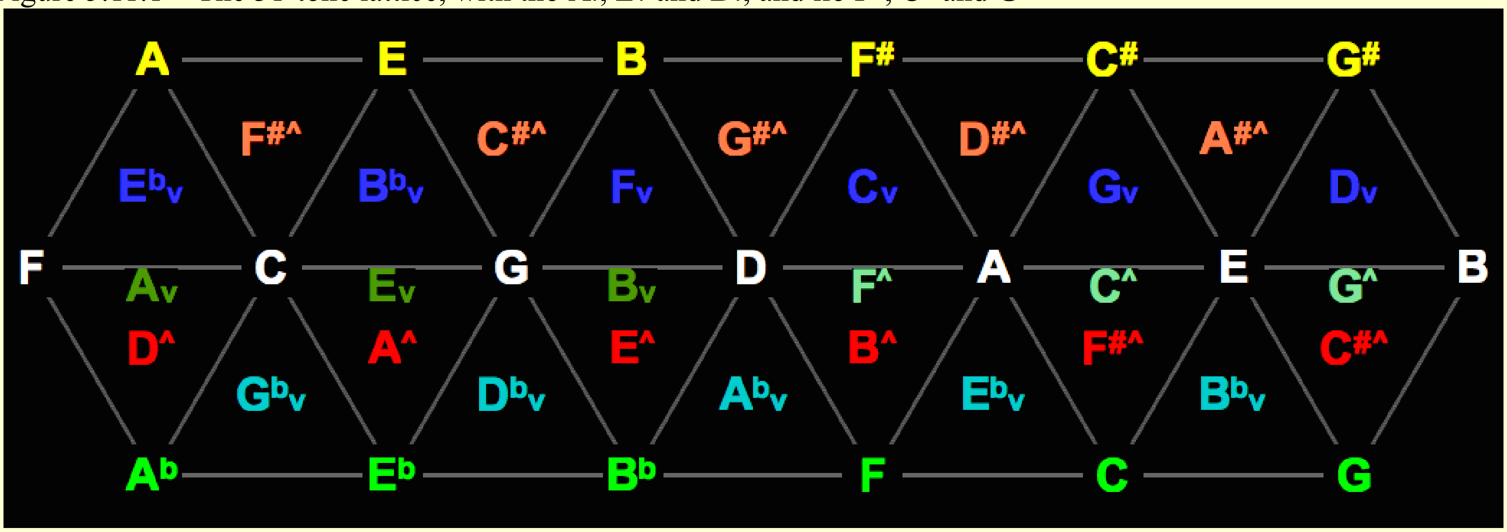
Chapter 5.11 – Selected Large Frameworks: 31, 41, 53 and 72 *

Every framework above 35 is a diatonic framework. All JI rungs are accurately represented.

31-tone is one of the most JI-friendly frameworks. It is the first one with less than 5¢ discrepancy for wa, ya and za. La and tha have a 10¢ discrepancy. Like 17-tone and 24-tone, it's a sharp-2 framework. Color associations: perfect = wa, upmajor = ru, major = yo, mid = ilo/lu/purple, minor = gu, downminor = zo.

The lattice looks the same as the 24-tone one, but unlike 24-tone, notes such as B^ and Cv are different notes.

Figure 5.11.1 – The 31-tone lattice, with tho Av, Ev and Bv, and ilo F^, C^ and G^



Like 17-tone and 22-tone, there are two approaches to assigning key signatures, using ups and downs vs. using double-sharps and double-flats. But the second approach requires triple-sharps and triple-flats for a few keys. For example, F♭♭ major has a B♭^{3}. So using ups and downs is reccomended.

Table 5.11.1 – 31-tone notation (asterisks indicate key signatures requiring triple-sharps or triple-flats)

steps	edo cents	JI	interval (with ^/v)	major keys	minor keys	interval (no ^/v)	major keys	minor keys
0	0¢	w1	P1	**D**	"	P1	**D**	"
1	39	1o1	^1	D^	"	d2	E♭♭	"
2	77	z2	A1 / vm2	E♭v	D♯	A1	D♯	"
3	116	g2	m2	E♭	"	m2	E♭	"
4	155	1u2	~2	E♭^ / Ev	Ev	AA1 / dd3	F♭♭ *	D^X
5	194	y2, w2	M2	**E**	"	M2	**E**	"
6	232	r2	^M2	E^	"	d3	F♭	"
7	271	z3	vm3	Fv	"	A2	E♯	"
8	310	g3	m3	**F**	"	m3	**F**	"
9	348	1o3	~3	F^	F♯v / F^	AA2 / dd4	G♭♭	E^X *
10	387	**<u>y3</u>**	M3	F♯	"	M3	F♯	"
11	426	r3	^M3 / d4	G♭	F♯^	d4	G♭	"

12	465	z4	v4	Gv	"	A3	F^X	"
13	503	w4	P4	**G**	"	P4	**G**	"
14	542	**1o4**	^4	G^	"	AA3 / dd5	A♭♭	"
15	581	zg5	A4 / vd5	A♭v	G♯	A4	G♯	"
16	619	ry4	d5	A♭	A♭ / G♯^	d5	A♭	"
17	658	1u5	v5	Av / A♭^	Av	AA4 / dd6	G^X *	"
18	697	**w5**	P5	**A**	"	P5	**A**	"
19	735	r5	^5	A^	"	d6	B♭♭	"
20	774	z6	A5 / vm6	B♭v	A♯ / B♭v	A5	A♯	"
21	813	g6	m6	B♭	"	m6	B♭	"
22	852	**3o6,** 1u6	~6	B♭^	Bv	AA5 / dd7	C♭♭	A^X
23	890	y6	M6	**B**	"	M6	**B**	"
24	929	r6	^M6	B^ / C♭	B^	d7	C♭	"
25	968	**z7**	vm7	Cv	"	A6	B♯	"
26	1006	w7, g7	m7	**C**	"	m7	**C**	"
27	1045	1o7	~7	C^	C^ / C♯v	AA6 / dd8	D♭♭	" *
28	1084	y7	M7	C♯ / D♭v	C♯	M7	C♯	"
29	1123	r7	^M7 / d8	D♭	C♯^	d8	D♭	"
30	1161	1u8	v8	Dv	"	A7	C^X	"
31	1200	**w8**	P8	**D**	"	P8	**D**	"

31-edo is near the limit of how many frets can be fit onto a guitar. See chapter 5.x for an example.

Figure 5.11.2 – The 31-tone guitar fretboard (asterisks indicate frets marked with dots)

E	E^	Fv	**F**	F^	F♯	G♭	Gv	**G ***	G^	G♯	A♭	Av	**A ***
B	B^	Cv	**C**	C^	C♯	D♭	Dv	**D ***	D^	D♯	E♭	Ev	**E ***
G	G^	G♯	A♭	Av	**A**	A^	A♯	B♭*	Bv	**B**	B^	Cv	**C ***
D	D^	D♯	E♭	Ev	**E**	E^	Fv	**F ***	F^	F♯	G♭	Gv	**G ***
A	A^	A♯	B♭	Bv	**B**	B^	Cv	**C ***	C^	C♯	D♭	Dv	**D ***
E	E^	Fv	**F**	F^	F♯	G♭	Gv	**G ***	G^	G♯	A♭	Av	**A ***

Table 5.11.2 – Various 31-tone chords

JI chord	JI ratio	EDOsteps	notes	name	spoken name
Dz	1/1 – 7/6 – 3/2	0-7-18	D Fv A	D.vm	D downminor
Dg	1/1 – 6/5 – 3/2	0-8-18	D F A	Dm	D minor
D1o	1/1 – 11/9 – 3/2	0-9-18	D F^ A	D~	D mid
Dy	1/1 – 5/4 – 3/2	0-10-18	D F♯ A	D	D or D major
Dr	1/1 – 9/7 – 3/2	0-11-18	D F♯^ A	D.^	D upmajor

Dh7	1/1 – 5/4 – 3/2 – 7/4	0-10-18-25	D F♯ A Cv	D(v7)	D down-seven
Ds6	1/1 – 6/5 – 3/2 – 12/7	0-8-18-24	D F A B^	Dm(^6)	D minor up-six
Dg7(zg5)	1/1 – 6/5 – 7/5 – 9/5	0-8-15-26	D F A♭v C	Dm7(v♭5)	D half-dim down-five
D1o7	1/1 – 11/9 – 3/2 – 11/6	0-9-18-27	D F^ A C^	D.~7	D dot mid seven

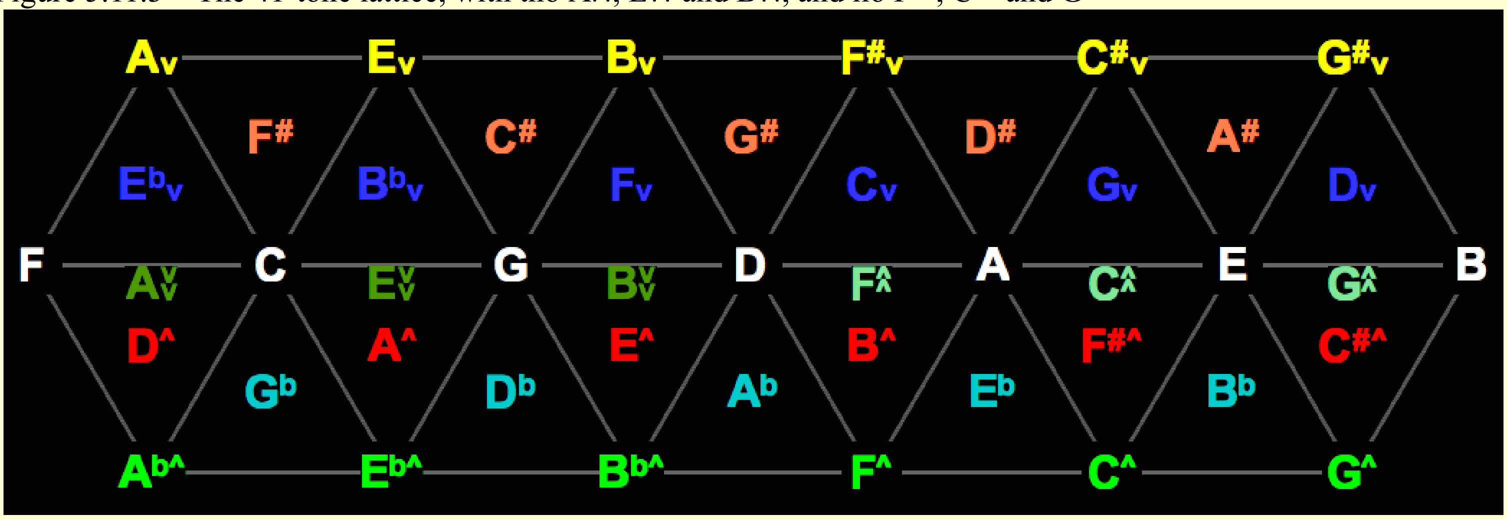
Table 5.11.3 – 31-tone key signatures using ups and downs, in chain-of-fifths order

key signature	major key	major scale	minor key	minor scale
♭ ♭ ♭ ♭ (^)	(A♭^ major)	(A♭^ B♭^ C^ D♭^ E♭^ F^ G^ A♭^)	(F^ minor)	(F^ G^ A♭^ B♭^ C^ D♭^ E♭^ F^)
♭ ♭ ♭ (^)	E♭^ major	E♭^ F^ G^ A♭^ B♭^ C^ D^ E♭^	C^ minor	C^ D^ E♭^ F^ G^ A♭^ B♭^ C^
♭ ♭ (^)	B♭^ major	B♭^ C^ D^ E♭^ F^ G^ A^ B♭^	G^ minor	G^ A^ B♭^ C^ D^ E♭^ F^ G^
♭ (^)	F^ major	F^ G^ A^ B♭^ C^ D^ E^ F^	D^ minor	D^ E^ F^ G^ A^ B♭^ C^ D^
(^)	C^ major	C^ D^ E^ F^ G^ A^ B^ C^	A^ minor	A^ B^ C^ D^ E^ F^ G^ A^
♯ (^)	G^ major	G^ A^ B^ C^ D^ E^ F♯^ G^	E^ minor	E^ F♯^ G^ A^ B^ C^ D^ E^
♯ ♯ (^)	D^ major	D^ E^ F♯^ G^ A^ B^ C♯^ D^	B^ minor	B^ C♯^ D^ E^ F♯^ G^ A^ B^
♯ ♯ ♯ (^)	A^ major	A^ B^ C♯^ D^ E^ F♯^ G♯^ A^	F♯^ minor	F♯^ G♯^ A^ B^ C♯^ D^ E^ F♯^
♯ ♯ ♯ ♯ (^)	E^ major	E^ F♯^ G♯^A^ B^ C♯^ D♯^ E^	C♯^ minor	C♯^ D♯^ E^ F♯^ G♯^ A^ B^ C♯ ^
♯ ♯ ♯ ♯ ♯ (^) ♭ ♭ ♭ ♭ ♭ ♭ ♭	B^ major C♭ major	B^ C♯^ D♯^ E^ F♯^ G♯^ A♯^ B^ C♭ D♭ E♭ F♭ G♭ A♭ B♭ C♭	G♯^ minor A♭ minor	G♯^ A♯^ B^ C♯^ D♯^ E♯^ F^ G♯^ A♭ B♭ C♭ D♭ E♭ F♭ G♭ A♭
♭ ♭ ♭ ♭ ♭ ♭	G♭ major	G♭ A♭ B♭ C♭ D♭ E♭ F G♭	E♭ minor	E♭ F G♭ A♭ B♭ C♭ D♭ E♭
♭ ♭ ♭ ♭ ♭	D♭ major	D♭ E♭ F G♭ A♭ B♭ C D♭	B♭ minor	B♭ C D♭ E♭ F G♭ A♭ B♭
♭ ♭ ♭ ♭	A♭ major	A♭ B♭ C D♭ E♭ F G A♭	F minor	F G A♭ B♭ C D♭ E♭ F
♭ ♭ ♭	E♭ major	E♭ F G A♭ B♭ C D E♭	C minor	C D E♭ F G A♭ B♭ C
♭ ♭	B♭ major	B♭ C D E♭ F G A B♭	G minor	G A B♭ C D E♭ F G
♭	F major	F G A B♭ C D E F	D minor	D E F G A B♭ C D
no sharps or flats	C major	C D E F G A B C	A minor	A B C D E F G A
♯	G major	G A B C D E F♯ G	E minor	E F♯ G A B C D E
♯ ♯	D major	D E F♯ G A B C♯ D	B minor	B C♯ D E F♯ G A B
♯ ♯ ♯	A major	A B C♯ D E F♯ G♯ A	F♯ minor	F♯ G♯ A B C♯ D E F♯
♯ ♯ ♯ ♯	E major	E F♯ G♯ A B C♯ D♯ E	C♯ minor	C♯ D♯ E F♯ G♯ A B C♯
♯ ♯ ♯ ♯ ♯	B major	B C♯ D♯ E F♯ G♯ A♯ B	G♯ minor	G♯ A♯ B C♯ D♯ E♯ F G♯
♯ ♯ ♯ ♯ ♯ ♯	F♯ major	F♯ G♯ A♯ B C♯ D♯ E♯ F♯	D♯ minor	D♯ E♯ F♯ G♯ A♯ B C♯ D♯
♯ ♯ ♯ ♯ ♯ ♯ ♯ ♭ ♭ ♭ ♭ ♭ (v)	C♯ major D♭v major	C♯ D♯ E♯ F♯ G♯ A♯ B♯ C♯ D♭v E♭v Fv G♭v A♭v B♭v Cv D♭v	A♯ minor B♭v minor	A♯ B♯ C♯ D♯ E♯ F♯ G♯ A♯ B♭v Cv D♭v E♭v Fv G♭v A♭v B♭v
♭ ♭ ♭ ♭ (v)	A♭v major	A♭v B♭v Cv D♭v E♭v Fv Gv A♭v	Fv minor	Fv Gv A♭v B♭v Cv D♭v E♭v F
♭ ♭ ♭ (v)	E♭v major	E♭v Fv Gv A♭v B♭v Cv Dv E♭v	Cv minor	Cv Dv E♭v Fv Gv A♭v B♭v Cv
♭ ♭ (v)	B♭v major	B♭v Cv Dv E♭v Fv Gv Av B♭v	Gv minor	Gv Av B♭v Cv Dv E♭v Fv Gv
♭ (v)	Fv major	Fv Gv Av B♭v Cv Dv Ev Fv	Dv minor	Dv Ev Fv Gv Av B♭v Cv Dv

(v)	Cv major	Cv Dv Ev Fv Gv Av Bv Cv	Av minor	Av Bv Cv Dv Ev Fv Gv Av
♯ (v)	Gv major	Gv Av Bv Cv Dv Ev F♯v Gv	Ev minor	Ev F♯v Gv Av Bv Cv Dv Ev
♯ ♯ (v)	Dv major	Dv Ev F♯v Gv Av Bv C♯v Dv	Bv minor	Bv C♯v Dv Ev F♯v Gv Av Bv
♯ ♯ ♯ (v)	Av major	Av Bv C♯v Dv Ev F♯v G♯v Av	F♯v minor	F♯v G♯v Av Bv C♯v Dv Ev F♯v
♯ ♯ ♯ ♯ (v)	(Ev major)	(Ev F♯v G♯v Av Bv C♯v D♯v Ev)	(C♯v minor)	(C♯v D♯v Ev F♯v G♯v Av Bv C♯v)

41-tone is even more JI-friendly than 31-tone. It has 7-banded rainbows: upmajor = ru, major = ruyo, downmajor = yo, mid = ilo/lu/purple, upminor = gu, minor = zogu, downminor = zo, perfect = wa. Almost every mid-sized comma has a keyspan of one. Most rows on the lattice have ups or downs.

Figure 5.11.3 – The 41-tone lattice, with tho Avv, Evv and Bvv, and ilo F^^, C^^ and G^^



Most of these edos can have their JI discrepancy made even less by treating them as frameworks tuned as rank-2, with the w5 as a generator. 72 can't, because it's ringy. All ratios can be notated as extended pythagorean. Each rung is mapped to the nearest EDOstep, which is mapped to the nearest note in the genchain of 5ths. Thus each rung has a genspan.

	yo 3rd	zo 7th	ilo 4th	tho 6th
31	M3 = 4 gens	vm7 = A6 = 10g	v4 = dd5 = -13g	~6 = AA5 = 15g
41	vM3 = d4 = -8 gens	vm7 = dd8 = -14g	vv4 = dd6 = -18g	~6 = $A^3$4 = 20g
53	vM3 = d4 = -8 gens	vm7 = dd8 = -14g	vv4 = $A^3$2 = 23g	v~6 = $A^3$4 = 20g

For example, 41-edo's JI discrepancy can be reduced by tuning the fifth justly. This reduces the yo rung's flatness from 5.8¢ to Ly-2 = 2¢. Zo goes from 3.0¢ flat to 3.8¢ sharp. However, in certain keys, these intervals are worsened. Analogous to meantone's wolf 5th in the key of G♯, and wolf major 3rd in B. To use all the colors, the tonic must be near the center of the 41-note genchain. For example, z3 is -15 gens, so if the tonic were any of the first 14 notes in the genchain, z3 would be about 16¢ flat.

Notes for a future chapter:

Note names:

Sharp-1 frameworks:	*12:*	*D * E F * G * A * B C * D*
	19:	*D * * E * F * * G * * A * * B * C * * D*
	26:	*D * * * E * * F * * * G * * * A * * * B * * C * * * D*
Sharp-2 frameworks:	*17:*	*D * * E F * * G * * A * * B C * * D*
	24:	*D * * * E * F * * * G * * * A * * * B * C * * * D*
	31:	*D * * * * E * * F * * * * G * * * * A * * * * B * * C * * * * D*
Sharp-3 frameworks:	*22:*	*D * * * E F * * * G * * * A * * * B C * * * D*
	29:	*D * * * * E * F * * * * G * * * * A * * * * B * C * * * * D*
	36:	*D * * * * * E * * F * * * * * G * * * * * A * * * * * B * * C * * * * * D*

There is a pentatonic counterpart to sharpness, defined as the number of edosteps spanned by <u>five</u> fifths. The edos in each category tend to end in the same digit: penta-sharp-0 edos are the pentatonic edos 5, 10, 15, etc., and end in 0 or 5. Penta-sharp-1 edos are 17, 22, 27, 32, etc. We've seen that sharp-1 edos are easily notated heptatonically without ups and downs, and are intrinsically "heptatonic-friendly". Likewise penta-sharp-1 edos, the "2 & 7" edos, are intrinsically pentatonic-friendly.

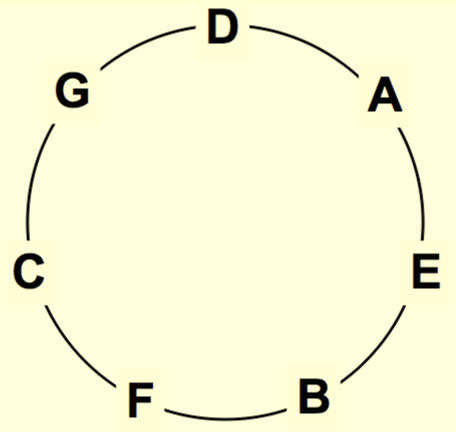
"2 & 7" edos:	*12*	*D * E F * G * A * B C * D*
	17	*D * * E F * * G * * A * * B C * * D*
	22	*D * * * E F * * * G * * * A * * * B C * * * D*
"4 & 9" edos:	*19*	*D * * E * F * * G * * A * * B * C * * D*
	24	*D * * * E * F * * * G * * * A * * * B * C * * * D*
	29	

36: upmajor = ru, major = yo, upmid = thu, downmid = tho, minor = gu, downminor = zo

Chapter 5.12 – Perfect Frameworks: 7, 14, 21, 28 and 35 *

All perfect frameworks have a circle of fifths of only 7 notes:

Figure 5.12.1 – The 7-edo circle of fifths



If each fifth is tuned identically, they are a slightly flat 686¢. This forms the equiheptatonic scale:

Table 5.12.1 – The equiheptatonic (7-edo) scale

unison	0¢
flattish major 2nd	171¢
neutral 3rd	343¢
slightly sharp 4th	514¢
slightly flat 5th	686¢
neutral 6th	857¢
sharpish minor 7th	1029¢
octave	1200¢

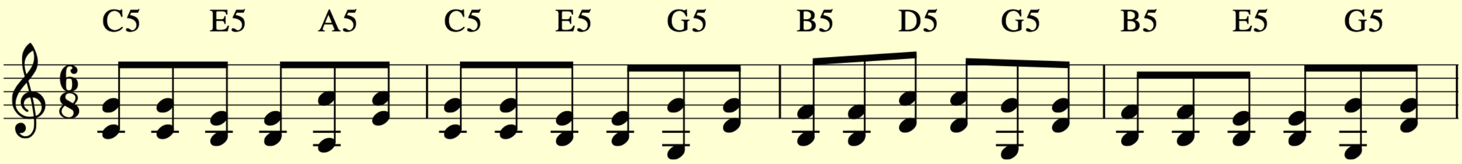
7-edo is one of only three edos well established in world music. The others are 5-edo and of course 12-edo. In chapter 5.1 I talked about my musical culture shock in Ghana with 5-edo. I had a similar experience in Africa with 7-edo. I play the mbira dzavadzimu from the Shona people of Zimbabwe. I already knew quite a few mbira songs before arriving there in 1990. But they sounded completely different on the equiheptatonic mbira. When we hear a diatonic melody in a Western tuning, we use the presence of semitones to get our bearings. But every scale step in 7-edo sounds like a flattish major 2nd. Playing a scale in 7-edo, after a few notes, the lack of semitones creates a feeling of free-falling, similar to the whole tone scale. But unlike that scale, there is a recognizable fifth. Somehow three major 2nds add up to a fourth, and four add up to a fifth.

Traditional mbira tunings aren't exactly 7-edo, but they're close, and the music works well with 7-edo. Mbira music uses mostly dyads, rarely triads, so 7-edo's neutral 3rd isn't a problem. And the JI pull discussed in chapter 5.1 is absent from mbira music. The occasional vocal harmonies are mostly in octaves or sometimes fifths. On an instrument with an inharmonic timbre, like the mbira or the marimba, the 7-edo scale is very consonant. The lack of semitones and tritones makes melodies very graceful. An mbira song written out in G:

Figure 5.12.2 – "Kariga Mombe", a traditional mbira dzavadzimu song

The B5 chord, even though notated as B and F, is <u>not</u> a diminished chord. The 7-edo scale sounds the same no matter which note you start on. There are no modes like major or minor or dorian. This makes the tonal center very ambiguous. Because mbira music is so circular, the barlines can shift, and the key of Kariga Mombe can "flip" from G to C:

Figure 5.12.3 – "Kariga Mombe" with different barlines



Whether thought of as in G or C, the chord progression has a similar feel. The root movement is by ascending 3rds and 4ths:

<u>Kariga Mombe in G</u>	<u>Kariga Mombe in C</u>
I – III – VI	I – III – VI
I – IV – VI	I – III – V
II – IV – VI	VII – II – V
I – III – V	VII – III – V

The symmetrical nature of mbira chord progressions causes the tonal center to shift, even when the mbira is tuned to a diatonic scale. But when tuned to 7-edo, almost any note can become the tonal center. Mbira players have been known to not recognize a song they know well, because it was started at an unusual point, implying a different tonal center. Although mbira music works in other tunings, 7-edo is perhaps the ideal tuning for it.

The West African balafon is also tuned nearly 7-edo. It also uses dyads. Traditional chord progressions tend to have root movements of three descending 3rds and one descending 2nd: I – VII – V – III or I – VI – V – III or I – VI – IV – III or I – VI – IV – II. Sometimes each half is repeated: I – VI – I – VI – V – III – V – III.

notate "s y" balafon part here?

While medieval Europeans avoided the tritone by flattening the B natural to B flat, many African cultures seem to have avoided it by slightly flattening all the fifths. They avoided the minor 2nd as well, in effect tempering out Lw1.

Of course, 7-edo music can use triads or tetrads. But there's no major triad or minor triad, just one triad that sounds the same no matter what note is the root. No augmented or diminished triads, but there is a sus4 and a sus2 triad. There are far fewer tetrads. There's only one 6th chord, and only one 7th chord, and they are homonyms.

As we'll see in chapter 5.x, "Non-Fifth-Based Notations", the concept of major and minor originated with the genchain of fifths. The major 3rd is formed by stacking four 5ths, and the minor 3rd is formed by stacking three 4ths. But in the perfect frameworks, because the circle of fifths has only seven notes, the two 3rds are the same. Thus there is no major or minor, or augmented or diminished. All intervals have the only other possible quality, perfect. Hence the name, perfect framework. If the white keys are taken from the circle of fifths, the keyboard is very symmetrical (the asterisks represent the black keys):

7	D E F G A B C D
14	D * E * F * G * A * B * C * D
21	D * * E * * F * * G * * A * * B * * C * *D
28	D * * * E * * * F * * * G * * * A * * * B * * C * * * D
35	D * * * * E * * * * F * * * * G * * * * A * * * * B * * * * C * * * * D

7-edo is the easiest edo to notate, because there is only one "version" (sharp, flat, up, down, etc.) of each note. Sharps and flats aren't needed because C and C♯ are the same note, just like C and B♯ are the same note in 12-edo.

The other perfect frameworks, 14, 21, 28 and 35, are multi-ring and must be notated with ups and downs.

Table 5.12.2 – The five perfect frameworks

prime	ratio	cents	framework / edo				
2	2/1 = w8	1200¢	7	14	21	28	35
3	3/2 = w5	702¢	4 (-16¢)	8 (-16¢)	12 (-16¢)	16 (-16¢)	20 (-16¢)
5	5/4 = y3	386¢	2 (-43¢)		7 (+14¢)		
7	7/4 = z7	969¢	6 (35%)				
11	11/8 = 1o4	551¢	3 (-22%)				
13	13/8 = 3o6	841¢	5 (10%)				

7-edo and 14-edo don't represent ya very well, so we'll concentrate on 21-edo.

Figure 5.12.5 – 21-tone notation

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
0¢	57	114	171	229	286	343	400	457	514	571	629	686	743	800	857	914	971	1029	1086	1143	1200
w1	1o1	g2	y2	r2	z3	1o3	**y3**	r3	w4	zg5	ry4	**w5**	z6	g6	**3o6**	r6	**z7**	g7	y7	1u8	w8
1	^1	v2	2	^2	v3	3	^3	v4	4	^4	v5	5	^5	v6	6	^6	v7	7	^7	v8	8
D	D^	Ev	E	E^	Fv	F	F^	Gv	G	G^	Av	A	A^	Bv	B	B^	Cv	C	C^	Dv	D

Every interval is perfect, so "perfect" is implied and can be omitted. Instead of P3 or vP4 we write 3 or v4. Instead of saying "up-perfect second", we simply say "upsecond", "downthird", etc.

Table 5.12.3 – 21-edo guitar fretboard (asterisks indicate frets marked with dots)

E	E^	Fv	**F**	F^	Gv	**G ***	G^	Av	**A ***
B	B^	Cv	**C**	C^	Dv	**D ***	D^	Ev	**E ***
G	G^	Av	**A**	A^	Bv	**B ***	B^	Cv	**C ***
D	D^	Ev	**E**	E^	Fv	**F ***	F^	Gv	**G ***
A	A^	Bv	**B**	B^	Cv	**C ***	C^	Dv	**D ***
E	E^	Fv	**F**	F^	Gv	**G ***	G^	Av	**A ***

There are only three key signatures in 21-edo. All 7 notes are either up, down, or plain. The global ups or downs symbol is placed where the sharps or flats would go.

add link to "Bueno pa gozar"

C.v - F.^ - D^.v - G^.^ - Fv.v - Bv.^ - G.v - G.^ - C.v
I.v7 – IV.^(v7) – ^II.v7 – ^V.^(v7) – ^^III=vIV.v7 – vVII.^(v7) – V.v7 – V.^(v7)

Ag7 – Dh7 – yBg7 – yEh7 – yyC#=gDg7 – gGh7 – Eg7 – Eh7

21-edo chord names:

C E G = C = C or C perfect
C E^ G = C.^ = C dot up
C^ E^ G^ = C^ = C up or C up perfect
Cv E Gv = Cv.^ = C-down dot up
Cv Evv Gv = Cv.v = C-down dot down
C Fv G = C.v4 or Csusv4 = C down-four or C sus down-four

C E G^ = C(^5) = C up-five
C E Gv = C(v5) = C down-five
C Ev Gv = C.v(v5) = C dot down down-five

C E G A = C6 = C six
C Ev G A = C6(v3) = C six down-three
C E G Av = C(v6) = C down-six
C Ev G Av = C.v6 = C dot down-six
Cv Ev Gv Av = Cv.6 = C-down six

C E G B = C7 = C seven
C Ev G B = C7(v3) = C seven down-three
C E G Bv = C(v7) = C down-seven
C Ev G Bv = C.v7 C dot down-seven

C E Gv B = C7(v5) = C seven down-five
C Ev Gv B = C7(v3,v5) = C seven down-three down-five
C Ev Gv Bv = C.v7(v5) = C dot down-seven down-five
C E G B^ = C(^7) = C up-seven
C Ev G B^ = C.v(^7) = C dot down up-seven

C D E G = C(9) = C add nine
C D Ev G = C.v(9) = C dot down add nine
C D^ E G = C(^9) = C add up-nine
C D^ Ev G = C.v(^9) = C dot down add up-nine

Chapter 5.13 – Pentatonic Frameworks: 5, 10, 15, 20, 25 and 30 *

All pentatonic frameworks have a very small circle of fifths, only five notes. If tuned identically, all fifths are a slightly sharp 720¢. This creates the equipentatonic scale discussed in chapter 5.1.

Table 5.13.1 – The equipentatonic (5-edo) scale

unison	0¢
sharpish major 2nd or flattish minor 3rd	240¢
slightly flat 4th	480¢
slightly sharp 5th	720¢
sharpish major 6th or flattish minor 7th	840¢
octave	1200¢

5-edo represents wa and zo very well for such a small edo. The smallest pentatonic edo that represents ya well is 15-edo. Yo is downmajor and gu is upminor.

Table 5.13.2 – The six pentatonic frameworks

prime	ratio	cents	framework / edo					
2	2/1 = w8	1200¢	5	10	15	20	25	30
3	3/2 = w5	702¢	3 (+18¢)	6 (+18¢)	9 (+18¢)	12 (+18¢)	15 (+18¢)	18 (+18¢)
5	5/4 = y3	386¢						
7	7/4 = z7	969¢						
11	11/8 = 1o4	551¢						
13	13/8 = 3o6	841¢						

In pentatonic frameworks, E and F are the same note, just like C♯ and D♭ are the same note in 12-edo. It's E in some contexts, but F in others. B and C are also the same. This is awkward, but as we'll see in chapter 5.14, it's essential for interval arithmetic to work normally. If the white keys are taken from the circle of fifths, the keyboard is very symmetrical (the asterisks represent the black keys):

```
5     D  E/F  G  A  B/C
10    D * E/F * G * A * B/C * D
15    D * * E/F * * G * * A * * B/C * * D
20    D * * * E/F * * * G * * * A * * * B/C * * * D
25    D * * * * E/F * * * * G * * * * A * * * * B/C * * * * D
30    D * * * * * E/F * * * * * G * * * * * A * * * * * B/C * * * * * D
```

In practice, every key has at least three names:

Figure 5.13.1 – Absolute notation in the 15 + 7 system

C♯ D E♭	C♯^ D^ E♭^	D♯v Ev Fv G♭v	D♯ E F G♭	D♯^ E^ F^ G♭^	F♯v Gv A♭v	F♯ G A♭	F♯^ G^ A♭^	G♯v Av B♭v	G♯ A B♭	G♯^ A^ B♭^	A♯v Bv Cv D♭v	A♯ B C D♭	A♯^ B^ C^ D♭^	C♯v Dv E♭v	C♯ D E♭

The minor 2nd is a unison, which means that the major 2nd is also a minor 3rd, the major 3rd is also a 4th, etc.

Figure 5.13.2 – Relative notation in the 15 + 7 system

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0¢	80	160	240	320	400	480	560	640	720	800	880	960	1040	1120	1200
P1 m2	^P1 ^m2	vA1 vM2	M2 m3	^m3 ^d4	vM3 vP4	M3 P4	^P4 ^d5	vA4 vP5	P5 m6	^P5 ^m6	vA5 vM6	M6 m7	^m7 ^d8	vM7 vP8	M7 P8

Since the interval between the 4th and the 5th is a major 2nd, two major 2nds add up to a 4th, and five of them add up to an octave. Pentatonic frameworks lend themselves to thinking pentatonically, hence their name. The pentatonic notation of chapter 5.3 fits them well. Similar to heptatonic notation for the perfect frameworks, every pentatonic interval is perfect, and there are no sharps or flats:

Figure 5.13.3 – Absolute notation in the 15 + 5 system

D	D^	Fv	F	F^	Gv	G	G^	Av	A	A^	Cv	C	C^	Dv	D

Figure 5.13.4 – Relative notation in the 15 + 5 system

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0¢	80	160	240	320	400	480	560	640	720	800	880	960	1040	1120	1200
1	^1	vs3	s3	^s3	v4d	4d	^4d	v5d	5d	^5d	vs7	s7	^s7	v8d	8d

Link to or quote from Igs's 5n-edo paper?

cover 15-edo key signatures and color associations

Table 5.13.3 – 15-tone guitar fretboard (asterisks indicate frets marked with dots)

E / F		E^ / F^	F♯v / Gv	F♯ / **G** *	G^ / A♭^	G♯v / Av	**A** *
B / C		B^ / C^	C♯v / Dv	C♯ / **D** *	D^ / E♭^	Ev / Fv	**E** / **F** *
G		G^ / A♭^	G♯v / Av	G♯ / **A** / B♭	A^ / B♭^	Bv / Cv	**B** / **C** *
D		D^ / E♭^	Ev / Fv	**E / F** *	E^ / F^	F♯v / Gv	F♯ / **G** *
A		A^ / B♭^	Bv / Cv	**B / C** *	B^ / C^	C♯v / Dv	**D** *
E / F		E^ / F^	F♯v / Gv	F♯ / **G** *	G^ / A♭^	G♯v / Av	**A** *

Chapter 5.14 – Supersharp Frameworks: 8, 13 and 18

All supersharp frameworks have a fifth larger than 5-edo's fifth of 720¢. They approximate 3/2 quite poorly, and mostly approximate 5/4 and 7/4 poorly as well. There are some JI ratios that are fairly accurate, such as 18-edo's 9/4 and 7/6. And there are some chords that are fairly consonant, like 18-edo's 5:6:7 chord. But most chords aren't.

The supersharp 4th is smaller than 480¢. Five 4ths minus three octaves normally makes a minor 2nd. But the narrow 4th makes a minor 2nd that is actually descending. F would be to the left of E on the keyboard. This makes normal fifth-based heptatonic notation impossible. Supersharp frameworks are the hardest to notate. Here are some options:

The best option is to use a notation based on an alternate 5th. This preserves conventional interval arithmetic, staff notation, and chord names. The second best 5th is used, which is always a narrower 5th. The supersharp framework becomes a superflat framework. These are also difficult to notate, as we'll see in the next chapter.

The second best option is to notate the framework as a subset of a larger framework, ideally a diatonic one. Here too, most conventional music theory is preserved. The trivial frameworks 2-tone, 3-tone, 4-tone and 6-tone, some of which are technically supersharp, are notated as subsets of conventional 12-tone.

One could use a notation that isn't fifth-based. 8-tone and 13-tone work with 2nd-based notation, and 18-tone with 3rd-based. Staff notation looks conventional, with sharps and flats, and without ups and downs. But interval arithmetic is completely different. The 4th and 5th are imperfect, and come in major and minor versions. And conventional chord naming methods completely break down. What do you call a chord with a major 3rd, a minor 5th, and a perfect 7th?

There are several non-heptatonic options. The pentatonic notation of chapter 5.3 (sub3rd, 4thoid, 5thoid, sub7th and octoid) can be used with any supersharp framework. This requires learning new interval arithmetic and new staff notation, and makes naming chords problematic.

Or, one could use octotonic notation. The eighth letter must be J, because H stands for B in many countries, and I is a roman numeral. This requires even more relearning than pentatonic notation. The A – J interval is an 8th. 3/2 becomes a 6th, and 2/1 becomes a nonave. The notation isn't fifth-based, but it's what might be called 3/2-based, meaning that it's still based on the approximate 3/2 being a perfect interval, and a chain of them still generates the notation.

There are many more possibilities. One could use a non-heptatonic notation that isn't 3/2-based. As we'll see in chapter 5.x, there are uses for these alternate notations. Let's examine the options for each of the three supersharp frameworks.

8-tone can't use the narrow 5th option, because that 5th would be half an octave, and the genchain of 5ths would collapse down to only two notes. The half-octave interval would simultaneously be a m3, a P4, a P5 and a M6!

Table 5.14.1 – 8-tone notation options

8-edo		24-edo subset		16-edo subset		perfect 2nd and 7th		pentatonic		octotonic	
		relative	absolute	relative	absolute	relative	absolute	relative	absolute	relative	absolute
0	0¢	P1	D	P1	D	P1	D	P1	D	P1	D
1	150¢	~2	E♭^ / Ev	M2	E	A1 / P2	D♯ / E	min sub3	E♭	P2	E
2	300¢	A2 / m3	E♯ / F	M3	F♯	A2 / m3	E♯ / F	maj sub3	E	P3	F
3	450¢	^M3 / v4	F♯^ / Gv	d3 / A4	F♭ / G♯	M3 / m4	F♯ / G	perf 4oid	G	P4	G
4	600¢	A4 / d5	G♯ / A♭	d4 / A5	G♭ / A♯	M4 / m5	G♯ / A♭	A4d / d5d	G♯ / A♭	P5	J
5	750¢	^5 / vm6	A^ / B♭v	d5 / A6	A♭ / B♯	M5 / m6	A / B♭	perf 5oid	A	P6	A
6	900¢	M6 / d7	B / C♭	m6	B♭	M6 / d7	B / C♭	min sub7	C	P7	B
7	1050¢	~7	C^ / C♯v	m7	C	P7 / d8	C / D♭	maj sub7	C♯	P8	C
8	1200¢	P8	D	P8	D	P8	D	perf 8oid	D	P9	D

8-tone forces us to use subset notation. The disadvantage of subset notation is that it constantly refers to notes which aren't actually there. If there's a major 3rd, there's no minor 3rd, and vice versa. There's never a perfect 4th or 5th. Roughly half of the natural notes are missing. Upon reading G♯, one has to imagine where G would be, then play sharp of there. Furthermore, if ups and downs are used, one key is represented by several up symbols.

However, if one is playing an 8-edo piece on a 16-edo or 24-edo guitar or keyboard, it's easier if the piece is notated as a subset of 16-edo or 24-edo. In this case, subset notation makes sense, because the "missing notes" are actually physically present.

How to choose between 16-edo and 24-edo if neither one is physically present? Since subset notation requires imagining an edo that isn't really there, choose the edo that's easier to imagine. 24-edo is a multiple of familiar 12-tone.

In 24-tone subset notation, half of the intervals are exactly what one would expect them to be: A2/m3, A4/d5, M6/d7 and P8. The other half are named by what they fall between. The 150¢ interval, midway between the minor 2nd and the major 2nd, is a mid 2nd. The 450¢ interval, midway between the major 3rd and the 4th, is either an upmajor 3rd or a down 4th. In contrast, the 16-tone subset notation requires thinking of 300¢ as a major 3rd, which only makes sense in the context of a 16-edo guitar.

In 24-edo subset notation, possible chord components are:

(P1) ~2 m3 ^M3/v4 A4/d5 ^5 M6/d7 ~7 (P8) ~9 A9 v11 ♯11 (^12) M13

There is no perfect 5th, just an up-5th. Thus many chords would have an "up-five" tacked on the end: Cm(^5), C.^(^5), C.^7(^5), etc. For the sake of brevity, the 8-edo 5th defaults to upped, not perfect. Likewise, the 3rd defaults to upmajor, not major. This only happens with 8-edo, because only 8-edo requires subset notation. The default 8-edo qualities are:

2nd = ~2, 3rd = ^M3, 4th = v4, 5th = ^5, 6th = M6, 7th = ~7, 9th = ~9, 11th = v11, 13th = M13

8-edo chords are very ambiguous, with many chord homonyms. Even the major and minor triads are homonyms.

Table 5.14.2 – Various examples of 8-edo chords

Chord edosteps	Chord notes	Proper name	Abbreviated name	Homonyms
0 – 3 – 5	D F♯^ A^	D.^(^5)	D	F♯^.m or Gv.m
0 – 2 – 5	D F A^	Dm(^5)	Dm	A^ or B♭v
0 – 3 – 5 – 7	D F♯^ A^ C^	D.^7(^5)	D7	F♯^.m♯11 or Gv.m♯11
0 – 3 – 5 – 6	D F♯^ A^ B	D6(^3,^5)	D6	Bm7 and Gv,♯9
0 – 2 – 5 – 7	D F A^ C^	Dm~7(^5)	Dm7	F6 and B♭v,♯9
0 – 2 – 5 – 6	D F A^ B	Dm6(^5)	Dm6	Bm7(♭5) and A^,9
0 – 2 – 4 – 7	D F A♭ C^	Ddim~7	Dm7(♭5)	Fm6
0 – 2 – 4 – 6	D F A♭ C♭	Ddim7	Ddim7	Fdim7, A♭dim7 and Bdim7
0 – 3 – 5 – 7 – 9	D F♯^ A^ C^ Ev	D.^7v9(^5)	D9	Gv.m6♯11, Ev.dim7,9 etc.
0 – 3 – 5 – 6 – 9	D F♯^ A^ B Ev	D6v9(^3,^5)	D6,9	A^.6,9(♭5), Gv.6♯9, B7♯9 and Ev.m7♯11
0 – 3 – 6 – 9 – 12	D F♯^ B Ev A♭	D6v9(♭5,^3)	D6,9(♭5)	Gv.6,9 etc.
0 – 3 – 5 – 6 – 10	D F♯^ A^ B E♯	D6♯9(^3,^5)	D6♯9	A^.6,9 etc.
0 – 3 – 5 – 7 – 10	D F♯^ A^ C^ E♯	D.^7♯9(^5)	D7♯9	F6,9 etc.
0 – 2 – 5 – 7 – 12	D F A^ C^ G♯	Dm~7♯11(^5)	Dm7♯11	C^.6,9 etc.

13-tone is best notated using the alternate, narrower 5th. Ups and downs are required. What is notated as the perfect 5th is 56¢ flat of 3/2, and sounds distinctly diminished. The up 5th is only 36¢ sharp of 3/2, not quite augmented, more of a sharp perfect. 13-tone becomes a superflat framework, which also requires reversing major/minor, aug/dim and sharp/flat. This is covered in detail in the next chapter.

Table 5.14.3 – 13-tone notation options

13-tone		narrow 5th		26-tone subset		perfect 2nd and 7th		pentatonic		octotonic	
0	0¢	P1	D	P1	D	P1	D	P1	D	P1	D
1	92¢	^1 / M2	D^ / E	d2	E♭♭	A1 / d2	D♯ / E♭	aug 1	D♯	m2	E♭
2	185¢	^M2 / M3	E^ / F♯	M2	E	P2	E	min sub3	E♭	M2	E
3	277¢	vm2 / ^M3	E♭v / F♯^	d3	F♭	A2 / m3	E♯ / F♭	maj sub3	E	m3	F
4	369¢	m2 / vm3	E♭ / Fv	M3	F♯	M3	F	Asub3 / d4oid	E♯ / G♭	M3	F♯
5	462¢	m3 / v4	F / Gv	d4	G♭	A3 / m4	F♯ / G♭	perf 4oid	G	P4	G
6	554¢	P4 / v5	G / Av	A4	G♯	M4	G	aug 4oid	G♯	m5	J♭
7	646¢	^4 / P5	G^ / A	d5	A♭	m5	A	dim 5oid	A♭	M5	J
8	738¢	^5 / M6	A^ / B	A5	A♯	M5 / d6	A♯ / B♭	perf 5oid	A	P6	A
9	831¢	^M6 / M7	B^ / C♯	m6	B♭	m6	B	A5oid / dsub7	A♯ / C♭	m7	B♭
10	923¢	vm6 / ^M7	B♭v / C♯^	A6	B♯	M6 / d7	B♯ / C♭	min sub7	C	M7	B
11	1015¢	m6 / vm7	B♭ / Cv	m7	C	P7	C	maj sub7	C♯	m8	C
12	1108¢	m7 / v8	C / Dv	A7	C^X	A7 / d8	C♯ / D♭	dim 8oid	D♭	M8	C♯
13	1200¢	P8	D	P8	D	P8	D	perf 8oid	D	P9	D

18-tone's second-best 5th is only 4¢ further from 3/2 than its best 5th. Narrow-5th notation makes 18-tone be a two-ring framework, a superset of 9-tone. Note that mid is upmajor/downminor, not upminor/downmajor.

36 notes to the octave is barely practical on a guitar, and not playable on a keyboard. Octotonic notation requires ups and downs. Another possible notation is nine-tone (nonotonic). Similar to heptatonic 14-tone, there is no major or minor, and every other note uses ups and downs.

Table 5.14.4 – 18-tone notation options

18-tone		narrow 5th		36-tone subset		perfect 3rd and 6th		pentatonic		octotonic	
0	0¢	P1	D	P1	D	P1	D	P1	D	P1	D
1	67¢	^1 / vM2	D^ / Ev	vm2	E♭v	A1 / d2	D♯ / E♭♭	aug 1	D♯	m2	E♭
2	133¢	d1 / M2	D♭ / E	^m2	E♭^	m2	E♭	dim sub3	E♭♭	~2	Ev
3	200¢	~2 / vM3	E^ / F♯v	M2	E	M2	E	min sub3	E♭	M2	E
4	267¢	m2 / M3	E♭ / F♯	vm3	Fv	A2 / d3	E♯ / F♭	maj sub3	E	m3	F
5	333¢	^m2 / ~3	E♭^ / Fv	^m3	F^	P3	F	aug sub3	E♯	~3	F^
6	400¢	m3 / A4	F / G♯	M3	F♯	A3 / d4	F♯ / G♭♭	dim 4oid	G♭	M3	F♯
7	467¢	^m3 / v4	F^ / Gv	v4	Gv	m4	G♭	perf 4oid	G	P4	G
8	533¢	P4 / A5	G / A♯	^4	G^	M4	G	aug 4oid	G♯	m5	J♭
9	600¢	^4 / v5	G^ / Av	A4 / d5	G♯ / A♭	A4 / d5	G♯ / A♭	AA4d / dd5d	GX / A♭♭	~5	Jv
10	667¢	d4 / P5	G♭ / A	v5	Av	m5	A	dim 5oid	A♭	M5	J
11	733¢	^5 / vM6	A^ / Bv	^5	A^	M5	A♯	perf 5oid	A	P6	A
12	800¢	d5 / M6	A♭ / B	m6	B♭	A5 / d6	AX / B♭	aug 5oid	A♯	m7	B♭
13	867¢	~6 / vM7	B^ / C♯v	vM6	Bv	P6	B	dim sub7	C♭	~7	Bv
14	933¢	m6 / M7	B♭ / C♯	^M6	B^	A6 / d7	B♯ / C♭	min sub7	C	M7	B
15	1000¢	^m6 / ~7	B♭^ / Cv	m7	C	m7	C	maj sub7	C♯	m8	C
16	1067¢	m7 / A8	C / D♯	vM7	C♯v	M7	C♯	aug sub7	CX	~8	C^
17	1133¢	^m7 / v8	C^ / Dv	^M7	C♯^	A7 / d8	CX / D♭	dim 8oid	D♭	M8	C♯
18	1200¢	P8	D	P8	D	P8	D	perf 8oid	D	P9	D

Chapter 5.15 – Superflat Frameworks: 9, 11, 13b, 16, 18b and 23 *

All superflat edos have a fifth smaller than 7-edo's fifth of 686¢. If the major 3rd is formed by stacking four 5ths, and the minor 3rd is formed by stacking three 4ths, the minor 3rd is wider than the major 3rd. Likewise the minor 2nd would be wider than the major 2nd. If the white keys form a chain of fifths, the keyboard has more black keys between E and F than between F and G:

9	D E * F G A B * C D
11	D E * * F G A B * * C D
13b	D E * * * F G A B * * * C D
16	D * E * * F * G * A * B * * C * D
18b	D * E * * * F * G * A * B * * * C * D
23	D * * E * * * F * * G * * A * * B * * * C * * D

Table 5.15.1 – The six superflat frameworks

prime	ratio	cents	framework / edo					
2	2/1 = w8	1200¢	9	11	13b	16	18b	23
3	3/2 = w5	702¢						
5	5/4 = y3	386¢						
7	7/4 = z7	969¢						
11	11/8 = 1o4	551¢						
13	13/8 = 3o6	841¢						

There are two approaches to notating superflat frameworks. The first approach preserves the melodic meaning of sharp/flat, major/minor and aug/dim, in that sharp is higher pitched than flat, and major/aug is wider than minor/dim. The disadvantage to this approach is that conventional interval arithmetic no longer works. e.g. M2 + M2 isn't M3, and D + M2 isn't E. Chord names are different because C – E – G isn't P1 – M3 – P5.

Figure 5.15.1 – Absolute notation in the 16 + 7 system, with sharp higher than flat

D	D♯ E♭	E	E♯	F♭	F	F♯ G♭	G	G♯ A♭	A	A♯ B♭	B	B♯	C♭	C	C♯ D♭	D

Figure 5.15.2 – Relative notation in the 16 + 7 system, with major/aug wider than minor/dim

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0¢	75	150	225	300	375	450	525	600	675	750	825	900	975	1050	1125	1200
P1	A1 d2	m2	M2	m3	M3	A3 d4	P4	A4 d5	P5	A5 d6	m6	M6	m7	M7	A7 d8	P8

The second way preserves the harmonic meaning of sharp/flat, major/minor and aug/dim, in that the former is always further fifthwards on the chain of fifths than the latter. Sharp is lower in pitch than flat, and major/aug is narrower than minor/dim. While this approach may seem bizarre at first, interval arithmetic and chord names work as usual. Furthermore, conventional 12-edo music can be directly translated to 16-edo "on the fly".

Figure 5.15.3 – Absolute notation in the 16 + 7 system, with sharp lower than flat

D	D♭ E♯	E	E♭	F♯	F	F♭ G♯	G	G♭ A♯	A	A♭ B♯	B	B♭	C♯	C	C♭ D♯	D

Figure 5.15.4 – Relative notation in the 16 + 7 system, with major/aug narrower than minor/dim

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0¢	75	150	225	300	375	450	525	600	675	750	825	900	975	1050	1125	1200
P1	d1 A2	M2	m2	M3	z3	d3 A4	P4	d4 A5	P5	d5 A6	M6	m6	M7	m7	d7 A8	P8

ru	was major	becomes minor
yo	major	minor
gu	minor	major
zo	minor	major

Chapter 5.16 – Notating Rank-2 Tunings, Part I: Triple Yo *

As noted at the end of chapter 4.1, rank-2 tunings represent a middle ground between JI tunings and edos. They mostly can be notated either with color notation, or with ups and downs. In chapter 4.2 we saw how color notation is used for adaptive JI: Cy – yAg – y=wDg – Gy – Cy. However, for rank-2 tunings, ups and downs are preferred over colors. Colors can be stacked indefinitely, but ups and downs can't; at some point they add up to something simpler. Also, a tempered interval can often be interpreted as several JI intervals, and using colors limits those interpretations.

The simplest rank-2 tuning is the 3-limit pythagorean tuning, a simple chain of fifths that makes a scale of all wa notes. This is the tuning for which conventional notation was first devised. If a rank-2 temperament has a period of an octave and a generator of a fifth, conventional notation works perfectly. When the West moved from pythagorean to meantone, notation didn't change. The gu temperament, the ru temperament, and even the gu and ru temperament can all be notated without any extra accidentals, just sharps and flats. Chords and scales are named conventionally.

Recall from chapter 4.6 that a deep comma splits either the octave or some voicing of the fifth, creating either a fractional period or a fractional generator. Let's examine such a single-comma rank-2 temperament.

The triple yo temperament y^{3}T ("porcupine") splits the wa 4th into three yo 2nds. The Ty2 is the generator.

> 3 · y2 = w4 + y^{3}1
> 3 · Ty2 = Tw4 + Ty^{3}1 = Tw4

To eliminate colors from the equation, map the intervals to perfect, major, etc. intervals. Thus w4 is a P4. The y2 is normally notated as a major 2nd, which would give us 3·M2 = P4. But this is wrong, because 3·M2 = A4, and A4 is larger than P4. So we add in a down to make the equation work:

> 3 · vM2 = P4 + comma
> comma = 3 · vM2 - P4 = v^{3}A4 - P4 = v^{3}A1 = 0 gens

This is a new use of ups and downs, indicating not keyspan but **genspan**, and as we'll see, also a comma. In a rank-2 temperament, every interval is defined by its genspan. Here are the genspans of the generator, the 4th and the 5th:

> vM2 = 1 gen
> P4 = 3 gens
> P5 = -3 gens

The genspan of an interval an octave away is the same, so Ww5 also equals -3 gens. Genspans add up, so genspan (M2) = genspan (M9) = genspan (P5 + P5) = 2 · genspan (P5) = -6 gens.

We can respell any note or interval by adding or subtracting a v^{3}A1 (the comma). Let's use this trick to find an alternate generator. Adding v^{3}A1 to vM2 would make a quadruple-down interval, so instead we subtract to make a double-up interval. Subtracting means adding the inverse ^^{3}d1, which means adding three ups and diminishing (flattening):

> alternate generator = gen - comma = vM2 - v^{3}A1 = vM2 + ^^{3}d1 = ^^m2

We can find the ratio for the alternate generator the same way:

> alternate generator = gen - comma = Ty2 - Ty^{3}1 = Tgg2

We can find the genspan of the up symbol by comparing the vM2 generator to a plain M2:

> up symbol = ^1 = M2 - vM2 = 2 · P5 - vM2 = 2 · (-3 gens) - 1 gen = -7 gens

We can find the ratio of the up symbol similarly. This ratio is usually a different comma:

> up symbol = ^1 = M2 - vM2 = Tw2 - Ty2 = Tg1

The gu comma is not tempered out, but it's still tempered, as a side effect of tempering out the triple yo comma. It grows to become roughly 50¢. Alternate ratios can be found by adding or subtracting the triple yo comma:

> alternate up symbol = up symbol + comma = Tg1 + Ty^{3}1 = Tyy1 = yoyo semitone 25/24

To summarize, the generator is notated as either vM2 or ^^m2 and the comma as v^{3}A1. The genchain is constructed from the generator, using the alternate generator every third time, to avoid double-ups and double-downs:

...**B** – C♯v – D^ – **E** – F♯v – G^ – **A** – Bv – C^ – **D** – Ev – F^ – **G** – Av – B♭^ – **C** – Dv – E♭^ – **F**...

This genchain is actually three intertwined chains of 4ths, a plain one, an up one, and a down one. But lattices usually have chains of 5ths, not 4ths. So we reverse the order:

...**F** – E♭^ – Dv – **C** – B♭^ – Av – **G** – F^ – Ev – **D** – C^ – Bv – **A** – G^ – F♯v – **E** – D^ – C♯v – **B**...

Separating out the three fifth-chains makes a lattice. I'll explain in a bit why the down row is above the up row:

Dv Av Ev Bv F♯v C♯v

F ———— **C** ———— **G** ———— **D** ———— **A** ———— **E** ———— **B**

E♭^ B♭^ F^ C^ G^ D^

No two notes in the lattice have the same name, essential for rigorous notation. In relative notation:

v8 v5 vM2 vM6 vM3 vM7

m3 ———— **m7** ———— **P4** ———— **P1** ———— **P5** ———— **M2** ———— **M6**

^m2 ^m6 ^m3 ^m7 ^4 ^1

In figure 1.3.10, we shifted the rectangular JI lattice into a triangular one, to make consonant intervals closer together, and dissonant ones further apart. We can do the same here, to move the vM2 and the v5 further from the P1, and the vM3 closer:

Dv Av Ev Bv F♯v C♯v G♯v

F ———— **C** ———— **G** ———— **D** ———— **A** ———— **E** ———— **B**

A♭^ E♭^ B♭^ F^ C^ G^ D^

Even though the intervals are tempered, their relative consonance isn't changed much. In relative notation:

v8 v5 vM2 vM6 vM3 vM7 vA4

m3 ———— **m7** ———— **P4** ———— **P1** ———— **P5** ———— **M2** ———— **M6**

^d5 ^m2 ^m6 ^m3 ^m7 ^4 ^1

The rows have been shifted so that the vM3, which is a tempered yo 3rd, occupies the same place as the y3 does in the ya JI lattice. This is why the down row was placed above the up row, to mimic the placement of yo and gu in the JI lattice. The advantage to this layout is that the triple yo lattice now has the same shape as the JI lattice. In particular, chords have the same shape. The disadvantage is that it's harder to trace the genchain, because it zigzags around. Thus it's harder to tell if a scale is formed by a continuous genchain or not. A continuous chain of the right length is guaranteed to be a MOS scale. So the unshifted layout makes it easier to find MOS scales, but the shifted layout makes it easier to find chords. Once again, melody vs. harmony!

Chords and scales are notated as discussed in chapter 5.8:

D – F^ – A = D.^m = D upminor
Ev – G – Bv = Ev.^m = E-down upminor

But because we avoided double-ups and double-downs, this same chord on an upped root is notated differently. The note two steps away on the genchain from the root is not an upminor 3rd, but a double-down major 3rd. This makes both an awkward chord name and, if using the shifted layout, an awkward shape in the lattice.

F^ – Av – C^ = F^.vv = F up dot double-down

Analogous to notating the 3rd of a D♯ major chord as F𝄪 instead of G, this chord can be notated with double-ups, to keep the chord names consistent:

F^ – A♭^^ – C^ = F^.^m = F-up upminor

As with conventional notation, tonal music with well-defined chords might be written with occasional double accidentals, to clarify the harmonies, whereas atonal music without well-defined chords might be written with only single accidentals, to reduce clutter.

The Av was converted to a double-up note via subtracting the comma v^{3}A1, equivalent to adding ^^{3}d1. The Av was upped three times, then diminished (flattened). Another solution would be to notate the root and fifth differently:

F♯vv – Av – C♯vv = F♯vv.^m = F-sharp double-down upminor

Relative chord notation is as usual. In D, the F^.^m chord would be ^♭III.^m, and the F♯vv.^m chord would be vvIII.^m.

The lattice can be expanded to include double ups and downs. All the unbolded notes appear twice (e.g. F♯vv and F^).

F♯vv C♯vv G♯vv D♯vv A♯vv E♯vv B♯vv

Av Ev Bv F♯v C♯v G♯v

F —— **C** —— **G** —— **D** —— **A** —— **E** —— **B**

A♭^ E♭^ B♭^ F^ C^ G^

F♭^^ C♭^^ G♭^^ D♭^^ A♭^^ E♭^^ B♭^^

The extra rows make a redundant lattice, in which some notes are duplicated. The relative notation lattice is also redundant, with some intervals duplicated (e.g. ^^m2 and vM2):

vvM3 vvM7 vvA4 vvA1 vvA5 vvA2 vvA6

v5 vM2 vM6 vM3 vM7 vA4

m3 —— **m7** —— **P4** —— **P1** —— **P5** —— **M2** —— **M6**

^d5 ^m2 ^m6 ^m3 ^m7 ^4

^^d3 ^^d7 ^^d4 ^^d8 ^^d5 ^^m2 ^^m6

As noted above, the up symbol equals both Tg1 and the yoyo semitone Tyy1 = 25/24. Yet another possible lattice avoids double ups and downs by equating a vertical step with an up:

—— **A** —— **E** —— **B** —— **F♯** —— **C♯** —— **G♯** ——

F^ C^ G^ D^ A^ E^ B^

Av Ev Bv F♯v C♯v G♯v

F —— **C** —— **G** —— **D** —— **A** —— **E** —— **B**

A♭^ E♭^ B♭^ F^ C^ G^

Fv Cv Gv Dv Av Ev Bv

—— **A♭** —— **E♭** —— **B♭** —— **F** —— **C** —— **G** ——

Staff notation looks much like 15-edo or 22-edo notation. Key signatures are constructed conventionally.

The "Mizarian Porcupine Overture" by Herman Miller pumps the triple yo comma in the closing section. This piece gave the tuning its nickname, the Porcupine temperament. On the last beat of the 2nd measure, the Evv in the bass clef should be its enharmonic equivalent E♭^. But it can't be, because it's tied to the previous note.

Figure 5.16.1 – Triple yo temperament staff notation: The closing section of the Mizarian Porcupine Overture

Like meantone, or any rank-2 tuning, there is a spectrum of possible tunings for the triple yo temperament, defined by the size of the generator (and the period, if the octave is tempered). To minimize the mistuning of yo and gu chords, it's best to keep the generator within a fairly narrow range of about 160-167¢. This makes a fifth of about 700-720¢ and a major 3rd of about 368-400¢. As the generator grows, the 5th and the 3rd shrink. As with meantone or the ru temperament, the generator's size can be described as a fraction of a comma from a just interval. In this case, the interval is y2 = 10/9. The generator could also be thought of as other intervals, e.g. gg2 = 27/25, but we'll use y2 because it has a lower odd limit. 160¢ is a little less than half a comma flat of y2, and 166¢ is third-comma triple yo. As we saw in Chapter 4.3, the just baseline can be found from the comma fraction. Third-comma tuning makes w5 just, and two-fifths tuning makes y3 just.

Certain generator sizes in this range coincide with simple octave fractions and reduce the tuning to a rank-1 edo. For example, 160¢ equals 2\15, and 166.67¢ equals 5\36.

(link to mp3s of the MPO in various tunings)

The comma is a v^{3}A1, thus A1 = ^^{3}1, and augmenting is the same as adding three ups. Thus frameworks that support the triple yo temperament <u>must</u> have a sharpness that is a multiple of 3: sharp-0, sharp-3, sharp-6, etc. However, not all such frameworks support this temperament. The triple yo comma equals the large wa semitone minus three gu commas: $y^3 1$ = Lw1 - 3·g1. The keyspans add up similarly: K($y^3 1$) = K(Lw1) - 3·K(g1). Since K(Lw1) is the edo's sharpness, for K($y^3 1$) to be zero, K(g1) must be 1/3 of the sharpness. The sharpness and K(g1) of each framework are shown in Figures 5.7.2 and 5.7.4 respectively. Thus a sharp-0 (i.e. heptatonic) framework must also have K(g1) = 0, i.e. fall in the "y3 = M3" region. Comparing the two figures, we find that 7-edo is the only such framework.

A sharp-3 framework will work only if it also falls in the "y3 = vM3" region, as 8, 15, 22 and 29 do. 36-edo is just over the red line, and will only temper out $y^3 1$ if tweaked to 36c-edo. The 2nd best approximation of y3 is used. Thus a $y^3 1$ comma pump won't work in 36-edo if you use major chords with 400¢ 3rds and minor chords with 300¢ 3rds. Instead, you must use downmajor and upminor chords with 3rds of 367¢ and 333¢. In fact, because of the "y3 = vM3" restriction, a $y^3 1$ comma pump in any sharp-3 framework requires downmajor and upminor chords.

A sharp-6 framework must also fall in the "y3 = vvM3" region, as 30, 37, 44 and 51 do. 58-edo requires tweaking y3 from the vM3 = 393¢ to the nearly as sweet vvM3 = 372¢. 23bc-edo would be possible but very dissonant. The 5th is 14 edosteps and the 3rds are 6 or 8 edosteps. Sharp-9 frameworks with y3 = v^3M3 include 45wy, 52w, 59, 66 and 73y. To summarize:

sharp-0 frameworks: 7
sharp-3 frameworks: 8, 15, 22, 29 and 36c
sharp-6 frameworks: 23bc, 30, 37, 44, 51 and 58c
sharp-9 frameworks: 45bc, 52b, 59, 66 and 73c

Of all these frameworks, the most practical are 15, 22 and 29. These are all sharp-3, and if the Mizarian Porcupine Overture were played in any of these frameworks, the notation would be unchanged. However, using a sharp-6 framework like 37-edo would require that every up and down be doubled. Thus the first chord would be E♭.vv, with a Gvv note.

Chapter 5.17 – Notating Rank-2 Tunings, Part II *

Any wa temperament creates a single-ring edo, a rank-1 tuning. Adding an untempered rung or a bicolored comma creates a rank-2 temperament. The lattice consists of parallel rings. For example, the small wa plus ya temperament sw+yT or 5-edo+y (aka "Blackwood"), contains 5-edo, which is a ring of 5 wa notes. 5-edo+y has a wa ring, a yo ring, a gu ring, a yoyo ring, a gugu ring, etc. The non-wa rings can be notated with ups and downs instead of colors. But they shouldn't, because while colors can be "stacked" indefinitely, ups and downs can't. At some point, ups and downs always add up to something simpler. Furthermore, if the edo requires ups and downs, e.g. 17-edo+y, the rings must use colors.

Any rank-2 temperament without any deep or wa commas doesn't split anything. It has a period of an octave and a generator of a 5th, and can be notated without ups and downs. However, it's often preferable to do so if the comma is not a unison, to avoid negative intervals.

For example, the large yo temperament LyT tempers out Ly-2 = (-15, 8, 1) = 2¢. Without ups and downs, the yo 3rd is notated as a wa dim 4th, and there are many negative intervals. For example, y2 = 10/9 is a dim 3rd, and w2 is a M2. Also, the yo chord must be awkwardly spelled as a dim-4 chord. Ups and downs can be used for convenience, much like 22-edo. The up symbol represents a tempered gu comma, and y3 is notated as a vM3.

Likewise, the large ru temperament LrT tempers out Lr-2. Without ups and downs, the zo 7th is notated as an aug 6th, and the ru 6th as a dim 7th. Again, ups and downs aren't needed but are desirable. The up symbol represents a tempered ru comma, and z7 is notated as a vm7.

The yoyo temperament yyT has an octave period and a generator of a yo 3rd. Proceeding as before:

> 2 · y3 = w5 + yy1
> 2 · **vM3** = P5 + comma = 2 gens
> comma = 2 · vM3 - P5 = vvA5 - P5 = **vvA1** = 0 gens
> alternate generator = gen - comma = vM3 - vvA1 = **^m3** = Ty3 - Tyy1 = Tg3
> up symbol = M3 - vM3 = 4 · P5 - vM3 = 8 gens - 1 gen = 7 gens
> up symbol = M3 - vM3 = 4 · Tw5 - Ty3 = Tg1

Again, the up symbol equals the gu comma. This is because the gu comma is the "invisible comma", the only ya comma that maps to a P1. In general, for ya rank-2 tunings, ^1 = Tg1, gu intervals become upped intervals, and yo ones become downed. But not always, see g^4T below.

> ^1 = Tg1 = Tg1 + Tyy1 = TLy1

The yoyo temperament's genchain is two intertwined chains of 5ths. The genchain is constructed with alternating downmajor and upminor 3rds, to avoid double-ups and double-downs:

> ...F – Av – C – Ev – G – Bv – D – F♯v – A – C♯v – E – G♯v – B...

With ups instead of downs:

> ...F – A♭^ – C – E♭^ – G – B♭^ – D – F^ – A – C^ – E – G^ – B...

The same genchain, with both ups and downs, with the down row above the up row, to mimic the placement of yo and gu in the JI lattice. The up notes are placed very close to the down notes, to show that they are the same note:

	Av		Ev		Bv		F♯v		C♯v		G♯v	
F		**C**		**G**		**D**		**A**		**E**		**B**
	A♭^		E♭^		B♭^		F^		C^		G^	

Because the comma is a vvA1, sharpening is equivalent to double-upping, and supporting frameworks must be sharp-0, sharp-2, sharp-4, etc. These frameworks all have mid intervals. The relative genchain, with mids:

> v5 ~2 ~6 ~3 ~7 ^4

m3 ——— **m7** ——— **P4** ——— **P1** ——— **P5** ——— **M2** ——— **M6**

Using the mid quality avoids some of the naming difficulties:

C – Ev – G – Bv = C.~7 = "C dot mid seven"
Ev – G – Bv – D = Ev.~7 = "E-down dot mid seven"

Mids can't be used on the staff. There, the downmajor chord and the upminor chord look different, but they sound the same. Again, tonal music might use double accidentals, to ensure that chords are named consistently. In the next example, the G and the D in the second chord are converted to double-down notes by adding the comma, a vvA1.

C – Ev – G – Bv = C.vM7 = "C dot downmajor seven"
Ev – G – Bv – D = Ev.^m7 = "E-down dot upminor seven" (same chord on a downed root, new name)
Ev – G♯vv – Bv – D♯vv = Ev.vM7 = "E-down dot downmajor seven" (chord respelled, to keep the old name)

As discussed at the end of the last chapter, we can use Figures 5.7.2 and 5.7.4 to find which frameworks support this temperament. The yoyo semitone equals the large wa semitone minus two gu commas, so K(yy1) = K(Lw1) - 2·K(g1) and K(g1) must be 1/2 of the sharpness.

sharp-0 frameworks with y3 = M3: 7-edo and 14c
sharp-2 frameworks with y3 = vM3: 10, 17 (barely), and 24c
sharp-4 frameworks with y3 = vvM3: 13 (not 13b), 20 and 27c
sharp-6 frameworks with y3 = v^3M3: 23b and 30c
flat-2 frameworks with y3 = ^M3: 11c and 18bc

The zozo temperament zzT splits the wa 4th into two zo 3rds.

2 · z3 = w4 + zz2
2 · **vm3** = P4 + comma = 2 gens
comma = 2 · vm3 - P4 = vvd5 - P4 = **vvm2**
alternate generator = gen - comma = vm3 - vvm2 = **^M2** = Tz3 - Tzz2 = Tr2
up symbol = m3 - vm3 = 3 · P4 - vm3 = 6 gens - 1 gen = 5 gens
up symbol = m3 - vm3 = 3 · Tw4 - Tz3 = Tr1

Again, the up symbol equals the "invisible" comma that is a P1. In 7-limit, this is the ru comma. In 11-limit, it depends on the ilo keyspan. If 1o4 is perfect, it's 1o1 = 33/32. If 1o4 is augmented, it's L1u1 = 729/704.

The genchain steps alternate between vm3 and ^M2, making two intertwined chains of 4ths. Again, we reverse the order, with generators going right to left, to get a familiar chain of 5ths running left to right. The down row is placed above the up row, to mimic the placement of zo and ru in the JI lattice.

	E♭v		B♭v		Fv		Cv		Gv		Dv	
F	——	**C**	——	**G**	——	**D**	——	**A**	——	**E**	——	**B**
	D^		A^		E^		B^		F♯^		C♯^	

In relative notation:

	vm2		vm6		vm3		vm7		v4		v1	
m3	——	**m7**	——	**P4**	——	**P1**	——	**P5**	——	**M2**	——	**M6**
	^1		^5		^M2		^M6		^M3		^M7	

Again, double-ups and double-downs may sometimes be needed for chord names. Chord spellings are corrected by adding or subtracting a vvm2.

D – Fv – A – Cv = D.vm7 = "D dot downminor seven"
Fv – G – Cv – D = Fv(^2)^6 = "F down, up-two, up-six" (same chord , awkward new name)
Fv – A♭vv – Cv – E♭vv = Fv.vm7 = "F down dot downminor seven" (same chord, respelled)

Fractional period rank-2 temperaments have multiple genchains running in parallel. This is different than the previous examples of a single genchain formed from intertwined chains of 5ths. Multiple genchains occur because a rank-2 genchain is really a 2-dimensional "genweb", running in octaves (or whatever the period is) vertically and fifths (or whatever the generator is) horizontally. Here's a simple rank-2 lattice, generated by 5ths, showing the octaves:

F2 — C3 — G3 — D4 — A4 — E5 — B5

F1 — C2 — G2 — D3 — A3 — E4 — B4

F0 — C1 — G1 — D2 — A2 — E3 — B3

When the period is an octave, the genweb octave-reduces to a single horizontal genchain:

F — C — G — D — A — E — B

But if the period is a half-octave, the genweb has vertical half-octaves, which octave-reduces to two parallel genchains. Temperaments with third-octave periods reduce to a triple-genchain, and so forth.

Ups and downs are used to distinguish between the genchains. This is yet another new use of ups and downs. The up has both a genspan and a **period-span**.

The small gugu temperament sggT splits the octave into two gu 5ths. The g5 is a dim 5th, but is it upped or downed? Choose so that the up symbol means up in pitch. In sggT, the tempered wa 5th tends to be sharp of just, so the wa dim 5th = Tsw5 tends to be less than a half-octave, therefore g5 must be upped. The generator is a Tw5. Adding or subtracting not only a comma but also a period from a generator makes another generator. Adding or subtracting a comma makes a gugu or yoyo interval, an obscure ratio without much relevance. However subtracting a period makes a gu interval, and subtracting a comma makes a yo one.

2 · g5 = w8 + sgg2
2 · **^d5** = P8 + comma = 2 periods
comma = 2 · ^d5 - P8 = ^^d9 - P8 = **^^d2**
alternate period = period - comma = ^d5 - ^^d2 = **vA4** = Tg5 - Tsgg2 = Ty4
up symbol = ^d5 - d5 = ^d5 - (-6 · P5) = 1 per - (-6 gens) = 6 gens + 1 per
up symbol = ^d5 - d5 = Tg5 - Tsw5 = Tg1
generator = P5 = Tw5
small generator = generator - period = P5 - vA4 = ^m2 = Tw5 - Ty4 = Tg2
alternate small generator = small generator - comma = ^m2 - ^^d2 = vA1 = Tg2 - Tsgg2 = TLy1

With octaves, the lattice looks like this:

C♭^3 — G♭^3 — D♭^4 — A♭^4 — E♭^5 — B♭^5 — F^6

F2 — C3 — G3 — D4 — A4 — E5 — B5

C♭^2 — G♭^2 — D♭^3 — A♭^3 — E♭^4 — B♭^4 — F^5

F1 — C2 — G2 — D3 — A3 — E4 — B4

C♭^1 — G♭^1 — D♭^2 — A♭^2 — E♭^3 — B♭^3 — F^4

F0 — C1 — G1 — D2 — A2 — E3 — B3

This octave-reduces to two genchains:

F — C — G — D — A — E — B

C♭^ — G♭^ — D♭^ — A♭^ — E♭^ — B♭^ — F^

F and C♭^ are exactly half an octave apart. The lattice could instead be constructed using the small generator, Tg2 = ^m2, equivalent to TLy1 = vA1. The genchains alternate between them, to avoid double ups and downs. This kind of lattice makes it harder to see the chords, but easier to trace the melody.

F —— G♭^ — G —— A♭^ — A —— B♭^ — B

C♭^ — C —— D♭^ — D —— E♭^ — E —— F^

Chord names can be awkward with only ups:

Dy = D – G♭^ – A = D(^d4) = "D, up-dim-four" (simple chord, complex name)
Dy = D – F♯v – A = D.v = "D dot down" (same chord, respelled with downs)

The original lattice, with the up chain also shown as a down chain (a redundant lattice):

Bv —— F♯v — C♯v —— G♯v — D♯v — A♯v —— E♯v

F —— C —— G —— D —— A —— E —— B

C♭^ — G♭^ — D♭^ — A♭^ — E♭^ —— B♭^ —— F^

In relative notation:

vM6—— vM3— vM7— vA4 — vA1 — vA5 —— vA2

m3 —— m7 —— P4 —— P1 —— P5 —— M2 —— M6

^d7 —— ^d4 — ^d8 —— ^d5 — ^m2 — ^m6 —— ^m3

As with the triple yo temperament, the lattice rows can be shifted to more closely resemble the JI lattice. However, it becomes less clear which up notes are equivalent to which down notes. Here A♭^ equals G♯v, and ^d5 equals vA4.

Dv —— Av —— Ev —— Bv —— F♯v — C♯v —— G♯v

F —— C —— G —— D —— A —— E —— B

A♭^ —— E♭^ — B♭^ —— F^ —— C^ —— G^ —— D^

v8 —— v5 —— vM2 —— vM6— vM3— vM7— vA4

m3 —— m7 —— P4 —— P1 —— P5 —— M2 —— M6

^d5 —— ^m2 —— ^m6 — ^m3 — ^d7 —— ^d4 —— ^d8

Double ups and downs can often be avoided by respelling the chord root:

D – F♯v – A – C = D7(v3) = "D seven, down-three"
G♯v – C – D♯v – F♯v = G♯v.7(^d4) (same chord, awkward new name...)
A♭^ – C – E♭^ – G♭^ = A♭^.7(v3) (...root respelled for a better name)

But not always. In this example, respelling the 2nd chord's root makes it appear different than the 1st chord:

D – F♯v – A	to	D – F^ – A	to	A – C♯v – E	(IV.v – IV.^m – I.v)
A♭^ – C – E♭^	to	G♯v – B – D♯v	to	E♭^ – G – B♭^	(same progression, looks like IV.v – vv♯III.^m – I.v)
A♭^ – C – E♭^	to	A♭^ – C♭^^ – E♭^	to	E♭^ – G – B♭^	(...better to use double-ups)

The <u>triple gu temperament</u> divides the octave into three yo 3rds. The generator is a wa 5th. The 5th is just, so the M3 is a Lw3 = 408¢, so the 400¢ period is a downed M3.

3 · y3 = w8 - $g^3$2
3 · **<u>vM3</u>** = P8 - comma
comma = P8 - 3 · vM3 = P8 - v^3A7 = **<u>^{3}d2</u>**
alternate period = period + comma = vM3 + ^^{3}d2 = **<u>^^d4</u>** = Ty3 + T$g^3$2 = Tgg4
double period = 2 · vM3 = vvA5 = Tyy5, alternate double period = vM3 + ^^d4 = ^m6 = Tg6

up symbol = M3 - vM3 = 4 · P5 - vM3 = 4 gens - 1 per
up symbol = M3 - vM3 = 4 · Tw5 - Ty3 = Tg1
generator = P5 = Tw5
small generator = generator - period = P5 - vM3 = ^m3 = Tw5 - Ty3 = Tg3
alternate small generator = small generator - comma = ^m3 - ^^{3}d2 = vvA2 = Tg3 - Tg^{3}2 = Tyy2

The lattice has three genchains, each a third of an octave apart:

Av —— Ev —— Bv —— F♯v —— C♯v —— G♯v —— B♯v

F —— C —— G —— D —— A —— E —— B

D♭^ —— A♭^ —— E♭^ —— B♭^ —— F^ —— C^ —— G^

In relative notation:

v5 —— vM2 —— vM6 —— vM3 —— vM7 —— vA4 —— vA1

m3 —— m7 —— P4 —— P1 —— P5 —— M2 —— M6

^d8 —— ^d5 —— ^m2 —— ^m6 —— ^m3 —— ^m7 —— ^4

With shifted rows:

—— Av —— Ev —— Bv —— F♯v —— C♯v —— G♯v

F —— C —— G —— D —— A —— E —— B

—— A♭^ —— E♭^ —— B♭^ —— F^ —— C^ —— G^

—— v5 —— vM2 —— vM6 —— vM3 —— vM7 —— vA4

m3 —— m7 —— P4 —— P1 —— P5 —— M2 —— M6

—— ^d5 —— ^m2 —— ^m6 —— ^m3 —— ^m7 —— ^4

The quadgu temperament g^{4}T splits the octave into four gu 3rds. The comma is fifthward, so the fifth tends to be flat, the wa minor 3rd tends to be > 300¢, and the period is a vm3. The comma is gu, so the yo 3rd tends to be sharp. The flat w5 and sharp y3 combine to make the gu comma become a descending interval. The up symbol always represents an increase in pitch, so ^1 = the ascending inverse of Tg1 = Ty1 = tempered 80/81. Thus gu is not up but down. Tg3 becomes vm3 and Ty3 becomes ^M3.

4 · g3 = w8 + g^{4}2
4 · **vm3** = P8 + comma
comma = 4 · vm3 - P8 = v^{4}d9 - P8 = **v^{4}d2**
alternate period = period - comma = vm3 - v^{4}d2 = **^^{3}4** = Tg3 - Tg^{4}2 = Ty^{3}4
double period = 2 · vm3 = vvd5 = Tgg5, alternate double period = P8 - vvd5 = ^^A4 = Tyy4
up symbol = m3 - vm3 = -3 · P5 - vM3 = -3 gens - 1 per
up symbol = m3 - vm3 = -3 · Tw5 - Tg3 = negative Tg1 = inverse of Tg1 = Ty1
generator = P5 = Tw5
small generator = generator - period = P5 - vm3 = ^M3 = Tw5 - Tg3 = Ty3

The quadru temperament r^{4}T is notated exactly the same way, with Tz3 = vm3 and ^1 = Tr1. So is g^4 & r^{4}T. The lattice can have up above down, to mimic the yo/gu placement, or down above up, to mimic zo/ru placement.

C♭vv / B^^ G♭vv / F♯^^ D♭vv / C♯^^ A♭vv / G♯^^ E♭vv / D♯^^

A♭v E♭v B♭v Fv Cv

—— F —— C —— G —— D —— A ——

D^ A^ E^ B^ F♯^

C♭vv / B^^ G♭vv / F♯^^ D♭vv / C♯^^ A♭vv / G♯^^ E♭vv / D♯^^

Examples of splitting w5, w4, and Ww5?

A quadruple comma will usually split either w8 or w4 or w5 or Ww4 or Ww5 into quarters. But sometimes a quadruple comma splits the octave into halves, and the w2 into four quartertones. This happens when the comma's wa factor is even, but not a multiple of 4, and its octave factor is odd, e.g. (2a+1, 4b+2, 4c). In this case, both the period and the generator are fractional. The period is the half-octave, and the generator is the quartertone. A major 9th is two half-octaves plus four quartertones, and the fifth is half a 9th = the half-octave plus two quartertones. The fifth is not a multiple of the generator. As a result, the lattice is triangular, not rectangular.

For example, the large quadlo comma L1o^4-2 = (-17, 2, 0, 0, 4) = 9¢ splits the octave into two lolo 4ths (1oo4 = 363/256), and splits the wa 2nd into four ilo quartertones (1o1 = 33/32). With la, we must first decide whether 1o4 is notated as a P4 or an A4. With this temperament, choosing the former avoids double sharps and flats. The up symbol is the invisible comma, the ilo quartertone. Thus 1o4 is an ^4.

4 · 1o1 = w2 + L1o^4-2
4 · **^1** = M2 + comma
comma = 4 · ^1 - M2 = ^^{4}1 - M2 = -(M2 - ^^{4}1) = -(M2 + v^{4}1) = **-v^{4}M2**
period = T1oo4 = **^^4**
alternate period = octave - period = P8 - ^^4 = **vv5** = Tw8 - T1oo4 = T1uu5
generator = up symbol = ^1 = T1o1
large generator = period + generator = vv5 + ^1 = v5 = T1uu5 + T1o1 = T1u5
inverse generator = period - generator = ^^4 - ^1 = ^4 = T1oo4 - T1o1 = T1o4

The triangular lattice is shown here with one row repeated:

B♭^^ = Cvv F^^ = Gvv C^^ = Dvv G^^ = Avv D^^ = Evv
F^ C^ G^ D^
—— **F** —————— **C** —————— **G** —————— **D** —————— **A** ——
Cv Gv Dv Av
B♭^^ = Cvv F^^ = Gvv C^^ = Dvv G^^ = Avv D^^ = Evv

Had we chosen to notate 1o4 as an A4, the comma would be a -^^{4}d^{3}2, and the top row would start with B♯vv = C♭♭^^!
The same lattice, with the natural notes repeated an octave up, and the quartertone generator running downwards:

—— **F** —————— **C** —————— **G** —————— **D** —————— **A** ——
F^ C^ G^ D^
B♭^^ = Cvv F^^ = Gvv C^^ = Dvv G^^ = Avv D^^ = Evv
Cv Gv Dv Av
—— **F** —————— **C** —————— **G** —————— **D** —————— **A** ——

Another example: the small quadzo temperament sz^{4}T tempers out (11, -14, 0, 4) = 48¢. The period is a half-octave = TLrr3 = (-5, 7, 0, -2). The generator is a zo 2nd, written as a vm2, or a zo 5th = vd5. The period is a ^^A3 or a vvd6. The comma is a v^{4}d^{3}4. Because the quartertone is a 2nd not a unison, the notation is messier:

D♭♭vv = A♯^^ A♭♭vv = E♯^^ E♭♭vv = B♯^^ B♭♭vv = FX^^ F♭vv = CX^^
G♭v D♭v A♭v E♭v
—— **F** —————— **C** —————— **G** —————— **D** —————— **A** ——
B^ F♯^ C♯^ G♯^

D♭♭vv = A♯^^ A♭♭vv = E♯^^ E♭♭vv = B♯^^ B♭♭vv = F^{X}^^ F♭vv = C^{X}^^

Use high/lows to clean up the notation?

A **sixfold comma** will sometimes split the octave into thirds and the re-voicied 5th into halves, or vice versa.

1o^{6}T (1o4 = ^4): comma = 1o^6-2 = (-16,-3,0,0,6) = -v^{6}A2

period = w8/3 = 1oo3 = (-5,-1,0,0,2) = ^^m3 = 1u^{4}4 = (11,2,0,0,-4) = v^{4}A4

gen = w4/2 = 1o^{3}2 = (-7,-2,0,0,3)= ^^{3}m2 = 1u^{3}3 = (9,1,0,0,-3) = v^{3}M3

period - gen = alt-gen = 1u2 = 12/11 = vM2

2 periods - gen = alt-gen = 1o4 = 11/8 = ^4

w5 = 3 periods - 2 gens = 1oo3 + 1oo3 + 1oo3 - 1o^{3}2 - 1o^{3}2

—— **F** —— D♭^^{3}=Ev^3 —— **C** —— A♭^^{3}=Bv^3 —— **G** —— E♭^^{3}=F♯v^3 —— **D** —— B♭^^{3}=C♯v^3 —— **A** ——

Dvv B♭^ Avv F^ Evv C^ Bvv G^ F♯vv

A♭^^ Gv E♭^^ Dv B♭^^ Av F^^ Ev C^^

—— **F** —— D♭^^{3}=Ev^3 —— **C** —— A♭^^{3}=Bv^3 —— **G** —— E♭^^{3}=F♯v^3 —— **D** —— B♭^^{3}=C♯v^3 —— **A** ——

Triangular lattice, using 1u2 and 1o4 as generators:

D♭^^{3}=Ev^3 A♭^^{3}=Bv^3 E♭^^{3}=F♯v^3 B♭^^{3}=C♯v^3

A♭^^ E♭^^ B♭^^ F^^ C^^

B♭^ F^ C^ G^

—— **F** —————— **C** —————— **G** —————— **D** —————— **A** ——

Gv Dv Av Ev

Dvv Avv Evv Bvv F♯vv

D♭^^{3}=Ev^3 A♭^^{3}=Bv^3 E♭^^{3}=F♯v^3 B♭^^{3}=C♯v^3

LL1o^{6}T (1o4 = ^4):

comma = LL1o^6-3 = (-35,9,0,0,6) = -v^{6}m3

period = w8/3 = s1uu4 = (12,-3,0,0,-2) = vv4 = L1o^{4}2 = (-23,6,0,0,4) = ^^{4}M2

gen = w5/2 = s1u^{3}4 = (17,-4,0,0,-3) = v^{3}4 = L1o^{3}2 = (-18,5,0,0,3) = ^^{3}M2

alt-gen = per - gen = w3/6 = 1o1 = 33/32 = ^1

—— **F** —— G^^{3}=B♭v^3 —— **C** —— D^^{3}=Fv^3 —— **G** —— A^^{3}=Cv^3 —— **D** —— E^^{3}=Gv^3 —— **A** ——

C^^ Fv G^^ Cv D^^ Gv A^^ Dv E^^

B♭vv C^ Fvv G^ Cvv D^ Gvv A^ Dvv

—— **F** —— G^^{3}=B♭v^3 —— **C** —— D^^{3}=Fv^3 —— **G** —— A^^{3}=Cv^3 —— **D** —— E^^{3}=Gv^3 —— **A** ——

The lattice written out triangularly using the alt-gen:

G^^{3}=B♭v^3 D^^{3}=Fv^3 A^^{3}=Cv^3 E^^{3}=Gv^3

C^^ G^^ D^^ A^^ E^^

C^ G^ D^ A^

— **F** — **C** — **G** — **D** — **A** —

Fv Cv Gv Dv

B♭vv Fvv Cvv Gvv Dvv

G$^{\wedge 3}$=B♭v^3 D$^{\wedge 3}$=Fv^3 A$^{\wedge 3}$=Cv^3 E$^{\wedge 3}$=Gv^3

<u>Multi-comma tempers</u>: So far, we've managed to notate every rank-2 temperament with only ups and downs and conventional accidentals (except for 17-edo+y). But if one comma splits the octave, and another splits some voicing of the fifth, we must resort to using colors as well.

The <u>double ruyo temperament</u> rryyT is a rank-3 temperament. But sgg&rryyT is rank-2. It's notated the same way as sggT. The comma is three-less, so the wa 5th is just. The wa aug 4th is more than half an octave, so the ruyo aug 4th must be downed.

2 · ry4 = w8 + rryy-2
2 · **<u>vA4</u>** = P8 + comma = 2 periods
comma = 2 · vA4 - P8 = vvA7 - P8 = -(P8 - vvA7) = **<u>-^^d2</u>**
alternate period = period - comma = vA4 - (-^^d2) = vA4 + ^^d2 = **<u>^d5</u>** = Try4 - Trryy-2 = Tzg5
up symbol = A4 - vA4 = Lw4 - ry4 = descending Tsry1 = Tg1 - Tr1
up symbol = A4 - vA4 = 6 · P5 - vA4 = 6 · Tw5 - Ty4 = 6 gens - 1 per

sw&rryyT

g3&r4T

More examples, from my notes:

Ly^4-2 = (-14, 3, 4) = ^^{4}-dd2. Gen = g2 = ^m2, alt-gen = Ly^{3}1 = v^{3}AA1. P4 = 4 gens. ^1 = g1 = -19 gens

z^{4}gg3 = pp1 = (-5, -1, -2, 4) = vvdd3. Gen = zzg4 = vd4, alt-gen = rry2 = ^A2. P5 = 2 gens. Gen2 = r2. ^1 = ??

sg^{3}3 = (15, -5, -3) = v^{3}dd3. Gen = y4 = ^A4, alt-gen = sgg6 = vvd6. Ww5 = 3 gens, P5 = 3g - 1p. ^1 = y8 = -17 gens.

Lr^3-3 = (-9, 11, 0, -3) = ^^{3}-dd3. Gen = r4 = 81/56 = ^A4, alt-gen = zz6 = vvd6. Ww4 = 3 gens, P5 = -3 gens. ^1 = r1 = -17 gens.

"min 2" really means -5 steps on the chain of 5ths, not 1 semitone. But we can just use familiar 12-edo interval arithmetic in all our calculations. The answer is the same.

rank-2 with non-5th gen and non-8ve per (quartertone tempers like L1o^{4}T, or multi-comma tempers)

quadruple commas (quarter-8ve, quarter-fifth, or quartertone-half-8ve)
sixfold commas (sixth-8ve, sixth-fifth, semitone-third-8ve, or thirdtone-half-8ve)

rank-3 with per = 8ve and gen1 = 5th, gen2 is a rung – – – ryy-2
rank-3 with per = 8ve and gen1 = 5th, gen2 is not a rung
rank-3 with non-fifth gen1
rank-3 with non-8ve per – – – rryy-2
rank-3 with non-5th gen1 and non-8ve per (multi-comma)

All other rank-2 tempers with an octave period can be notated without colors, using only ups and downs.

Chapter 5.18 – Alternate Keyboards, Alternate Notations *

One of the aims of this book is to provide a single notation for JI and near-JI temperaments (color notation), another notation for all edos (ups and downs), and another for all rank-2 temperaments (pergens). Both ups/downs and lifts/drops can be viewed as virtual colors whose exact meaning depends on the context. Thus the 3 notations merge into one universal notation.

This notation is backwards-compatible with conventional notation. Only three things are required for this: that the notation be octave-equivalent, heptatonic and generated by fifths.

The value of this is that every aspect of microtonal music theory can be expressed in this notation: sheet music, chord names, progressions, temperament names, MOS scales, etc. Thus all any musician needs to learn is this one notation. But notation is not just for musicians, it's for theorists and composers too. They need a way to express their ideas to themselves, and for "thinking outside the box". Also, every piece of music is a lesson from one composer to all future composers. For these non-performance-related issues, non-standard notations are useful.

There's a parallel with 12edo composers that use unusual scales like the whole-tone scale, the harmonic minor scale, the LsLsLsLs octotonic scale or the LssLssLss Tcherepnin scale. They may invent their own notation to compose in, perhaps hexatonic or nonatonic, but they don't impose it on others. When they present their music to other musicians, they use conventional notation. The 12edo world has one universal notation for communication, and alternate notations for non-heptatonic and non-fifth-generated music. The microtonal world should have the same.

In chapter 5.4 we saw how the keys of a midi controller can be rearranged to have more than 12 keys per octave. If there are 15 keys per octave, how do you decide which are white and which are black? it would be logical to look at 15edo notation, and let the natural notes be white. But the resulting layout D * * E/F * * G * * A * * B/C * * D is hard to play. There are way too many black keys, but that can be solved by reversing black and white, and having the black keys be the natural notes: b w w b w w b w w b w w b w w b. Now there are too many white keys! It would be best to have a more equal number of black and white. 7 white and 8 black would be ideal, since we're accostomed to 7 natural notes. But the larger problem is that since the b w w pattern repeats every three keys, and there are no landmarks to orient you.

Most 15-tone keyboards use this layout: W * W * W * W * W * W * W * * W. It has 7 white and 8 black keys, and the two adjacent black keys make a nice landmark. It can be described as 7w-8b, or since it's a MOS scale, as 1L6s. Where did this layout come from? Where did the familiar 7w-5b layout come from?

Western notation is based on the familiar chain of fifths:

...D♭ A♭ E♭ B♭ F C G D A E B F♯ C♯ G♯ D♯...

Here's the relative notation genchain, the quality-chain of chapter 3.2:

...d8 d5 m2 m6 m3 m7 P4 P1 P5 M2 M6 M3 M7 A4 A1...

Every genchain, heptatonic or not, generated by the 5th or not, with a period of an 8ve or not, shares certain properties. The period, the generator, and their difference are always perfect, and everything else is imperfect. Thus there are always exactly three perfect degrees. The rest are imperfect. One side of the genchain is major and augmented, and the other side is minor and diminished.

Applying this genchain to a 12-tone keyboard creates the familiar C * D * E F * G * A * B C layout. Applying this genchain to 15-tone makes an awkward layout. Is there another genchain more suited to 15-tone? In chapter 5.3, we examined the pentatonic chain of fifths. But this genchain applied to 15-tone creates the exact same awkward layout. And pentatonic names are unfamiliar. Is there any other heptatonic alternative? Yes, if the genchain is a chain not of fifths but of some other interval.

We've seen that every sizing framework has a natural naming framework, which is always fifth-based, but may not be heptatonic. Every sizing framework also has a **natural heptatonic generator**, which is always heptatonic, but may not be the fifth. For 15-tone, it's the downmajor 2nd = 2\15. When using a non-fifth generator, how does one decide what note to start the genchain on? A genchain of 2nds has sequential letters. If the letters runs alphabetically from A to G, we have ABCDEFG. Every 2nd between two adjacent letters of the alphabet is perfect. G–A is an augmented 2nd. Only the central 3 intervals of any genchain are perfect. Thus the 4th and 5th can be major or minor.

...D♯ E♯ F♯ G♯ A B C D E F G A♭ B♭ C♭ D♭...

...A1 A2 M3 M4 M5 M6 P7 P1 P2 m3 m4 m5 m6 d7 d8...

Seven natural generators always add up to some number of octaves, plus or minus a single step. With 15-tone, it's minus, so the righthand side of the two genchains have flats and minors, not sharps and majors. The single step ensures that natural notation never needs ups and downs. There are actually two natural generators, the other one is the octave inverse. The smaller of the two is usually the preferred generator, but the fifth is preferred for historical reasons. Applying the absolute genchain to 15-tone creates this layout:

Figure 5.16.1 – Absolute notation in the 15 + 7 system with the natural generator (perfect 2nd = 2\15)

D	D♯ E♭	E	E♯ F♭	F	F♯ G♭	G	G♯	A♭	A	A♯ B♭	B	B♯ C♭	C	C♯ D♭	D

In alt-tuner, this corresponds to the even distribution placement, as opposed to the chain of 5ths placement. Applying the relative genchain to 15-tone creates this:

Figure 5.16.2 – Relative notation in the 15 + 7 system with the natural generator (perfect 2nd = 2\15)

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0¢	80	160	240	320	400	480	560	640	720	800	880	960	1040	1120	1200
P1	A1 d2	P2	A2 d3	m3	M3	m4	M4	m5	M5	m6	M6	A6 d7	P7	A7 d8	P8

The cents are for 15-edo, and represent the average cents of that keyspan. The usual perfect 4th and 5th become the min 4th and maj 5th. M4 and m5 are tritones. The 2nd-based M3 is ~400¢ and the m3 is ~320¢, corresponding well to yo and gu. Likewise the 6ths do as well. The perfect 2nd is neutral-sounding, and the A2 sounds more major.

The minor 5th results from stacking four perfect 2nds, and the major 5th results from subtracting three perfect 2nds from an octave. A – E is minor, B – F♯ is major, C – G is minor, and D – A is major. There's no easy way to tell if a fifth is major or minor. Using a non-fifth-based notation completely changes interval arithmetic. Everything you've spent years memorizing, e.g. that C – E is a major 3rd, is suddenly all wrong. Using 15-tone's natural heptatonic notation has disadvantages. Why use it? Because a notation implies a keyboard layout, and vice versa. If we used 5th-based notation with a 2nd-based keyboard, we would get this:

Figure 5.16.3 – Absolute notation in the 15 + 7 system with the 5th generator (perfect 5th = 9\15)

D	D^ E♭^	Ev	E F	F^	F♯v Gv	G	G^ A♭^	G♯v Av	A	A^ B♭^	Bv	B C	C^	C♯v Dv	D

These note names don't seem to fit as well. It's much easier to navigate an unusual keyboard if you can count on two conventions: the natural notes are white keys, and adding a sharp or flat moves you one key higher or lower.

Figure 5.16.4 – Relative notation in the 15 + 7 system with the 5th generator (perfect 5th = 9\15)

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
P1	^1 ^m2	vM2	M2 m3	^m3	vM3 v4	M3 P4	^4 ^d5	vA4 v5	P5 m6	^5 ^m6	vM6	M6 m7	^m7	vM7 v8	P8

Here is the genchain in both 5th-based and 2nd-based notation:

5th-based	D^	E	F♯v	G^	A	Bv	C^	**D**	Ev	F^	G	Av	B♭^	C	Dv
	^1	M2	vM3	^4	P5	vM6	^m7	**P1**	vM2	^m3	P4	v5	^m6	m7	v8
2nd-	D♯	E♯	F♯	G♯	A	B	C	**D**	E	F	G	A♭	B♭	C♭	D♭

based	A1	A2	M3	M4	M5	M6	P7	**P1**	P2	m3	m4	m5	m6	d7	d8

The seven conventional modes are simply any 7 adjacent intervals in the conventional relative genchain. We can construct 7 analogous modes. The major scale uses only perfect and major intervals, i.e. P1 P2 M3 M4 M5 M6 P7 P8. Likewise the minor scale uses 3P, 3m and 1M, i.e. P1 P2 m3 m4 m5 M6 P7 P8. The 7 modes in both notations:

D lydian (2P 4M 1A)	D E♯ F♯ G♯ A B C D	D E F♯v G^ A Bv C^ D
D major (3P 4M)	D E F♯ G♯ A B C D	D Ev F♯v G^ A Bv C^ D
D mixolydian (3P 3M 1m)	D E F G♯ A B C D	D Ev F^ G^ A Bv C^ D
D dorian (3P 2M 2m)	D E F G A B C D	D Ev F^ G A Bv C^ D
D minor (3P 1M 3m)	D E F G A♭ B C D	D Ev F^ G Av Bv C^ D
D phrygian (3P 4m)	D E F G A♭ B♭ C D	D Ev F^ G Av B♭^ C^ D
D locrian (2P 4m 1d)	D E F G A♭ B♭ C♭ D	D Ev F^ G Av B♭^ C D

The relationship between the modes is preserved. Lydian is still the mode with the most sharps, amd locrian the one with the least. However, the meaning of major, dorian, etc. completely changes. The only mode which has both 4/3 and 3/2 (m4 and M5) is dorian. Minor, phrygian and locrian all have a minor 5th = ~640¢. The only modes with a major 3rd are lydian and major. There are no modes with a maj 3rd and a min 4th.

Here's the conventional modes written out in many keys, starting on the tonic, and also on C, to show how sharps gradually become flats.

E lyd, B maj, F♯ mix, C♯ dor, G♯ min, D♯ phr, A♯ loc	B C♯ D♯ E F♯ G♯ A♯ B	C♯ D♯ E F♯ G♯ A♯ B
A lyd, E maj, B mix, F♯ dor, C♯ min, G♯ phr, D♯ loc	E F♯ G♯ A B C♯ D♯ E	C♯ D♯ E F♯ G♯ A B
D lyd, A maj, E mix, B min, F♯ min, C♯ phr, D♯ loc	A B C♯ D E F♯ G♯ A	C♯ D E F♯ G♯ A B
G lyd, D maj, A mix, E min, B min, F♯ phr, C♯ loc	D E F♯ G A B C♯ D	C♯ D E F♯ G A B
C lyd, G maj, D mix, A dor, E min, B phr, F♯ loc	G A B C D E F♯ G	C D E F♯ G A B
F lyd, C maj, G mix, D dor, A min, E phr, B loc	C D E F G A B C	C D E F G A B
B♭ lyd, F maj, C mix, G dor, D min, A phr, E loc	F G A B♭ C D E F	C D E F G A B♭
E♭ lyd, B♭ maj, F mix, C dor, G min, D phr, A loc	B♭ C D E♭ F G A B♭	C D E♭ F G A B♭
A♭ lyd, E♭ maj, B♭ mix, F dor, C min, G phr, D loc	E♭ F G A♭ B♭ C D E♭	C D E♭ F G A♭ B♭
D♭ lyd, A♭ maj, E♭ mix, B♭ dor, F min, C phr, G loc	A♭ B♭ C D♭ E♭ F G A♭	C D♭ E♭ F G A♭ B♭

A similar table can be made for the 15-tone white-key heptatonic scale. Here they are, in both notations:

B lyd, A maj, G♯ mix, F♯ dor, E♯ min, D♯ phr, C♯ loc	A B C♯ D♯ E♯ F♯ G♯ A	A Bv C♯v D^ E F♯v G^ A Bv
C lyd, B maj, A mix, G♯ dor, F♯ min, E♯ phr, D♯ loc	B C D♯ E♯ F♯ G♯ A B	Bv C^ D^ E F♯v G^ A Bv
D lyd, C maj, B mix, A dor, G♯ min, F♯ phr, E♯ loc	C D E♯ F♯ G♯ A B C	C^ D E F♯v G^ A Bv C^
E lyd, D maj, C mix, B dor, A min, G♯ phr, F♯ loc	D E F♯ G♯ A B C D	D Ev F♯v G^ A Bv C^ D
F lyd, E maj, D mix, C dor, B min, A phr, G♯ loc	E F G♯ A B C D E	Ev F^ G^ A Bv C^ D Ev
G lyd, F maj, E mix, D dor, C min, B phr, A loc	F G A B C D E F	F^ G A Bv C^ D Ev F^
A♭ lyd, G maj, F mix, E dor, D min, C phr, B loc	G A♭ B C D E F G	G Av Bv C^ D Ev F^ G
B♭ lyd, A♭ maj, G mix, F dor, E min, D phr, C loc	A♭ B♭ C D E F G A♭	Av B♭^ C^ D Ev F^ G Av
C♭ lyd, B♭ maj, A♭ mix, G dor, F min, E phr, D loc	B♭ C♭ D E F G A♭ B♭	B♭^ C D Ev F^ G Av B♭^
D♭ lyd, C♭ maj, B♭ mix, A♭ dor, G min, F phr, E loc	C♭ D♭ E F G A♭ B♭ C♭	C Dv Ev F^ G Av B♭^ C

If you wanted to memorize all the major modes, which notation would you rather use?

A notation also implies various rank-2 temperaments with the same generator, and various chord progressions that pump the associated comma. The 2\15 generator is 160¢, implying 10/9 or 11/10. The former implies the triple yo temperament, and the triple yo comma pump. Composers who use this pump a lot may prefer to think in 2nd-based notation. There follows a triple yo pump that uses yo (downmajor) and gu (upminor) triads in both notations. The chords in the 2nd-based notation simply list the component intervals. By analogy, unaltered roman numerals represent a major scale.

5th-based: I.v – vVI.^m – vII.v – [vvVII=^♭VII].^m – ^♭III.v – I.^m – V7(v3) – I.v

G.v – Ev.^m – Av.v – [F♯vv=F^].^m – B♭^.v – G.^m – D7(v3) – G.v
G Bv D – Ev G Bv – Av C♯vv Ev – [F♯vv Av C♯vv = F^ A♭^^ C^] – B♭^ D F^ – G B♭^ D – D F♯v A C – G Bv D

2nd-based: I(M3M5) – VI(m3M5) – II(M3M5) – VII(m3M5) – ♭III(M3M5) – I(m3M5) – V(M3M5d7) – I(M3M5)
G(M3M5) – E(m3M5) – A♭(M3M5) – F(m3M5) – B♭(M3M5) – G(m3M5) – D(M3M5d7) – G(M3M5)
G B D – E G B – A♭ C E – F A♭ C – B♭ D F – G B♭ D – D F♯ A C♭ – G B D

Microtonalists call the triple yo comma the porcupine comma, and this 15-tone keyboard layout and notation porcupine. But the layout and notation don't really have anything to do with triple yo, or 250/243, or any ratio. It has to do with the fundamental question of how to notate 15 keys or frets with 7 letters. Even if your keyboard or guitar were using a non-octave scale, and 15 frets equaled 7/3 or 3/1, 2\15 would still be the natural heptatonic generator.

Is there any other heptatonic genchain possible? Yes, the one generated by 3rds. What letter does it start on? Both the ABCDEFG genchain and the FCGDAEB genchain have only 1 note whose position is unchanged, the center note D. If the genchain of 3rds also has a central D, it runs EGBDFAC. "Every Good Boy Deserves Fudge And Candy".

...D♯ F♯ A♯ C♯ E G B D F A C E♭ G♭ B♭ D♭...

...A8 A3 M5 M7 M2 M4 P6 P1 P3 m5 m7 m2 m4 d6 d1...

But using this genchain with 15-tone is the worse of both worlds – both unfamiliar and awkward. The generator is the 15-edo interval closest to 2\7 = 343¢, which is 4\15 = 320¢. Seven generators = 28\15, which is 2 edosteps short of the octave, thus the 3rdwards side of the genchain has flats and minors. Ups and downs are required, one sharp = 2 ups.

Every sizing framework will be compatible with one of these 3 genchains. This genchain is natural for 17-tone. The generator = 5\17 = perfect 3rd. Seven generators = 35\17, which is 1 edostep past the octave. Unlike 15-tone, the 3rdwards side of the genchain has sharps and majors.

...D♭ F♭ A♭ C♭ E G B D F A C E♯ G♯ B♯ D♯...

...d8 d3 m5 m7 m2 m4 P6 P1 P3 M5 M7 M2 M4 A6 A1...

This genchain gives us this notation and keyboard layout:

D * E * * F * G * * A * B * * C * D
P1 - A1/d2 - m2 - M2 - A2/d3 - P3 - A3/d4 - m4 - M4 - m5 - M5 - A5/d6 - P6 - A6/d7 - m7 - M7 - A7/d8 - P8

The diagram only shows part of the full scale tree, which extends sideways from 0¢ (0\1) to 1200¢ (1\1). The full tree contains four pentatonic kites and six heptatonic kites. Each kite is named after its head. The blue kite is the 4\7 kite; the others are the 1\7, 2\7 3\7, 5\7 and 6\7 kites. The 3\7 kite is the mirror image of the 4\7 kite, 5\7 mirrors 2\7, and 6\7 mirrors 1\7. The 4\7 kite contains frameworks best notated by heptatonic notation generated by the fifth (i.e., to sharpen or augment means to add seven fifths, octave-reduced). The octave inverse of the generator is also a generator, thus fourth-generated is equivalent to fifth-generated, and the 3\7 kite contains the exact same frameworks as the 4\7 kite. The 2\7 kite is for notation generated by thirds, and the 1\7 kite is for notation generated by seconds.

Every framework larger than 7-tone will appear on only one of these three mirror-pairs of kites. The only exception is perfect frameworks, which appear on the spine of every heptatonic kite. This means that every non-perfect edo above 7-edo has a "natural" (not requiring ups and downs) heptatonic notation, generated by either the 2nd, the 3rd, or the 5th. For example, the natural heptatonic notation for 22-tone is 2nd-based, known as triple yo notation or porcupine notation after the comma that also implies that generator.

Whatever interval generates the notation is by definition perfect. To determine which intervals in an edo are perfect, rather than using the polygon method of Figure 5.2.3, you can simply look for the heptatonic kite containing that edo.

There are only four octotonic kites in two mirror pairs. There are no 2\8, 4\8 or 6\8 kites because those nodes are spinal nodes. Every edo larger than 8-edo, other than those that are multiples of 8, has a natural octotonic notation generated by either 1\8 (octotonic 2nd) or 3\8 (octotonic 4th). There are only three nonatonic kite pairs, and every edo above 9 and not a multiple of 9 has a natural nonatonic notation generated by 1\9, 2\9 or 5\9.

15-edo: The only other logically consistent alternative is to use what I call the edo's "natural generator". Western notation is fifth-generated, but the fifth isn't the natural generator for most edos. For 15-edo, it's the 2\15 2nd. Natural because 2\15 is directly under 1\7 on the scale tree. It's the only x\15 interval that's directly under any y\7 interval, which makes 2\15 the only heptatonic-friendly generator in 15-edo. Besides its octave inverse, of course.

One fun thing to do with any yaza piece is "flip" it by exchanging yo for ru and vice versa, and zo for gu and vice versa. Wa, zogu and ruyo are unchanged. y3 becomes r3, g6 becomes z6, etc. That shifts some notes around by 36/35, maintaining a rough melodic similarity, and making otonal utonal and vice versa. Monzo-wise, (a, b, c, d) becomes (a + 2c + 2d, b + 2c + 2d, -d, -c).

In 14-edo, the best representation of y3 = 5/4 is the up-3rd = 429¢, but the best representation of the yo chord = 4:5:6 uses the plain 3rd = 343¢. This is because the best representation of a chord must represent all the intervals, not just the ones from the root. So the upper interval in close voicing, the minor 3rd, must be represented accurately too.

Table 5.8.8 – Mapping JI triads of odd limit 9 to EDOs 9-24

EDO	r	y	g	z	sus4	sus2	g(zg5)	z(zg5)
	9:7:6	4:5:6	6:5:4	6:7:9	6:8:9	9:8:6	5:6:7	7:6:5
9*	Cm	Cm	C	C	Csus4	Csus2	Caug	Caug
10	C	C~	C~	Cm	Csus4	Csus2	C~(v5)	Cm(v5)
11*	Cm	C~	C~	C	**	**	C~(v5)	C(v5)
12	C	C	Cm	Cm	Csus4	Csus2	Cdim	Cdim
13b*	Cm(^5)	C.vm(^5)	C.vm(^5)	C.^(^5)	**	**	C.^(v5)	C.^(v5)
14	C.^	C	C	C.v	Csus4	Csus2	C(v5)	C.v(v5)
15	C	C.v	C.^m	Cm	Csus4	Csus2	C.^dim(^5)	Cdim(^5)
16*	C(d3)	Cm	C	C(A3)	**	**	Caug	Caug
17	C	C	Cm	Cm	Csus4	Csus2	Cdim	Cdim
18b*	C.^m(^5)	Cm(^5)	C~(^5)	C(^5)	**	**	C~(v5)	C(v5)
19	C(A3)	C	Cm	C(d3)	Csus4	Csus2	Cm(dd5)	C(d3,dd5)
20	C.v	C.v	C.^m	C.^m	Csus4	Csus2	C.^m(vv5)	C.^m(vv5)
21	C.^	C.^	C.v	C.v	Csus4	Csus2	C.v(vv5)	C.v(vv5)
22	C	C.v	C.^m	Cm	Csus4	Csus2	C.^dim(^5)	Cdim(^5)
23*	C(d3)	Cm	C	C(A3)	**	**	C(AA5)	C(A3,AA5)
24	C.^	C	Cm	C.vm	Csus4	Csus2	Cdim(v5)	C.vdim(v5)

** major is narrower than minor*

*** there are two equally valid representations, e.g. for 16-edo, Csus4(d5) or Csus4(A4)*

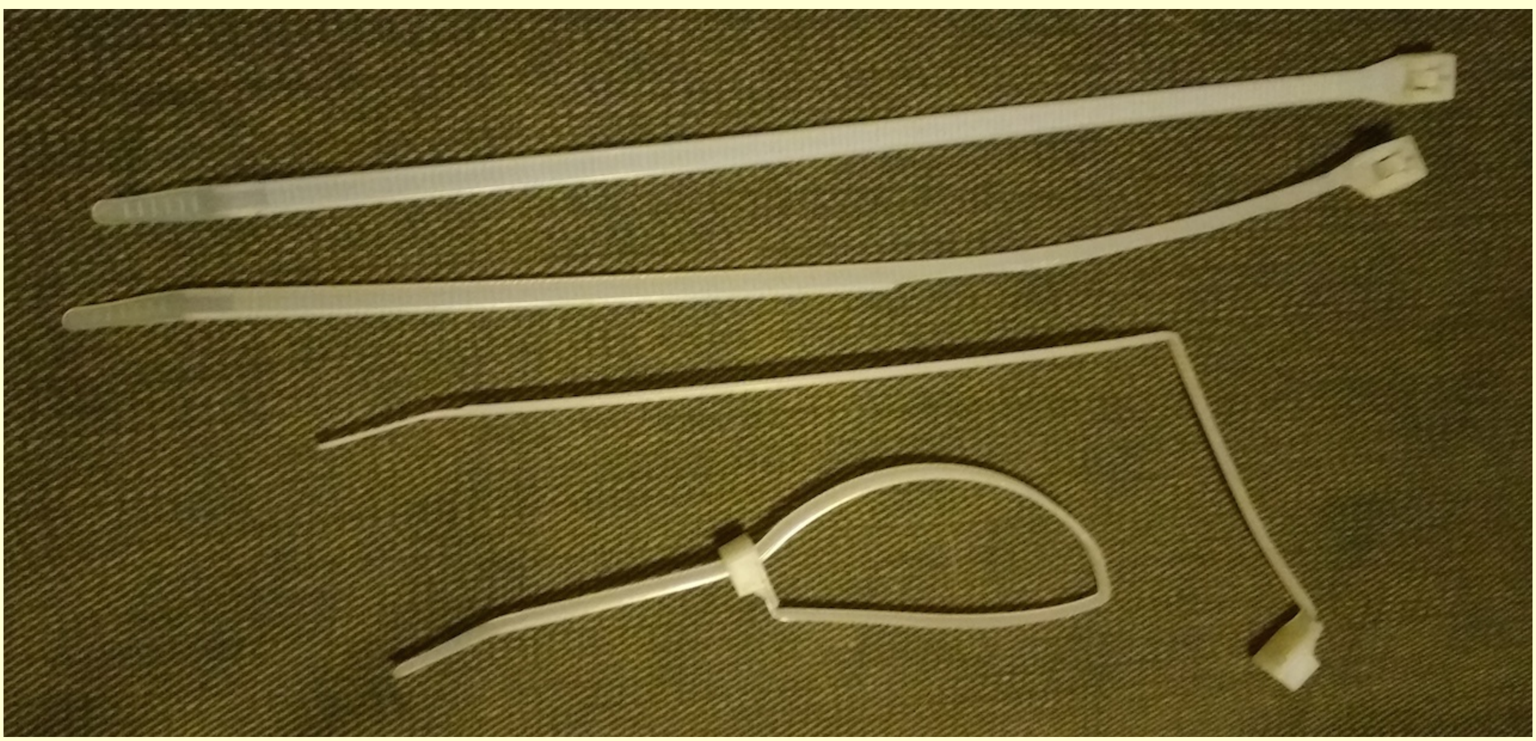
Chapter 5.19 – Guitar Frettings: Edos and Udos

A guitar or other string instrument can be refretted by removing the frets and attaching cable ties (zip ties) to the neck. These frets are cheap and adjustable, allowing easy experimentation. This guitar is tuned to 15-edo.

A close-up of the frets. The original 12-edo frets are still visible.

Another close-up of the cable ties.

The cable ties are trimmed in half to make a narrower fret, then crimped before attaching.



To minimize buzz, get the cable ties to fit as tightly as possible. If the neck has a taper, move the cable tie to a narrow part of the neck, tighten it, then slide it to its final spot. The extra material can be trimmed off with nail clippers.

One can estimate fret placement by looking at the original 12edo frets. Halfway between two frets is very close to 50¢ from the lower fret, at 49.28¢. This website calculates exact fret placements: www.ekips.org/tools/guitar/fretfind2d/

The simplest way to fret a guitar is to use straight frets to create an edo. A JI or rank-2 tuning with straight frets creates lots of wolf intervals. For example, a meantone tuning might run like this:

Figure 5.x.1 – A meantone[12] fretting using P1 - m2 - M2 - m3 - M3 - P4 - d5 - P5 - m6 - M6 - m7 - M7 - P8

		1	2	3	4	5	6	7	8	9	10	11	12
E		F	F♯	G	G♯	A	B♭	B	C	C♯	D	D♯	E
B		C	C♯	D	D♯	E	F	F♯	G	G♯	A	A♯	B
G		A♭	A	B♭	B	C	D♭	D	E♭	E	F	F♯	G
D		E♭	E	F	F♯	G	A♭	A	B♭	B	C	C♯	D
A		B♭	B	C	C♯	D	E♭	E	F	F♯	G	G♯	A
E		F	F♯	G	G♯	A	B♭	B	C	C♯	D	D♯	E

The shaded columns correspond to the frets with dots. E♭ and D♯, which would be a unison in 12-edo, are two different notes. As are A♭ and G♯, B♭ and A♯, etc. These mistuned unisons have both advantages and disadvantages. The advantage is you get more than 12 notes per octave, and more options while playing. You can play both a B major chord and a C minor chord, not possible with only 12 notes of meantone.

The disadvantage is that mistuned unisons create wolf intervals. On a keyboard, meantone has a few wolf intervals. However, these wolves are easily avoided because they only occur in remote keys. On a guitar, there are many more wolves. In addition there are wolf octaves, the worst wolves of all. Any chord outside of the keys of C major and G major can contain wolves. To avoid the wolves, any such chord can only be played at certain places on the neck. For example, playing an E♭ major chord as x11343 uses both E♭ and D♯, a wolf octave. There is also a wolf 4th from B♭ to D♯. E♭ major must be played as 668886. E major played as 022100 has a wolf 3rd. It must be played as x22454 or 779997.

A JI fretting, even if only ya, will contain even more wolf intervals.

Figure 5.x.2 – A ya JI fretting using w1 g2 w2 g3 y3 w4 y4 w5 g6 y6 g7 y7 w8

wE		gF	wF♯	gG	yG♯	wA	yA♯	wB	gC	yC♯	gD	yD♯	wE
wB		gC	wC♯	gD	yD♯	wE	yE♯	wF♯	gG	yG♯	gA	yA♯	wB
wG		gA♭	wA	gB♭	yB	wC	yC♯	wD	gE♭	yE	gF	yF♯	wG
wD		gE♭	wE	gF	yF♯	wG	yG♯	wA	gB♭	yB	gC	yC♯	wD
wA		gB♭	wB	gC	yC♯	wD	yD♯	wE	gF	yF♯	gG	yG♯	wA
wE		gF	wF♯	gG	yG♯	wA	yA♯	wB	gC	yC♯	gD	yD♯	wE

For example the Gy chord 320003 has both gG and wG, making a wolf y8. Its wD makes a wolf y5 with gG. Most open-string chords have wolves. Many chords are impossible to play on all 6 strings. Open tunings greatly reduce the wolves of rank-2 and JI frettings. To be playable in standard tuning, JI requires crooked frets, much more complicated.

There is some historical precedent to crooked frets. Renaissance lutes sometimes had tastini, "little frets", which were pieces of wood glued onto the fretboard just behind the fret, and only affected one or two strings.

Here's a beautiful crooked-fret guitar from Ron Sword:

Figure 5.x.3 – A JI guitar from Ron Sword at MetatonalMusic.com

To avoid wolves, straight-frets frettings usually are tuned to edos. For example, here's a 10-edo fretting. The open strings are all tuned in 4ths. Because B and C are the same note, G – B is a 4th.

Figure 5.x.4 – A 10-edo fretting, with cents from low E, octave-reduced

E = 0¢		120¢	240¢	360¢	480¢	600¢	720¢	840¢	960¢	1080¢	0¢
B = 720¢		840¢	960¢	1080¢	0¢	120¢	240¢	360¢	480¢	600¢	720¢
G = 240¢		360¢	480¢	600¢	720¢	840¢	960¢	1080¢	0¢	120¢	240¢
D = 960¢		1080¢	0¢	120¢	240¢	360¢	480¢	600¢	720¢	840¢	960¢
A = 480¢		600¢	720¢	840¢	960¢	1080¢	0¢	120¢	240¢	360¢	480¢
E = 0¢		120¢	240¢	360¢	480¢	600¢	720¢	840¢	960¢	1080¢	0¢

Another possibility is tuning to a subset of a larger edo:

Figure 5.x.5 – A 12-note subset of 19-edo fretting using P1 m2 M2 m3 M3 P4 d5 P5 m6 M6 m7 M7 P8

E			F	F♯		G	G♯		A		B♭	B		C	C♯		D	D♯		E
B			C	C♯		D	D♯		E		F	F♯		G	G♯		A	A♯		B
G			A♭	A		B♭	B		C		D♭	D		E♭	E		F	F♯		G
D			E♭	E		F	F♯		G		A♭	A		B♭	B		C	C♯		D
A			B♭	B		C	C♯		D		E♭	E		F	F♯		G	G♯		A
E			F	F♯		G	G♯		A		B♭	B		C	C♯		D	D♯		E

For consistency, the dots on the guitar neck always correspond to the F, G, A, B and D notes of the D string, assuming fifth-based notation. The missing frets create missing octaves, because each string (except the two E strings) plays a different edo-subset. For example, it's almost impossible to play two G♯ notes an octave apart.

Is there any straight-fret alternative to edos that has no wolves or missing octaves? Yes, but only if the edo is multi-ring (ring = an edo's circle of fifths). Because 10-edo is a 2-ring edo, we can adjust every other fret and still have only 10 notes per octave total, if we adjust them all consistently. For example, adjust every other fret (the odd-numbered ones, obviously) sharp by 26¢ so that 3\10 = 5/4. This creates 5-edo+y. This is an **udo** ("OO-doh"), which stands for "unequal division of an octave".

Figure 5.x.6 – A 10-udo fretting, with 3\10 = 5/4 = 386¢ (5-edo+y or Blackwood)

E = 0¢		146¢	240¢	386¢	480¢	626¢	720¢	866¢	960¢	1106¢	0¢
B = 720¢		866¢	960¢	1106¢	0¢	146¢	240¢	386¢	480¢	626¢	720¢
G = 240¢		386¢	480¢	626¢	720¢	866¢	960¢	1106¢	0¢	146¢	240¢
D = 960¢		1106¢	0¢	146¢	240¢	386¢	480¢	626¢	720¢	866¢	960¢
A = 480¢		626¢	720¢	866¢	960¢	1106¢	0¢	146¢	240¢	386¢	480¢
E = 0¢		146¢	240¢	386¢	480¢	626¢	720¢	866¢	960¢	1106¢	0¢

What differentiates an udo from JI or rank-2, which are also unequal? An udo is edo-like. It has straight frets and no wolf octaves. No unisons are mistuned, and the number of notes per octave equals the number of frets per octave.

Each ring in a multi-ring edo makes a **sub-edo**, and an udo is simply a collection of sub-edos, at least one mistuned from the original edo. They may or may not form a subset of a larger edo, a **super-edo**. For example, raising every other fret by 40¢ makes a tuning that's a 10-note subset of 15-edo.

Figure 5.x.7 – A 2-ring 10-udo fretting, with a sub-edo of 5 and a super-edo of 15

E = 0¢		160¢	240¢	400¢	480¢	640¢	720¢	880¢	960¢	1120¢	0¢
B = 720¢		880¢	960¢	1120¢	0¢	160¢	240¢	400¢	480¢	640¢	720¢
G = 240¢		400¢	480¢	640¢	720¢	880¢	960¢	1120¢	0¢	160¢	240¢
D = 960¢		1120¢	0¢	160¢	240¢	400¢	480¢	640¢	720¢	880¢	960¢
A = 480¢		640¢	720¢	880¢	960¢	1120¢	0¢	160¢	240¢	400¢	480¢
E = 0¢		160¢	240¢	400¢	480¢	640¢	720¢	880¢	960¢	1120¢	0¢

We've assumed that the open strings are always tuned by 4ths and 3rds. However, a mandolin is tuned in 5ths, and a sitar is tuned in alternating 4ths and 5ths. An udo requires that the intervals between open string notes all lie on one ring, i.e. are all in the sub-edo. Otherwise adjusting frets adds more notes per octave. For example, with a 14-edo guitar tuned E A D G B E, moving every other fret keeps you at 14 notes per octave. But if tuned E A D G B^ E, then all the open notes don't lie on the same ring (7-edo doesn't contain them all), and C on the B^ string will be different than C on the G string.

12-edo is a 1-ring edo, with a single ring of fifths. But it has four rings of major thirds, with a sub-edo of 3-edo. If the open strings are tuned to an augmented chord, moving every 4th fret would create a 4-ring 12-udo. If the open strings are tuned to a dim7 chord (4-edo), moving every 3rd fret would create a 3-ring 12-udo. If the open strings are tuned to a whole tone scale (6-edo), moving every other fret would create a 2-ring 12-udo.

An edo's 2nd best fifth can be used to turn a 1-ring edo into a multi-ring edo. For example, 22-edo has one ring of its best fifth, 13\22 = 709¢. But 22b-edo has two rings of 12\22 = 655¢. The sub-edo is 11-edo, which approximates JI poorly. But that's a good thing, because we can tune the 2nd ring to bring some of the other 22-edo notes closer to JI. For example, we can flatten the 2nd ring by 7¢ to get a just 3/2:

Figure 5.x.8 – A 2-ring 22b-udo fretting, with a sub-edo of 11, with 13\22 = 3/2 = 702¢

E = **0¢**		47¢	**109¢**	157¢	**218¢**	266¢	**327¢**	375¢	**436¢**	484¢	**545¢**
B = **655¢**		702¢	**764¢**	811¢	**873¢**	920¢	**982¢**	1029¢	**1091¢**	1138¢	**0¢**
G = **436¢**		484¢	**545¢**	593¢	**655¢**	702¢	**764¢**	811¢	**873¢**	920¢	**982¢**
D = **1091¢**		1138¢	**0¢**	47¢	**109¢**	157¢	**218¢**	266¢	**327¢**	375¢	**436¢**
A = **545¢**		593¢	**655¢**	702¢	**764¢**	811¢	**873¢**	920¢	**982¢**	1029¢	**1091¢**
E = **0¢**		47¢	**109¢**	157¢	**218¢**	266¢	**327¢**	375¢	**436¢**	484¢	**545¢**

11-edo notes are bolded. The open strings are tuned in 11-edo 4ths and 3rds. With edos, "ring" is a somewhat vague concept, because 22-edo and 22b-edo sound exactly the same. But applied to udos, "ring" is much more rigorous.

There's no advantage to udos for keyboard tunings, because keyboards need never have wolf octaves. But in order to analyze an udo, it's handy to recreate it in alt-tuner. Here's how:

On the keyboard screen, set the # of keys to 22. On the rungs screen, set rung #2's keyspan to 12.
On the linkages screen, enter two commas, one of 11 5ths and one of 11 zo 7ths, locking these two rungs to 11-edo.
On the rows screen, lengthen the ruyo row by setting "to" to 3.
On the lattice screen, set up two rings by selecting any 11 wa or zo notes, and any 11 yo or ruyo notes.
On the graph screen, move the yo slider to explore this udo's possibilities. 400¢ = 33-edo, 371¢ = 55-edo.

Every rank-2 tuning with a period that's a fraction of an octave can be an udo. The sub-edo is the reciprocal of the octave fraction. For half-octave temperaments, the sub-edo is 2-edo, and the open strings must be tuned in 600¢ tritones. All but two frets per octave are adjusted. This of course requires that the original edo be an even number.

Figure 5.x.9 – A 10-udo rank-2 ssLss-ssLss fretting with a 600¢ period and a 705¢ generator (sggT or pajara)

A = 600¢		705¢	810¢	990¢	1095¢	0¢	105¢	210¢	390¢	495¢	600¢
D = 0¢		105¢	210¢	390¢	495¢	600¢	705¢	810¢	990¢	1095¢	0¢
A = 600¢		705¢	810¢	990¢	1095¢	0¢	105¢	210¢	390¢	495¢	600¢
D = 0¢		105¢	210¢	390¢	495¢	600¢	705¢	810¢	990¢	1095¢	0¢
A = 600¢		705¢	810¢	990¢	1095¢	0¢	105¢	210¢	390¢	495¢	600¢
D = 0¢		105¢	210¢	390¢	495¢	600¢	705¢	810¢	990¢	1095¢	0¢

For third-octave temperaments like gggT, the sub-edo is 3-edo, and the open strings must be tuned in 400¢ major 3rds. All but three frets per octave are adjusted.

Blackwood in the most general sense is a fifth-octave rank-2 temperament. Tune the open strings to 5-edo and adjust all but 5 frets per octave. Note that for 15-edo, adjusting the 1st, 4th, 7th, etc. frets creates an udo, but not a rank-2 tuning. For that the 2nd, 5th, 8th, etc. frets also must be adjusted.

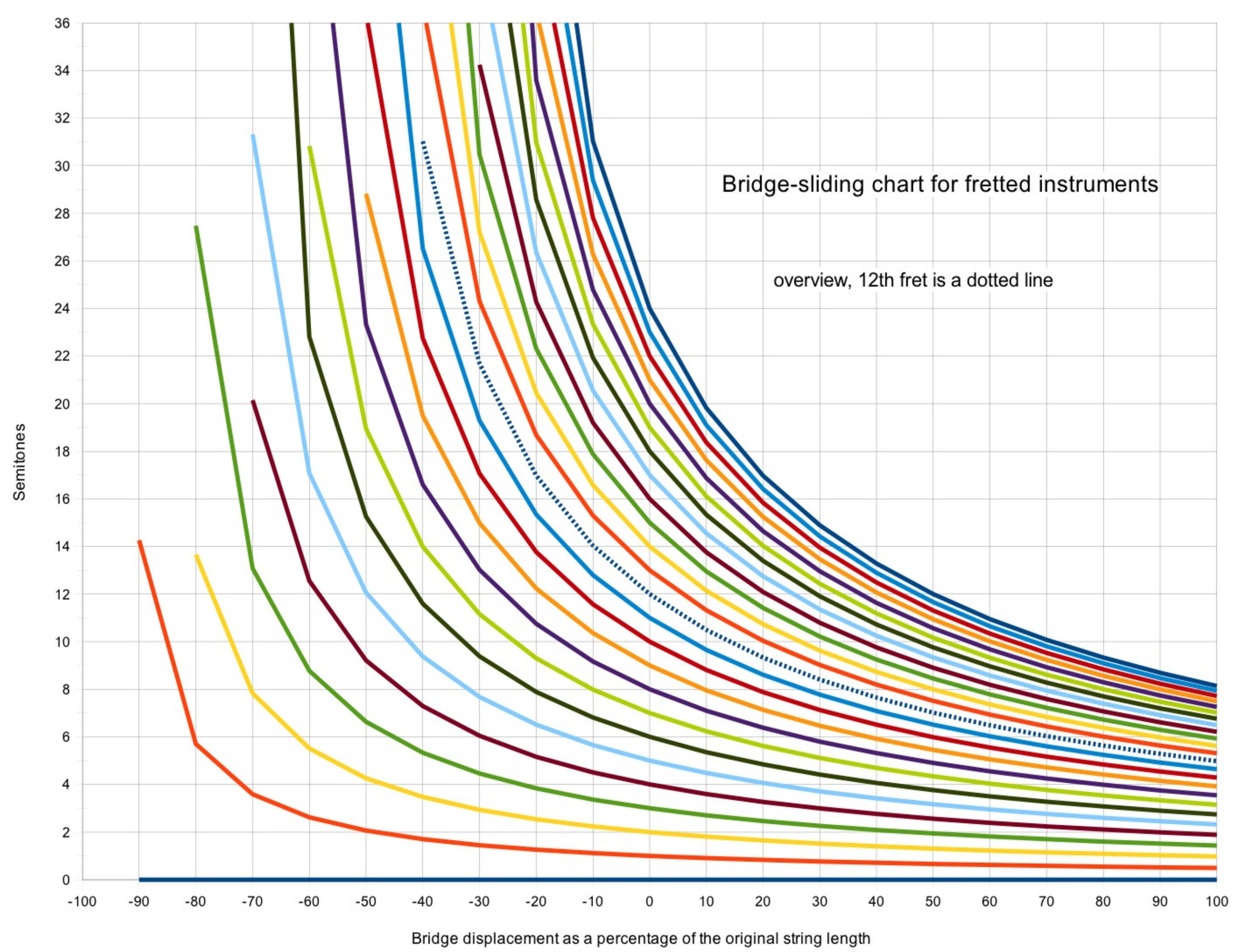
EDONOIs can be **UDONOIs** if you tune the open strings to the NOI or some multiple of it. For example, the Georgian tuning divides the 3/2 into 4 equal steps. Tune your mandolin's strings by just 5ths. There are four frets per 5th, and eight frets per 9th. Adjusting every 4th fret, or 3 out of every 4 frets, makes a 4-ring UDONOI. The sub-EDONOI is 1-ED(3/2). Adjusting every other fret makes a 2-ring UDONOI, with a sub-EDONOI of 2-ED(3/2).

There are no prime-number udos, because prime-number edos like 19-edo are never multi-ring. Thus while there are several ways to place 15 straight frets per octave to get only 15 notes (15-edo, 3-ring 15-udo, or 5-ring 15-udo), there's only one way to place 19 frets (19-edo). Except for the trivial case of tuning the strings in octaves, which makes every tuning an udo, with a sub-edo of 1-edo!

In Table 4.1.2, rank-2 MOS tunings are in Paul Erlich's words the "middle path" between JI and edos. Udos are also a middle path between JI and edos. But not all rank-2 tunings are udos, and not all udos are rank-2.

A quick easy way to get a new tuning on a guitar or banjo is to slide the bridge to lengthen or shorten the strings. This makes a strange scale that not only isn't an edo, it doesn't even repeat at the octave, or at any other interval! The next chart shows the scale produced as a function of how far you slide the bridge. The center line is zero displacement, the original tuning, assumed to be 12-ET, with a scale of 100¢, 200¢, etc.

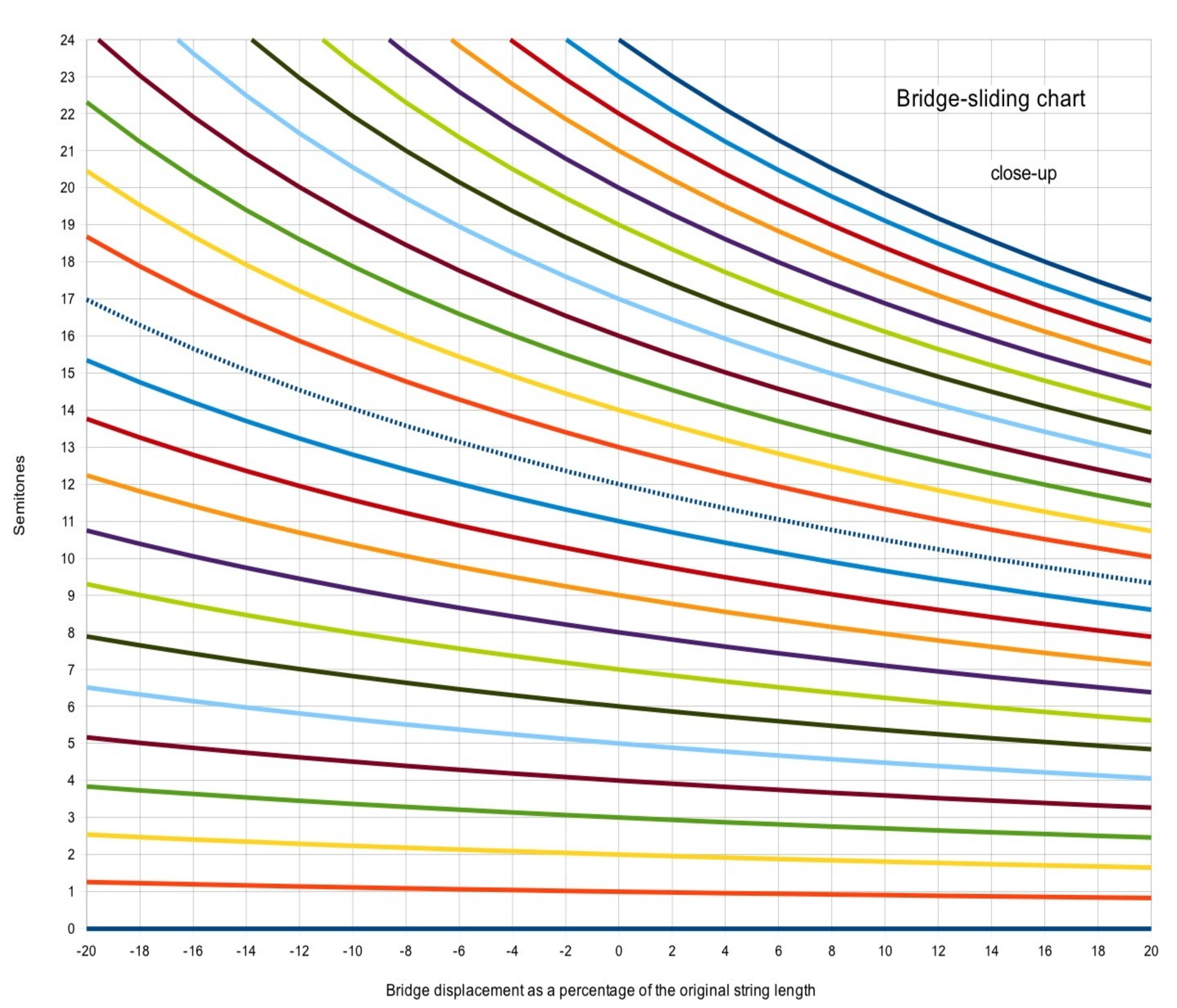
Figure 5.x.10 – Bridge sliding chart



The left-hand side of the graph corresponds to shortening the total string length by a certain percentage. -100% means moving the bridge over all the frets all the way to the nut, creating zero string length and infinitely high pitch. On the right, 100% means doubling the original string length. Doubling would of course drop the open-string tone by an octave, so we're assuming that the string is retuned/replaced after the bridge is moved.

The next chart is a close-up of the first one. Over on the far left, where the orange line crosses the 12-semitone grid line, there are 9 notes in the first octave, but only 7 in the next octave. The spacing between the colored lines is not constant. On the left, the gap is more than 100¢, and is even larger for the higher frets. On the right, it's less than 100¢, and is even smaller for the higher frets.

Figure 5.x.11 – Bridge sliding chart #2

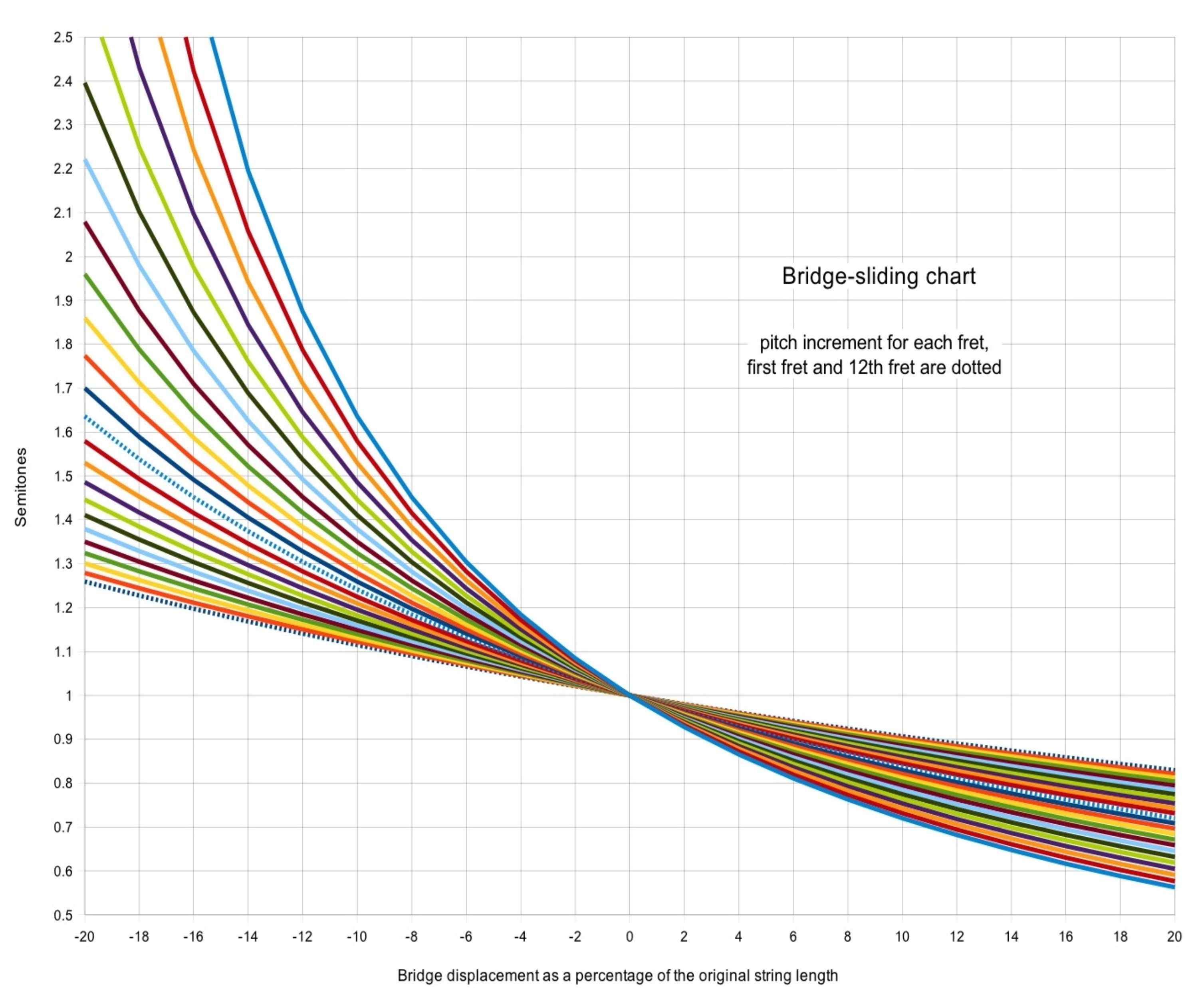


There are hardly any unisons or octaves from one string to the next. For example, if you tune the 3rd string to the 5th fret of the 4th string, the 3rd string's 1st fret and the 4th string's 6th fret will <u>not</u> be a unison. If you tune the 3rd string to be an octave below the 1st string's 2nd fret, the 3rd string's 1st fret will <u>not</u> be an octave below the 1st string's 3rd fret. In other words, moving any two notes up or down a fret changes the interval between them. A barre chord will not sound the same at different positions.

With an edo fretting, every interval is a multiple of the edostep. But when you slide the bridge, the number of intervals is much larger. The scale on one string has every step a different size. An example is the scale on the right-hand side of the graph above, with about 16 pitches in the first octave. Now imagine six copies of this scale, each one offset by about 4 or 5 semitones from the next. Over the guitar's whole range, only five pairs of pitches will coincide. Thus every fret has its own unique pitch, and the total number of pitches the guitar can produce is much greater. If playing up to the 16th fret (a three-octave range), there are about 96 different pitches. There are more in the middle range, the central octave has about 40.

The final chart looks at how much each fret increases the pitch. In 12-ET, all the frets are 100¢. Shortening the strings by 20% results in the 1st fret moving you up 126¢. But it only moves you up 84¢ for a lengthening of 20%.

Figure 5.x.12 – Bridge sliding chart #3



Chapter 5.20 – Smooth Voice Leading: 3-tone and 4-tone *

Much of edo theory applies to adaptive JI comma pumps as well. For example, suppose we're arranging 3-part JI vocal harmonies for Wimoweh in C. Easy: one voice goes C-C-C-B, above it is E-F-E-D, below is G-A-G-G, and we're done. The chords are G-C-E to A-C-F to G-C-E to G-B-E back to G-C-E, and so on.

The three voices could have been G-F-G-G and C-A-C-B and E-C-E-D, but we don't like the way the high voice jumps down to the note the middle voice was on. We want smooth voice leading, where the melody moves as little as possible from chord to chord, and if two chords share a note, the same voice must sing that note for both chords.

Next let's arrange 3-part harmonies for a song that goes C – Am – Dm – G – C. Tuned Cy – yAg – y=wDg – Gy – Cy, it's the classic g1 = 81/80 comma pump. This might be sung in adaptive JI or meantone or 12-edo or 19-edo, doesn't matter what as long as the tonic doesn't drift. Can we get smooth voice leading? Let's try: the chords are G-C-E to A-C-E to A-D-F to B-D-G to C-E-G. All three voices moved up by a 3rd or more. That might be OK if we only sing the chords once, but to sing them repeatedly, we must use awkward jumps in the melodies, perhaps G-A-A-G-G and C-C-D-B-C and E-E-F-D-E.

Why is this so? Because 3-edo doesn't temper out g1. We aren't singing in 3-edo, that would be ridiculous! But we can use 3-edo theory to analyze the 3-tone **voicing framework** that 3-part harmonies imply. Musicians are unconsciously thinking in 3-tone when they talk of "moving a harmony up a part", meaning moving it up a 3rd or a 4th.

Table 5.19.1 – 3-edo's obvious mapping of yaza JI

	rungs	ratios
0	w1	w1, g2, y2, z3, w3
1	y3	w2, r2, g3, y3, z4, w4, zg5, z6
2	w5, z7	r3, ry4, w5, r5, g6, y6, z7, w7
3	w8	w6, r6, g7, y7, w8

3-edo maps g1 to 1\3, one 3-EDOstep. But since we pumped g1 downwards, shouldn't smooth voice leading make the voices move down? No, because what's getting pumped isn't the pitch but the voice. The C note gets pumped downward one 3-EDOstep, from the middle voice down to the low voice. The pitch hasn't drifted, so the final C chord has the same C note, but now in the low voice, and the middle voice ends up on the E. So a downward pump moves the voices up!

An important corollary: <u>If a chord progression doesn't pump a comma, smooth voice leading is always possible</u>.

The C – Am – Dm – G – C progression could be tuned za. Using r3 = 9/7 and z3 = 7/6 for the chord's 3rds pumps r1 = 64/63 (Cr – rAz – rDz – rGr – rCr), which 3-edo does temper out. Wouldn't that mean the voices don't move? No, because 3-edo's obvious mapping puts r3 at 2\3, not 1\3. But for 3-edo theory to work, all triads must map to 3 different EDOsteps, i.e. 0–1–2. So we must map z7 = 7/4 to 3\3, not 2\3. This makes r3 and z3 both map to 1\3, and r1 map to -1\3. Since we're pumping r1 upwards, the notes are mapped down a part, and the voices move up a part.

Table 5.19.2 – 3-edo's non-obvious mapping of yaza JI (7/4 = 3\3)

	rungs	ratios
0	w1	w1, g2, y2, r2, w3
1	y3	w2, z3, g3, y3, r3, w4, ry4, r5
2	w5	z4, zg5, w5, z6, g6, y6, r6, w7
3	z7, w8	w6, z7, g7, y7, w8

Alternatively, we could leave z7 mapped to 2\3, avoid ru and zo chords, and instead use h7no5 chords (4:5:7), which map to 0-1-2. Now we can pump r1 with smooth voice leading: D – A7no5 – G – D can be tuned Dy – Ah7no5 – zGy – zDy, which makes A-D-F♯ to A-C♯-G to B-D-G to A-D-F♯.

Since w2 = 9/8 and w4 = 4/3 both map to 1\3, the theory works with sus2 and sus4 chords too. Csus4 – B♭sus4 – Cm makes G-C-F to B♭-E♭-F to C-E♭-G. As before, we pumped g1 down, making the voices move up.

Let's try pumping a comma that 3-edo does temper out, $g^3 2$ = 128/125. C – Em – E – G♯m – G♯=A♭– Cm – C. The chords are C-E-G to B-E-G to B-E-G♯ to B-D♯-G♯ to B♯-D♯-G♯=C-E♭-A♭ to C-E♭-G to C-E-G. No problem!

3-edo's nearest edo-mapping is (3, 5, 7). The dot product of any ratio with this mapping gives us the 3-edo-steps. The triple yo comma is $y^3 1$ = 250/243 = (1, -5, 3). The dot product is 3 - 25 + 21 = -1. Because it isn't zero, a $y^3 1$ pump will require awkward jumps. The quintyo comma is y^5-2 = 3125/3072 = (-10, -1, 5). The dot product is zero, and a y^5-2 pump won't need jumps.

What if we add a 4th part? Perhaps a bass line, for which we don't mind jumps nearly as much. As long as the chords are the same, i.e. still triads, voice leading issues are unchanged. So it's not really about 3-part harmony, but about <u>triadic</u> harmony.

For 4-part tetradic harmony, use 4-edo theory. Jazzy Wimoweh goes C7 – F7 – C7 – G7. We get G-B♭-C-E to F-A-C-E♭ to G-B♭-C-E to G-B-D-F to G-B♭-C-E. The voices move as smoothly as possible. The 3rd voice goes C-C-C-D-C, a bigger step than C-C-C-B-C, but going to B would cause even bigger jumps elsewhere.

Let's pump g1 with C7 – Am7 – Dm7 – G7 – C7. G-B♭-C-E to G-A-C-E to F-A-C-D to F-G-B-D to E-G-B♭-C. We moved down, not up, because 4-edo maps g1 to -1/4, and 3-edo maps it to 1/3.

Let's try g1 with different chords: C6 – A7 – D7 – GM7 – C6. G-A-C-E to G-A-C♯-E to F♯-A-C-D to F♯-G-B-D to E-G-A-C. The melodies changed, but because there are still common notes between all the chords, a smooth voice leading inevitably makes the voices move downwards.

The standard jazz progression CM7 – Dm7 – G7 – CM7 makes C-E-G-B to C-D-F-A to B-D-F-G to B-C-E-G, no good. What comma is this? The only root movement not by a fifth is in the chord change from CM7 to Dm7, linked by the C note. So it all depends on the tuning of Dm7's 7th. For g7 = 9/5, it's g1 descending = 1\4. For z7, it's r1 ascending = 1\4. Both map the notes upwards and the voices downwards.

But for w7 = 16/9, there's no comma to pump. Why did this happen? Because our tetrads must map to 0–1–2–3 in 4-edo for the theory to work. If 3/2 is 2\4, w7 must be 4\4. So we can <u>sing</u> chords with w7, but we can't use w7 in our JI theory to predict anything about voice leading. But because w7 is so close to g7 and z7, we know that the voices will move downwards.

Table 5.19.3 – 4-edo's obvious mapping of yaza JI

	rungs	ratios
0	w1	w1, w2
1	y3	g2, y2, r2, z3, g3, y3, r3, z4
2	w5	w3, w4, zg5, ry4, w5, w6
3	z7	r5, z6, g6, y6, r6, z7, g7, y7
4	w8	w7, w8

w2 maps to 0\4, and w4 maps to 2\4, so sus chords won't work. Unless thought of as using r2 and z4. Or y2 = 10/9 and g4 = 27/20. All four ratios map to 1\4.

Can we still pump r1 smoothly like the triadic example in D? D7 – A7 – G7 – D7 becomes Dh7 – Ah7 – zGh7 – zDh7 makes A-C-D-F♯ to A-C♯-E-G to B-D-F-G to C-D-F♯-A. No, because adding the 4th voice messed up the voice leading.

So pumping 81/80 or 64/63 prevents smooth voice leading in tetradic harmony. What comma pumps don't?

zz2 49/48

50/49 tritone substitution. Ch7 – zgG♭=ryF♯h7 – Ch7

A round that pumps a comma:

C a d G

G A A B

C C D D

E E F F

G G

G G F F

E E D D

C C B A

A G G

(Part V is unfinished.)

Glossary/Index

Terms from part I are conventional music theory terms; all others, unless otherwise noted, are original to the author

12-ET (ch. 1.2) – equal temperament, our standard tuning system

absolute notation (ch. 2.3) – notation of specific notes (e.g. G♯, yE, Am7 or gGy6) (a conventional music theory term)

all-odd (ch. 2.7) – two-less; the all-odd voicing of a chord is often its most consonant voicing

ambiguous (ch. 3.4) – refers to qualities like neutral or half-aug with an ambiguous keyspan

azul, azure (ch. 2.1) – 7-under, see zo

bicolored (ch. 2.4) – refers to chords, chord progressions, scales or melodies that use only two colors (including wa)

ca, clear (ch. 2.1) – refers to 2-limit ratios such as 1/1, 2/1, 4/1, etc., and to the clear rung

central (ch. 2.2, ch. 3.2) – not large or small; within 3 steps of the midpoint on a row

cents (ch. 1.2) – one cent is one hundredth of an equal-tempered semitone

child (ch. 4.8) – a multi-comma temperament is a child of each temperament implied by each comma, see family

" (ch 5.7) – refers to a node on the scale tree

class (ch. 3.1) – an approximate measure of remoteness, or dissonance, for intervals, chords, progressions and scales

clear, ca (ch. 2.1) – refers to 2-limit ratios such as 1/1, 2/1, 4/1, etc., and to the clear rung

color (ch. 2.1) – a label indicating the presence or absence of prime factors other than 2 or 3 in an interval's ratio

color depth (ch. 4.6) – the GCD of all the exponents of the comma's monzo, except the first two

color signature (ch. 2.6) – a tuning that assigns a default color to each of the twelve notes

comma (ch. 1.3) – a small difference in pitch between two JI intervals, often around 20-35¢

comma pump (ch. 4.2) – a chord progression that produces tonic drift if all chords are untempered

comma shift (ch 4.2) – a pitch shift of a full comma

comma warp (ch.4.2) – refers to two chords that warp an interval by a comma, e.g. Cy,9 – Fy6 warps the C–D interval

compound color (ch. 2.2) – a color whose ratios contain 2 or more primes besides 2 and 3, e.g. zogu

da-re-mu (ch. 2.3) – color solfege, an extension of solfege (movable or fixed) that incorporates color names

deep (ch. 4.6) – having a color depth of more than 1, such a comma splits either the period or the generator

degree (ch. 1.2) – the position of an interval in the scale, for example 3rd, 5th, etc.

detempered (ch 4.7) – midway between tempered and just, done in alt-tuner by setting the strength slider to < 100%

diatonic (ch 5.7) – refers to EDOs that have a fifth between 686¢ = 4\7 and 720¢ = 3\5

diminished unison (ch. 3.3) – a unison that diminishes the quality but raises the pitch

DOL (ch. 4.8) – see double odd limit

double (color) (ch. 2.2) – a color in which some prime number higher than 3 has an exponent of 2 or -2

double large/double small (ch. 3.2) – the central interval is increased/decreased by AA1 = $(2187/2048)^2$ = 227¢

double octave (ch. 2.7) – a 15th, usually but not always a wa 15th (a conventional music theory term)

double odd limit (ch. 3.2, 4.8) – the combined odd limit of both of the ratio's numbers, largest first; DOL (6/5) = (5, 3)

doubled (ch. 3.2) – see "squared"

down (ch. 5.5) – a symbol "v" used to lower a note or shrink an interval by one edo-step or mapping comma

drift (ch. 4.2) – see "tonic drift"

edo (ch. 1.2 and 4.1) – a tuning based on an equal division of an octave

EDONOI (ch. 4.1 and 4.8) – a tuning based on an equal division of a non-octave interval

edostep (ch. 4.1) – the step between notes in an edo, for example a 10-edo-step is 120¢

edomapping (ch. 4.1) – maps each prime number to an edo-step, e.g. 12edo's 8ve-reduced edomapping is (12, 7, 4, 10)

extra-augmented (ch. 3.4) – halfway between aug and double-aug, refers to intervals fifthward of a half-aug interval

extra-diminished (ch. 3.4) – halfway between dim and double-dim, refers to intervals fourthward of a half-dim interval

family (ch. 4.8) – adding any comma(s) to a temperament creates a family of temperaments, see child

fifthward, 5thwd (ch. 2.1) – rightward on the harmonic lattice, in the dominant direction

fourthoid, fifthoid (ch 5.3) – scale degrees in the pentatonic framework

fourthward, 4thwd (ch. 2.1) – leftward on the harmonic lattice, in the subdominant direction

fractional generator (ch. 4.6) – refers to a temperament's generator being some fraction of a wa interval

fractional period (ch. 4.6) – refers to a temperament's period being not an octave but some fraction of it
framework (ch. 5.1) – heptatonic, pentatonic, chromatic, 10-tone, 19-tone, etc.

generator (ch. 1.2, ch. 4.6) – an interval that can generate all the other intervals, see also "period"
genchain, generator-chain (ch. 4.8) – a chain of generators, and the scale formed by the chain when octave-reduced
gu, green (ch. 2.1) – 5-under (minor), having one 5 factor in the ratio's denominator, e.g. 6/5 = g3 = gu 3rd

half-augmented (ch. 3.4) – halfway between perfect and augmented, refers to intervals fifthward of a neutral interval
half-diminished (ch. 3.4) – halfway between perfect and diminished, refers to intervals fourthward of a neutral interval
harmonic lattice (ch. 1.3) – a set of rungs, see chapter 1.3
harmonic series (ch. 1.2) – the naturally occurring overtones in string and wind instruments
homonym (ch. 2.4) – refers to two chords with the same notes, e.g. Am7 and C6 (a conventional music theory term)

i- (ch. 3.6) – a prefix used for disambiguation, see ila, ilo, iso, ino and inu
ICC (ch. 4.2) – see "innate comma chord"
ila (ch. 3.6) – alternate form of la, to avoid confusion of "la" and the solfege syllable "La"
ilo (ch. 3.6) – alternate form of lo, to avoid confusion of "lo C" and "low C"
iso (ch. 3.7) – alternate form of so, to avoid confusion of "so" and the solfege syllable "So"
ino (ch. 3.7) – alternate form of no, to avoid confusion of "no 3rd" meaning either "19o 3rd" or "omit 3rd"
inu (ch. 3.7) – alternate form of nu, to avoid confusion of "the nu key" and "the new key"
innate comma chord (ch. 2.4) – a chord that inevitably contains a wolf interval when tuned justly, e.g. a 6/9 chord
integer-limit (ch. 1.2) – refers to the larger of the two numbers in a ratio, whether odd or even
interval of equivalence (ch. 1.2 and 4.6) – the period, or some multiple of it

JI (ch. 1.2) – just intonation

kite (ch. 5.7) – a kite-shaped region of the scale tree
keyspan (ch. 2.1) – the width of an interval in semitones; a perfect 4th has a keyspan of 5

la, ila (ch. 3.6) – 11-all, undecimal, the 2.3.11 prime subgroup, including lolo, triple lu, etc.
lavender (ch. 3.6) – the pseudocolor that equates ilo and lu, e.g. 11/9 and 27/22 = lavender 3rd
large (ch. 1.4, ch. 3.2) – the central interval is increased by Lw1 = 2187/2048 = seven 5ths minus 4 octaves = 114¢
limit (ch. 1.2) – see prime limit, odd limit
lo, ilo, lovender (ch. 3.6) – refers to ratios with an 11 over, written 1o (number 1 letter o), e.g. 11/8 = 1o4 = ilo 4th
lu, luvender (ch. 3.6) – refers to ratios with an 11 under, written 1u (number 1 letter u), e.g. 16/11 = 1u5 = lu 5th

magnitude (ch. 2.3) – large vs. small vs. central. For an interval's width in cents, see size
magnitude-chain (ch. 3.2) – a series of magnitudes on a row, usually 7ss – 7s – 7 central – 7L – 7LL
microcomma (ch. 3.2) – any comma less than 1¢
mid (ch. 5.7) – a relative quality midway between major and minor, written "~"
midpoint (ch. 3.2) – any ratio = $2^a\ 3^b\ 5^c\ 7^d$..., such that b + c + d ... = 0; used to define magnitude
minicomma (ch. 2.2) – any comma less than 10¢ and greater than 1¢
minisharp/miniflat (ch. 2.2) – raised/lowered by a minicomma
monzo – (ch. 1.2) – a JI ratio expressed as a list of prime exponents, e.g. (-2, 0, 1) is the monzo of 5/4
MOS, mossy, moment of symmetry (ch. 4.1) – refers to scales with only two types of intervals
MODMOS (ch. 4.1) – a scale derived from a MOS scale by raising or lowering one or more notes by L - s

na (ch. 3.7) – 19-all, the 2.3.19 prime subgroup, including nono, triple nu, etc.
" (ch. 3.7) – when after twenty, fifty, seventy, etc., nine-all, e.g. twenty-na = twenty-nine-all = 29a = 2.3.29
near (ch. 3.1) – being few steps away on the harmonic lattice, the opposite of remote
nearest edomapping (ch. 4.1) – The edo-mapping that most closely approximates JI
negative (ch. 2.2, ch. 3.3) – refers to a ratio that takes you down the scale to a higher pitch
neutral (ch. 3.4) – halfway between major and minor (a conventional music theory term)
no, ino (ch. 3.7) – refers to ratios with a 19 factor over, e.g. 19/15 = 19og4 = nogu 4th, 19/16 = 19o3 = ino 3rd
" (ch. 3.7) – when after twenty, fifty, etc., nine-over, e.g. twenty-no = twenty-nine-over = 29o
nowa, noca, nowaca (ch. 4.8) – refers to prime subgroups or temperaments that exclude clear and/or wa
-note (ch. 5.1) – steps per octave, refers to naming frameworks such as 7-note = diatonic, 5-note = pentatonic, etc.
noya, noza (ch. 3.6) – refers to prime subgroups that omit 5 or 7, a descriptive term not used in actual subgroup names

nu, inu (ch. 3.7) – refers to ratios with a 19 under, e.g. 21/19 = 19uz2 = nuzo 2nd, 24/19 = 19u3 = nu 3rd
" (ch. 3.7) – when after twenty, fifty, etc., nine-under, e.g. twenty-nu = twenty-nine-under = 29u

octave fraction (ch. 4.1) – an interval expressed as a fraction of an octave, using a backslash, e.g. 3\5 = 720¢
odd limit (ch. 1.2) – refers to the largest number in a ratio after factoring out all the twos
odd name, odd-limit name (ch. 4.8) – one of two possible names for a multi-comma temperament
-oid (ch. 5.1) – a suffix used for scale degrees in the pentatonic framework
otonal, otonality (ch. 1.2) – loosely speaking, refers to a chord in which the primes > 2 are on the tops of the ratios
over, overness (ch. 1.2) – analogous to otonal, refers to an interval for which the monzo's final number is positive

partial pump (ch. 4.2) – a comma pump that stops just short of returning to the original chord
pentatonic (ch 5.7) – besides the conventional meaning, refers to EDOs that have a fifth of 720¢ = 3\5
perchain, period-chain (ch. 4.8) – a chain of periods, especially non-octave periods
perfect (ch 5.7) – besides the conventional meaning, refers to EDOs that have a fifth of 686¢ = 4\7
period (ch. 1.2, ch. 4.6) – an interval within which the scale periodically repeats, see also "interval of equivalence"
pitch class (ch. 1.3) – the collection of notes separated by octaves that share the same name
pitch shift (ch. 4.2) – a small adjustment of pitch, usually by a comma or a fraction of a comma, see "comma shift"
plain (ch 5.5) – referring to notes or scale degrees, neither up nor down
plane (ch. 3.5) – in the 7-limit harmonic lattice, a set of rows with the same ya or za content
plus (ch. 4.8) – in temperament names, used to include an untempered rung in the temperament, e.g. g+zT
po (ch. 2.6) – an accidental that raises by a wa comma = (-19, 12), written p
positive (ch. 3.3) – not negative; the vast majority of ratios are positive intervals
primary color (ch. 2.2) – a color whose ratios contain at most one prime besides 2 and 3
prime limit (ch. 1.2) – refers to which primes are used in a ratio, more generally, which primes are deemed consonant
prime name, prime-limit name (ch. 4.8) – one of two possible names for a multi-comma temperament
pseudocolor (ch. 3.4) – a color equating two real colors only a mini- or microcomma apart, e.g. purple or lavender
purple (ch. 3.4) – the pseudocolor that equates zozogu and ruruyo, e.g. 49/40 or 60/49 = purple 3rd
purple quartertone (ch. 3.4) – an alternate name for the large purple unison = Lp1 = 57¢
pythagorean (ch. 1.3) – 3-limit, wa

qu (ch. 2.6) – an accidental that lowers by a wa comma = (-19, 12), written q, pronounced "ku"
quad (ch. 3.2) – an abbreviation for quadruple, as in quadgu comma = $g^4 2$ = 648/625
quadricolored (ch. 2.4) – refers to chords, chord progressions, scales or melodies that use four colors (including wa)
quality (ch. 1.2) – major, minor, perfect, augmented, diminished, etc.
quality-chain (ch. 3.2) – a series of qualities on a row
quartertone (ch. 3.4) – half of a semitone (a conventional music theory term)
quint (ch. 3.2) – an abbreviation for quintuple, as in small quintgu comma = $sg^5 3$

ratio (ch. 1.2) – a just musical interval expressed as a ratio of frequencies
reduced monzo (ch. 2.2) – a JI ratio expressed as the sum of octave-reduced rungs, e.g. {0, 0, 1} is the monzo of 5/4
relative notation (ch. 2.3) – notation of intervals or chords, e.g. m3, y2, Im7 or Vy6 (a conventional music theory term)
remote (ch. 2.2, ch. 3.1) – being many steps away on the harmonic lattice, with steps using larger primes being bigger
row (ch. 1.3) – a chain of fifths in the harmonic lattice; each row has its own unique color
ru, red (ch. 2.1) – 7-under (supermajor), having one 7 factor in the ratio's denominator, e.g. 8/7 = r2 = ru 2nd
rung (ch. 1.3) – a step in a given direction in the harmonic lattice; always the same interval

sa (ch. 3.7) – spoken form of 17a and 17-all, refers to the 2.3.17 prime subgroup
" (ch. 3.7) – when after thirty, forty, fifty, etc., seven-all, e.g. thirty-sa = thirty-seven-all = 37a = 2.3.37
scale tree (ch. 5.7) – a special form of the Stern-Brocot tree, used to compare frameworks
septimal (ch. 1.2) – having 7 as a prime factor
sharpness, sharp-N (ch. 5.7) – refers to how many keys or frets a sharp symbol represents in a given sizing framework
shift (ch. 4.2) – see "pitch shift" and "comma shift"
sign (ch. 3.3) – refers to whether a ratio is positive or negative
size (ch. 1.2, ch. 2.3) – an interval's width in cents. For large vs. small vs. central, see magnitude
small (ch. 2.2, ch. 3.2) – the central interval is decreased by Lw1 = 2187/2048 = seven 5ths minus 4 octaves = 114¢
so, iso (ch. 3.7) – spoken form of 17o, refers to ratios with a 17 factor in the numerator, e.g. 17/16 = 17o2 = iso 2nd

" (ch. 3.7) – when after thirty, forty, fifty, etc., seven-over, e.g. thirty-so = thirty-seven-over = 37o
split (ch. 4.6) – to create a fractional period or generator, e.g. the yoyo comma splits the wa fifth in half
squared (ch. 3.2) – refers to ratios that are an exact square of another ratio, e.g. the squared gu comma = $(81/80)^2$
stepspan (ch. 5.1) – describes how many steps a degree spans, useful in alternative frameworks
su (ch. 3.7) – spoken form of 17u, refers to ratios with a 17 factor in the denominator, e.g. 24/17 = 17u4 = su 4th
" (ch. 3.7) – when after thirty, forty, fifty, etc., seven-under, e.g. thirty-su = thirty-seven-under = 37u
sub- (ch. 5.1, ch. 5.2) – a prefix used in other frameworks like pentatonic and 10-tone
subgroup (ch. 1.2) – a list of the primes used in a JI tuning; the 2.3.7.11 subgroup excludes prime 5
subharmonic series (ch. 2.4) – the harmonic series inverted, as in C5, C4, F3, C3, gA♭2, F2, rD2, C2...
subminor (ch. 1.3) – flatter than minor, but sharper than diminished
subthird, subseventh (ch. 5.3) – scale degrees in the pentatonic framework
superflat (ch 5.7) – refers to EDOs that have a fifth narrower than 686¢ = 4\7
supermajor (ch. 1.3) – sharper than major, but flatter than augmented
supersharp (ch 5.7) – refers to EDOs that have a fifth wider than 720¢ = 3\5
system (ch. 5.1) – a combination of a naming framework and a sizing framework

temperament (ch. 1.2) – a tuning which approximates JI by tempering out one or more commas
tha (ch. 3.6) – 13-all, tridecimal, the 2.3.13 prime subgroup, including thotho, triple thu, etc.
" (ch. 3.7) – when after twenty, forty, fifty, etc., -three-all, e.g. twenty-tha = twenty-three-all = 23a = 2.3.23
tho (ch. 3.6) – 13-over, refers to ratios with a 13 factor in the numerator, written 3o, e.g. 13/8 = 3o6 = tho 6th
" (ch. 3.7) – when after twenty, forty, fifty, etc., -three-over, e.g. twenty-tho = twenty-three-over = 23o
thu (ch. 3.6) – 13-under, refers to ratios with a 13 factor in the denominator, written 3u, e.g. 16/13 = 3u3 = thu 3rd
" (ch. 3.7) – when after twenty, forty, fifty, etc., -three-under, e.g. twenty-thu = twenty-three-under = 23u
three-less (ch. 4.6) – refers to a ratio $2^a\ 3^b\ 5^c\ 7^d$... such that b is zero, i.e., no wa rungs
-tone (ch. 5.1) – keys per octave, refers to sizing frameworks such as 12-tone, 19-tone, etc.
tonic drift (ch. 4.2) – the drifting of a song sharp or flat by a comma, one possible effect of a comma pump
tricolored (ch. 2.4) – refers to chords, chord progressions, scales or melodies that use three colors (including wa)
triple (color) (ch. 3.1) – a color in which some prime number higher than 3 has an exponent of 3 or -3
triple large/triple small (ch. 3.1) – the central interval is increased/decreased by $L^3w1 = (2187/2048)^3$ = 341¢
triple warp (ch. 4.2) – refers to two chords that warp three intervals by a comma, e.g. Ch9 – Dh9
triple wide (ch. 2.7) – refers to an interval that has been widened by 3 wa octaves
tweak (ch. 4.8) – a letter added to the edo name indicating that a rung is not the nearest edo-mapping, e.g., 12e-edo
two-less (ch. 2.7) – refers to a ratio $2^a\ 3^b\ 5^c\ 7^d$... such that a is zero, i.e., both numbers in the ratio are odd numbers

under, underness (ch. 1.2) – analogous to utonal, refers to an interval for which the monzo's final number is negative
up (ch. 5.5) – a symbol ^ used to raise a note or widen an interval by one edo-step or mapping comma
upside-down (ch. 3.2) – refers to an ascending interval with a negative keyspan
utonal, utonality (ch. 1.2) – loosely speaking, refers to a chord in which the primes > 2 are on the bottoms of the ratios

wa, white (ch. 2.1) – 3-limit, pythagorean, e.g. 4/3 = w4 = wa 4th = white 4th
" (ch. 3.7) – when after thirty, forty, sixty, etc., one-all, e.g. thirty-wa = thirty-one-all = 31a
wide (ch. 2.7) – refers to an interval that has been widened by a wa octave, e.g. a 10th is a wide 3rd
wo (ch. 3.7) – when after thirty, forty, sixty, etc., one-over, e.g. thirty-wo = thirty-one-over = 31o
" (app. 2) – fifthward wa, e.g. 3/2, 9/8, 27/16, etc.
wolf (ch. 2.3) – an interval considered mistuned or unplayable (a conventional music term)
wolf chord (ch. 2.4) – a chord containing a wolf interval
wu (ch. 3.7) – when after thirty, forty, sixty, etc., one-under, e.g. thirty-wu for thirty-one-under = 31u
" (app. 2) – fourthward wa, e.g. 4/3, 16/9, 32/27, etc.

ya (ch. 2.1) – 5-all, 5-limit, the 2.3.5 prime subgroup, including yoyo, triple gu, etc.
yaza (ch. 2.1) – 5-all and 7-all, 7-limit, the 2.3.5.7 prime subgroup, including zogu, ruyoyo, etc.
yo, yellow (ch. 2.1) – 5-over (major), having one 5 factor in the ratio's numerator, e.g. 5/4 = y3 = yo 3rd

za (ch. 2.1) – 7-all, septimal, the 2.3.7 prime subgroup, including zozo, triple ru, etc.
zo, azure/blue (ch. 2.1) – 7-over (subminor), having one 7 factor in the ratio's numerator, e.g. 7/6 = z3 = zo 3rd

Appendix 1 – A Guide to Shorthand Notation

Format: optional elements are in brackets. Just as notes are assumed to be natural, intervals are assumed to be central, and perfect intervals are assumed to be wa.

<u>Absolute format for notes</u>: color note-name accidental, e.g. yo G-sharp = yG♯

<u>Relative format for intervals</u>: magnitude color [quality] scale-degree, e.g. large wa [major] 3rd = Lw3 or LwM3

In chord names, alterations are always enclosed in parentheses, and additions never are. If the chord root is wa, the color can be omitted, e.g. Cy is wCy, and Ig is wIg. However, imperfect scale degrees require colors: wIIIy not IIIy.

<u>Absolute format for chords</u>: root-color note-name accidental third-color [degree] [,additions] [(alterations)] [omissions] e.g. yAg or zE♭y,z7 or Cg7(zg5)zg9

<u>Relative format for chords</u>: magnitude root-color scale-degree third-color [degree] [,additions] [(alterations)] [omissions] e.g. yVIg or LwIIIy,z7 or Ig7(zg5)zg9

1-7 – diatonic scale degree, e.g. y3, w5
1o – lo, ilo, lovender, 11-over
1u – lu, luvender, 11-under
3o – tho, 13-over
3u – thu, 13-under
8-15 – used for larger intervals or larger frameworks
9, 11, 13 – used in chord names, e.g. Cm9, D11, V13 (a conventional music theory term)
17o, 17u, 17a, 19o, 19u, 19a, etc. – 17-over/under/all, 19-over/under/all, etc.
4d, 5d, 8d, 11d, 12d, etc. – "oid" scale degrees (pentatonic)

♯, ♭, ♮ – sharp, flat, natural
+ (plus sign) – used in temperament names (ch. 4.8) and system names (ch. 5.1)
- (minus sign) – used for negative intervals (ch 3.3) and temperament names (ch 4.8)
[ ] (brackets) – used to name generator chains (ch. 4.8), e.g. meantone [7] or gu [7]
also used to indicate comma pump equivalences, e.g. [wD♭=yC♯]h7
() (parentheses) – used to notate a JI ratio as a monzo
{} (curly brackets) – used to notate a JI ratio as a reduced monzo, also used for a set of generators
^ (caret) – up, an accidental that increases the keyspan by one, the opposite of "v"
~ (tilde) – mid, a quality midway between major and minor
/ (forward slash) – used in frequency ratios, e.g. 5/4 = y3, also for slash chords (C/B) and add chords (C6/9)
also used as a lift in pergen notation
\ (backslash) – used in octave fractions, e.g. 3\5 = 720¢, also used as a drop in pergen notation

A – when before a scale degree, augmented; otherwise the note A
a – all, i.e. both over and under, as in wa (2.3), ya (2.3.5), or 17a (2.3.17)

B – the note B
b – ♭

C – the note C
c – ca, clear, 2-limit

D – the note D
d – when before a scale degree, diminished; when after a pentatonic scale degree, "-oid"

E – the note E

F – the note F

G – the note G

g – gu, green, 5-under

H – used in German and other languages for the note B
h – harmonic-series chord, e.g. Ch7
hA – half-augmented
hd – half-diminished, not to be confused with the half-diminished tetrad

I – roman numeral 1, used in relative chord notation (e.g. I – IV – v)

i- – a disambiguation prefix

L – large

M – major
m – minor

n – neutral

o – over, as in yo, 17o or 19oo

P – perfect
p – po, raises by a wa comma, also purple, used informally in relative notation only

q – qu, lowers by a wa comma

r – ru, red, 7-under

s – when before a color name, small; when before a pentatonic scale degree, sub-; when in a chord, subharmonic

T – tempered, as in Tw5 = 700¢

u – under, as in gu, 17u or 19uu

V – roman numeral 5, used in relative chord notation (e.g. I – IV – V)
v – down, an accidental that lessens the keyspan by one, the opposite of ^

W – wide
w – wa, white, 3-limit

X – roman numeral 10, possibly used in relative chord notation in larger naming frameworks (e.g. I – IX – V)
x – double-sharp, when after a note name
xA – extra-augmented
xd – extra-diminished

y – yo, yellow, 5-over

z – zo, azure, 7-over

unused letters: J, K, N, O, Q, R, S, U, Y, Z, e, f, j, k, t

Appendix 2 – Possible Extensions to the Notation

The -o and -u suffixes could be applied to wa to make **wo** = 3-over = fifthward wa and **wu** = 3-under = fourthward wa. The terms 3o and 3u are unavailable for these meanings, as they already mean 13-over and 13-under. The wo intervals are 3/2, 9/8, 27/16, etc. 15/8 is fifthward but not wo. All wa intervals are either wo, wu or ca.

When spoken, double could be abbreviated as **bi-** ("bye"), as in biruyo. Triple could be abbreviated **tri-** ("try"), as in triyo.

In keeping with the bantu-like sound of colorspeak, bi- might be pronounced "bee" and tri- might be "tree". Quad and quint might become **kwa-** and **kwi-**. Sixfold becomes tribi-, and sevenfold becomes **sevi-**. 8 = kwabi-, 9 = tritri-, and 10 = kwibi-. Perhaps 11 = levi- and 13 = thiri-.

Large and small could be abbreviated as **la** and **sa**. Double large = lala, triple large = trila, etc.

529/512 = 23oo2 is a double-twenty-tho second. Double-twenty-tho, and similar terms for primes higher than 19, could be shortened by using **-h-** as in twenty-thoho. Note that twenty-thotho is unavailable, as it already means 23o3o.

The mid symbol ~ could be extended to the 4th and 5th. The **mid 4th** would be halfway between perfect and augmented, and the **mid 5th** would be halfway between diminished and perfect. The rationale for choosing A4 over d4 is that just as m2 and M2 are the two 2nds closest to P1 on the relative notation chain of 5ths, P4 and A4 are the two closest 4ths. There would be no mid unison or mid octave. This would simplify 72edo notation in the tritone region, replacing doubleup with downmid, triple-up with mid, and doubledown-aug with upmid:

without mids:	P4	^4	^^4	^^{3}4	vvA4	vA4	A4/d5	^d5	^^d5	v^{3}5	vv5	v5	P5
with mids:	P4	^4	v~4	~4	^~4	vA4	A4/d5	^d5	v~5	~5	^~5	v5	P5

In staff notation, it would be possible to have an additional accidental, a "plain sign", analogous to the natural sign, that cancels ups and downs without affecting sharps and flats. This might make it easier to notate a melody that moves by an edostep from down-flat to plain-flat.

Appendix 3 – A Guide to Widely-used Microtonal Terms

There is a significant body of microtonal writing that uses different conventions than this book. This chapter bridges the gap between conventions.

Microtonalists use many terms that could fairly be called jargon. Some are unnecessarily pedantic, using Latin or Greek words where English ones would suffice. Other names, like those for commas and temperaments, seem quite arbitrary and are difficult to memorize.

General terms:
val = edo-mapping
patent val = nearest edo-mapping
eigenmonzo = a ratio that lies on the just intonation baseline, an eigenvector with an eigenvalue of 1.
essentially tempered chord = an innate comma chord that's been tempered
pythagorean = 3-limit = wa, includes clear too
undecimal = either yazala or la
tridecimal = either yazalatha or tha
nonatonic = 9-tone or 9-note
decatonic = 10-tone or 10-note
hendecatonic = 11-tone or 11-note
dodecatonic = 12-tone = chromatic
enneadecatonic, nonadecatonic = 19-tone
icosatonic = 20-tone
icosihenatonic = 21-tone
icosiditonic = 22-tone
etc.

Wa intervals:
pythagorean comma = wa comma = LLw-2
limma = small wa 2nd = sw2
apotome = large wa semitone = Lw1
Mercator's comma = wa minicomma = wa-53 comma = eightfold-large wa minicomma = L^8w3

Ya intervals:
syntonic comma, Ptolemaic comma, comma of Didymus, diatonic comma, chromatic diesis = gu comma = g1
schisma = the yo minicomma Ly-2, or more generally, any minicomma
diaschisma = gugu comma = sgg2
major chroma, major limma, pelogic comma = large yo semitone = Ly1
diesis = triple gu comma = ggg2
maximal diesis, porcupine comma = triple yo comma = yyy1
kleisma = sixfold yo minicomma = y^6-2

Yaza intervals:
Archytas' comma, septimal comma = ru comma = r1
septimal diesis, slendro diesis = zozo comma = zz2
tritonic diesis, jubilisma = double ruyo comma = rryy-2
septimal kleisma, marvel comma = minicomma = ruyoyo minicomma = ryy-2
breedsma = deep purple microcomma = double zozogu microcomma = z^4gg3

Yazala and yazalatha intervals:
mothwellsma = loruru comma = 1orr
rastma = lulu minicomma = 1uu1
negustma = tholuru comma = 3o1ur1

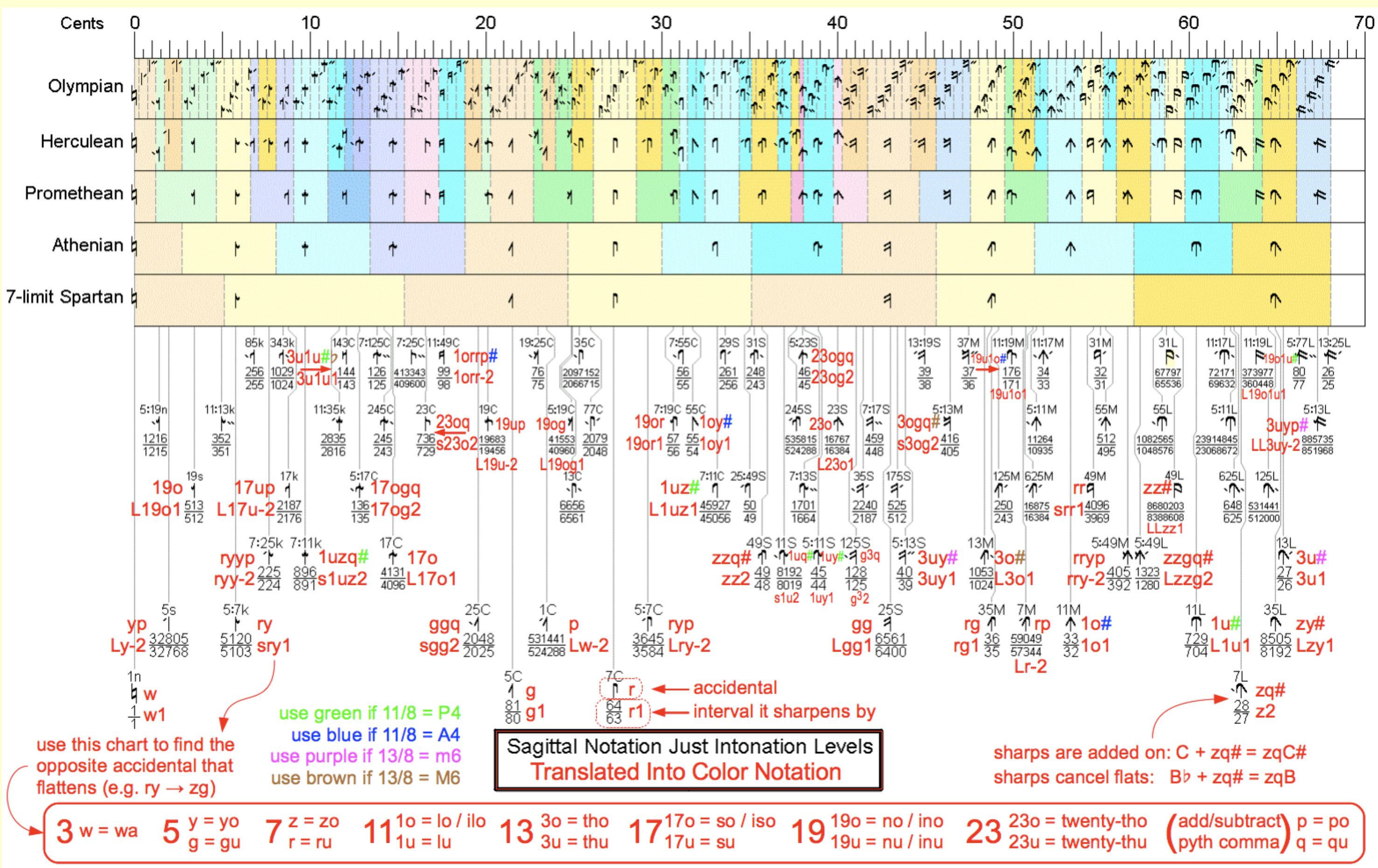
Sagittal notation uses arrow-like symbols to indicate small alterations of pitch. Despite this, it's much closer to color notation than to ups and downs notation, especially its mixed form. There are symbols that correspond directly to w, y, g, z, r, 1o, 1u, 3o and 3u. There are optional symbols that correspond to zg, ry, zy, gr, yy, ryy, etc., as well as 1og, 1uz, 3o1u, etc. Every likely combination of colors can be represented by a single symbol. In mixed Sagittal, these symbols are used alongside conventional sharps and flats. In pure Sagittal, sharps and flats aren't used, and there are additional symbols for y♯, z♭, etc. The stated objective is to reduce clutter, but the result is a very large set of symbols to memorize, especially with pure Sagittal. It's somewhat analogous to learning to read and write with a syllabary like Chinese uses, rather than an alphabet like European languages use. Furthermore, all these unfamiliar symbols are nameless. This makes learning them harder, and makes spoken communication between musicians almost impossible.

With Sagittal, notating edos 5-72 requires at least 17 pairs of accidental symbols. With mixed Sagittal, a single note can have up to 3 symbols, counting the double flat as two symbols. With ups and downs, only one pair of symbols is needed, and a single note can have up to 4 symbols (e.g. F♭♭^^). Less to memorize, and only slightly more on the page.

Sagittal notates many edos as subsets of a larger edo. With ups and downs, only edos 6 and 8 require subset notation. But Sagittal uses subset notation for an additional 19 edos: 11, 13, 14, 18, 20, 25, 28, 30, 32, 35, 37, 42, 44, 52, 54, 59, 61, 66 and 71. Many of the "parent edos" that the subsets are taken from are unreasonably large, such as 56 for 14-edo, or 176 for 44-edo.

The only advantage to subset notation is translating from one edo to another. For example, if you want to play an 11-edo piece on your 22-edo guitar, it's easier if the piece is notated as a subset of 22-edo. In this case, subset notation makes sense, because you really do have 22 notes at hand. But if your instrument has only 11 notes, e.g. a keyboard or a flute, subsets are confusing, because the notation refers to notes that aren't actually there. If you're playing in C, Sagittal's F, G, A and B notes are simply missing. When you read F♯, you have to imagine where F would be, then play sharp of there. With ups/downs, you have options. Most people can use non-subset notation, in which F, G, A and B are present. 22-edo guitarists can use subset notation. For the other 18 edos for which Sagittal uses subset notation, the "parent edo" is quite large, at least 36, and will almost never be physically present.

With p and q, color notation can do everything Sagittal can. This chart translates many Sagittal accidentals into colors. (high-res version at www.tallkite.com/misc_files/Sagittal-JI-Translated-To-Colors.png).



Temperaments: see chapter 4.5 for more.

3-limit (wa):
pythagorean temperament = wa linkage or wa temperament (implies 12-edo)
blackwood temperament = small wa temperament (implies 5-edo)
apotome temperament = large wa temperament (implies 7-edo)

ya:
meantone temperament = gu temperament
helmholtz or schismatic temperament = large yo temperament
diaschismic temperament = small gugu temperament
porcupine temperament = triple yo temperament
augmented temperament = triple gu temperament
dimipent temperament = quadgu temperament
father temperament = gu 2nd temperament
dicot temperament = yoyo temperament
bug temperament = gugu temperament

za:
archy temperament = ru temperament
semaphore temperament = zozo temperament

yaza:

rank-3 (one comma):
marvel temperament = ruyoyo temperament
pajara temperament = double ruyo temperament
starling temperament = zo triple gu temperament
breedsmic temperament = double zozogu temperament, deep purple temperament

rank-2 (two commas):
septimal meantone temperament = gu and zo triple gu temperament
dominant meantone temperament = gu and ru temperament
godzilla temperament = gu and zozo temperament
diminished temperament = rugu and double ruyo temperament
porcupine temperament = triple yo and ru temperament

Appendix 4 – Color Notation In Other Languages

Conventional staff notation is universal (language-independent). Every country uses the same staff, the same clefs, and the same sharp and flat signs. But music terminology isn't. Spanish has not D♯ and minor 7th, but Re sostenido and séptima menor.

Color notation must also be universal. Many terms can be translated into other languages, but a few terms can't be. Just as one must learn a few Italian words like allegro and andante to read sheet music, one must learn a few English words to read color notation. Fortunately, the full word needn't be learned, just the first letter.

The color accidentals w, y, g, z and r must not vary. Spanish speakers shouldn't translate yellow into amarillo, and then shorten it to amo or mo. In order for terms such as 1o, 3u, 17a, etc. to be universal, -o, -u and -a for over, under and all must not vary. Thus wa, yo, gu, zo and ru are also invariant. Po, qu, p and q are also invariant.

Not only staff notation but also written chord names must not vary. Ch7 and Cs7 are invariant, thus h and s are also invariant. The words harmonic and subharmonic can vary. B natural is called H in certain countries, e.g. Germany. "Natur sieben" = h7 = w1 y3 w5 z7, and "ha sieben" = H7 = H D♯ F♯ A.

All colors for primes 11 and higher can vary. In many European languages, tho/thu/tha becomes tro/tru/tra. Spanish for 11 is once, and lo/lu/la might become onco/oncu/onca. Or it might remain lo/lu/la, for conciseness. If so, a helpful mnemonic is lavender, since most Western and even some Asian languages have a word very similar to it. Italian for 11 is undici, suggesting uno/unu/una. But if 1o is uno, an "uno chord" would also be a "one chord". Thus 1o becomes either undo/undu/unda or perhaps unó/unu/una, with the accent distinguishing unó from uno.

The short form of temperament names and subgroup names must not vary, because they are likely to be written at the top of the score. In such names, primes 11 and higher must be written in their numeric form. Thus on the score the thulu temperament is written 3u1uT, and the yalatha subgroup is written ya1a3a.

The disambiguation prefix i- is invariant, and is used as needed in all languages. Disambiguation is only necessary if the other word needs to be used in a musical context. The note C sounds like sea, but there's no problem, because noone ever needs to discuss a "sea chord". But "no" as in no5 and nowa is invariant, therefore 19o must be ino in most European languages. Disambiguation is also needed if the other word is extremely common, like "the" or "and".

Sometimes one color needs disambiguation from another. In Latin American Spanish, z and s sound the same, and zo and so are a problem. The rule is to add i- to the higher prime's color. Zo is pronounced "so", and 17o is pronounced "iso". I- is used even when 17o is not alone, thus 17oz is isozo. 17o is written as iso not so, to match the pronunciation. Sa becomes isa, to differentiate it from za. Su needn't change to isu, but might for consistency.

Another example: the Dutch word for 17 begins with z, so Dutch might use zo/ru/za for 7 and izo/(i)zu/iza for 17. Or Dutch might borrow from nearby English (seventeen) and German (siebzehn), and use so/su/sa for 17.

Two colors might possibly sound alike and also sound like some musical term. If so, use i- for the higher prime as before, and reuse the final vowel to prefix the lower prime. If z and s sound the same, and the solfege syllable is So, 17o5 becomes iso So and z5 becomes ozo So.

Another use for i- is for when thick accents make communication difficult. In Castillian Spanish, zo sounds like "tho". A Spaniard pronounces 3o as "tro", so there's no conflict among Spaniards. But a Spaniard might be confused by an American saying 3o as "tho". Therefore the American says zo and itho, and the Spaniard says tho and itro.

In terms like twenty-tho and thirty-wu, the final digit is abbreviated similarly to -wo/-tho/-so/-no. Italian for 31 is trentuno, and 31u is trentunu. But 31o needs to be distinct from 31, and trentuno won't work. The solution is to accent the final syllable, so that 31o = trentunò or trentunó.

Roman numerals are invariant, for chord progressions. P, M, m, A and d are invariant, for chord names and pergens. The spoken terms are of course translated into the usual terms for perfect, major, minor, etc. Many countries have adopted jazz chord names such as CM7, even if their word for major is dur. Pergens are never written on the score as quarter-fifth, but as (P8, P5/4). A pergen's enharmonic interval is written as C^^ = C♯. Edos are indicated as ^1 = 1\31.

L and s are invariant, but the spoken words large and small can be translated. This is analogous to an English speaker seeing "f" or "p" on a score and thinking loud/soft, not forte/piano. The translated words must not have any musical connotations such as major/minor or augmented/diminished or largo (slow tempo).

The symbols ^ v / \ ~ are invariant, but the terms up, down, lift, drop and mid can vary. Up and down may possibly be translated as above/below or top/bottom. Lift/drop may be translated as raise/lower. Lift and drop should be translated into verbs, since ^ is high, but / starts low and goes high. Preferably transitive verbs, drop not fall. All five terms should be words not usually applied to notes or clefs or melodies or intervals, e.g. not high/low or treble/bass or rising/falling or neutral. In temperament names, both "and" and "plus" should have distinct names.

Clear, ca and noca are never used in interval names or chord names. They are never used on staff notation without a lengthy explanation, since staff notation assumes octaves. Thus they can be translated freely. Clear means transparent, not "easily understood". The words plain, central, double, triple, etc. can also be translated freely. Plain must be distinct from natural and clear, and may be translated as simple. Central must be distinct from mid and neutral.

Here are all the invariant color notation terms, with their English meanings:

wa, yo, gu, zo, ru	white, yellow, green, azure/azul, red
w, y, g, z, r	(the short forms)
-o, -u, -a, ya, za	over, under, all, yellow-all, azure-all
p, q, po, qu	pythagorean-over, pythagorean-under,
L, s, no, nowa	large, small, no (as in omit), no-white
h, s	harmonic series, subharmonic series
T, i-	i- for disambiguation

The following table summarizes possible translations. 1o refers to 11-over, and -1o refers to -wo as in thirty-wo and forty-wo. See en.xen.wiki/w/Color_Notation_Translations for more translations. (Thanks to Praveen Venkataramana for his general assistance with this section.)

Table A.4.1 – Suggested translations of color notation for Western European languages

	English	German	French	Spanish	Portuguese	Italian
11	l-	l-	onz-	onc-	onz-	un-? und-?
13	th-	dr-	tr-	tr-	tr-	tr-
17	s-	s-	s-	s-	s-	s-
19	n-	n-	n-	n-	n-	n-
-1	-w-	ein-	-un-	-un-	-um-	-un-
-3	-th-	dr-	-tr-	-tr-	-tr-	-tr-
-7	-s-	s-	-s-	-s-	-s-	-s-
-9	-n-	n-	-n-	-n-	-n-	-n-
L	large	groß	grand	grande	grande	grande
s	small	klein	petit	pequeña	pequena	piccolo
	central	zentral		central		
h	har	natur		armo		
s	sub	sub		sub		
^	up	oben? hoch?	haut?	arriba	cima?	su
v	down	neider	bas?	abajo	baixo?	giù
/	lift	heb		levante		alzare?
\	drop	tropf		soltando		cadere?
~	mid	mitte	milieu	medio	meio	medio
	plain	schlicht?		sencillo		
&	and	und		y		
+	plus	plus		mas		
W	wide			ancho		
ca	clear			claro		
2	double	doppel		doble		
3	triple	dreifach		triple		
4	quad	vierfach		cuad		
5	quint	funffach		quint		
6	sixfold	sechsfach				
	4thwd			a cuarta		
	5thwd			a quinta		

Disambiguations: (1o refers to 11-over, and -1o refers to -1-over, e.g. -wo in thirty-wo and forty-wo)

English: 1o = ilo ("low C"), 1a = ila (La solfege), 17o = iso (So solfege), 19o = ino, 19u = inu ("new key")
German: 19o = ino
French: 13a = itra (tra vs. trois), -3a = -itra-, 19o = ino
Spanish: -1o = -unó (31 vs. 31o), 19o = ino, Latin American Spanish only: 17 = is- (z and s sound the same)
Portuguese: 19o = ino
Italian: 17u = isu (su means ^), 19o = ino, -1o = -unò or -unó (31 vs. 31o)

Appendix 5 – The Ideal Microtonal Notation

One goal of this book is to create one universal performer-friendly microtonal notation, for communication between theorists, composers, arrangers and performers. This communication may be via not only sheet music but also chord chart.

This is in addition to various composer-friendly notations for non-8ve and non-5th tunings, and for non-12 keyboards.

The Ideal Universal Microtonal Notation:

1) Works for just intonation, EDOs and rank-2 temperaments

2) Works for staff notation, but is also typeable, speakable, and even singable (solfege)

3) Is backwards compatible: 8ve-equivalent, 5th-generated and heptatonic, to keep familiar interval arithmetic

4) Includes relative notation, essential for chord names

5) Minimizes memorization and learning time: uses a small vocabulary of familiar words and symbols

6) Minimizes calculations: avoids ratios, avoids counting edosteps

   a) Uses no numbers larger than 9 (except for chord names, e.g. 11th chords)

7) Balances simplicity vs. brevity

   a) Is simple enough to remember it after not using it for a year

   b) Is brief enough to speak the names of the chords as you strum them

   c) Tends to give simple things shorter names, and complex things longer names

8) Avoids subset notations: no "missing notes" (exceptions: 8-edo, non-8ve and non-5th tunings)

Appendix 6 – Various Mathematical Formulas and Proofs

Basic formulas:

The size in cents of a ratio R is ¢ (R) = 1200 · log (R) / log (2), which is approximately 1731 · ln (R).

For example, ¢ (3/2) = 1200 · log (3/2) / log (2) = 701.955¢, alternatively ¢ (3/2) ≈ 1731 · ln (3/2) = 701.860¢

This formula can be reversed to find a "ratio" that has a size of C cents.
Because the formula returns one number, not two, the "ratio" is in the form of a decimal number, not an integer ratio.
The "ratio" of an interval of C cents is $\mathbb{R}(C) = 2^{(C\,/\,1200)} = e^{(C\,\cdot\,\ln(2)\,/\,1200)} \approx e^{(C/1731)}$.

For example, the ratio that is 702¢ wide is $\mathbb{R}(702) = 2^{(702/1200)} = 1.50004$, which is approximately 3/2.

Approximation of the cents of narrow intervals:

A superparticular ratio has a numerator one greater than the denominator, for example 5/4, 10/9, 21/20, etc.
For a superparticular ratio of form (D + 1) / D, for large values of D, the ratio's cents approaches a simpler formula.
¢ ((D + 1) / D) = 1200 · ln ((D + 1) / D) / ln (2) ≈ 1731 · ln ((D + 1) / D) = 1731 · ln (1 + 1/D)
As D becomes large, 1/D approaches zero, ln (1 + 1/D) approaches 1/D, and the cents approaches 1731/D.
Therefore ¢ ((D + 1) / D) ≈ 1731/D

For example, ¢ (36/35) is approximately 1731/35 = 49.46¢, very close to the actual size of 48.77¢.
For ratios with larger numbers, the formula is even more accurate: ¢ (101/100) ≈ 17.31¢ (actual size 17.23¢)

Except for very large D (> 3700), this formula slightly overestimates the cents.
It has an error less than 1¢ for ratios narrower than 59¢, and less than 0.1¢ for ratios narrower than 19¢.

This formula can be generalized for any ratio (D + x) / D where x << D:
¢ ((D + x) / D) = ¢ (1 + x / D) ≈ 1731 · x / D

For example, ¢ (91/88) ≈ 1731 · 3 / 88 = 59.01¢. The actual size = 58.04¢ is within 1¢ because ¢ (91/88) < 60¢.

The formula can be reversed to find a ratio approximately C cents wide: C ≈ 1731 / D, therefore D ≈ 1731 / C.
The ratio is (D + 1) / D ≈ (1731 / C + 1) / (1731 / C) = (1731 + C) / 1731.
For integer values of C, this formula returns an integer ratio.

For example, a ratio of approximately 5¢ is (1731 + 5) / 1731 = 1736/1731 (the actual size of 1736/1731 is 4.99¢).

<u>The frequency of interference beats</u>:

To calculate the frequency of interference beats, take the difference of the two beating frequencies.
Consider a slightly mistuned unison near A-220, which is A below middle-C.
If the 2 notes have frequencies 220 cps and 221 cps, A-220 and A-221 will beat at 221 - 220 = 1cps.
But if the sounds are harmonic, their overtones will beat as well:
A-220 has harmonics at A-440, E-660, A-880, etc.
A-221 has harmonics at A-442, E-663, A-884, etc.
These harmonics beat at 2 cps, 3 cps, 4 cps, etc.
The higher overtones are often fainter, so the main impression will often be 1 cps beats.

The interval between A-220 and A-221 is the frequency ratio 221/220 = 7.9¢
That same 7.9¢ interval in a higher register will beat faster.

For example, A-440 plus 7.9¢ makes A-442, because ¢ (442/440) = ¢ (221/220).
A-440 and A-442 will beat at 442 - 440 = 2 cps, as well as 4 cps, 6 cps, etc.

Therefore higher-register intervals would seem to require more accurate tuning.
But lower frequencies often have more prominent overtones, which would increase the perceived beat frequency.

For example, A-110 and A-110.5 would seem to have a beat frequency of 0.5 cps.
But if they have prominent harmonics at A-220 and A-221, the perceived beat frequency might be 1 cps.

For a unison at frequency F cps mistuned by C cents which is beating at B cps, the two frequencies are F and F + B.
The interval between the 2 mistuned notes is (F + B) / F. This narrow interval can be approximated as shown above:
Its size C ≈ 1731 · B / F, therefore B ≈ C · F / 1731 (as well as 2 · C · F / 1731, 3 · C · F / 1731, etc.).
The beat frequency B is directly proportional to the mistuning C and the frequency F.

For example, a unison at A-440 mistuned by 4¢ will beat at ≈ 4 · 440 / 1731 ≈ 1 cps (as well as 2 cps, 3 cps, etc.).
A mistuning of 2¢ will beat at 0.5 cps. A unison an octave lower at A-220 mistuned by 2¢ will beat at 0.25 cps.

The term F / 1731 can be thought of as the interval from 1731 cps (about A6) down to the note with frequency F.
A6, which is two octaves and a sixth above middle-C, is A-1760, only 29¢ sharper than 1731 cps.
Therefore a rough estimate of the beat frequency is: F = C times the ratio of the descending interval from A6.

For example, F4 is roughly a 5/1 below A6, and a 3¢ mistuning of F4 would beat at roughly 3 · 1/5 = 0.6 cps.

The note a fifth above A-220 is 220 · 3/2 = E-330. Consider a mistuned fifth formed by A-220 and E-331.
The fundamentals won't beat because they are too far apart. Only the harmonics will beat.
A-220 has harmonics at A-440, E-660, A-880, C♯-1100, E-1320, G-1540, A-1760, B-1980, etc.
E-331 has harmonics at E-662, B-993, E-1324, G♯-1655, B-1986, etc.
Every 3rd harmonic of A-220 will beat with every 2nd harmonic of E-330 (with the fundamental as the 1st harmonic).
Harmonics E-660 and E-662 beat at 2 cps, E-1320 and E-1324 beat at 4 cps, B-1980 and B-1986 at 6 cps, etc.
The beat frequencies are 2 cps, 4cps, 6 cps, etc. The higher overtones are fainter, so the main impression will be 2 cps.

For an interval with the ratio N/D whose lower note is at frequency F cps, the upper frequency is (N / D) · F.
For an interval <u>near</u> the ratio N/D whose lower note is at frequency F cps, which is widened by C cents:
The upper frequency is ℝ (C) · (N / D) · F ≈ [(1731 + C) / 1731] · (N / D) · F = (1 + C / 1731) · (N / D) · F.
The lowest pair of harmonics that will beat are the Nth harmonic of the lower note and the Dth harmonic of the higher.
These two frequencies are N · F and D · (1 + C / 1731) · (N / D) · F = (1 + C / 1731) · N · F
The beat frequency is their difference, (1 + C / 1731) · N · F - N · F = N · C · F / 1731.
For a slight narrowing of N/D, C is negative, but the beat frequency is of course still positive.
Either widening or narrowing by C cents produce the same beat frequency, so we can speak of a mistuning of C cents.
The beat frequency B is directly proportional to the integer limit N, the mistuning C, and the frequency F.

For example, a fifth 3/2 on A-220 mistuned either sharp or flat by 5¢ will beat at about 3 · 5 · 220 / 1731 = 1.9 cps.
However, a tritone 7/5 on A-220 mistuned by 5¢ will beat at about 7 · 5 · 220 / 1731 = 4.4 cps.

<u>Therefore more complex ratios require more accurate tuning.</u>

(However, there are other factors besides interference beats, such as combination tones.)

A mathematical basis for just intonation keyspans, stepspans and cents:

TERMS:

k (R) = keyspan of ratio R, according to the sizing framework (keyspan is the number of semitones in an interval)
s (R) = stepspan of ratio R, according to the naming framework (stepspan is generally one less than the degree)
¢ (R) = size in cents of ratio R = 1200 · log (R) / log (2)
1 / R = descending form of musical interval R = mathematical inverse of ratio R
R · S = musical sum of intervals R and S = mathematical product of ratios R and S
R / S = musical difference of intervals R and S = mathematical quotient of ratio R divided by ratio S
For any integer N, R^N = musical sum of N intervals R = mathematical product of N intervals of ratio R

> For example, under the conventional 12 + 7 system:
> k (3/2) = 7 semitones, s (3/2) = 4 (a fifth), and ¢ (3/2) ≈ 702¢
> k (4/3) = 5 semitones, s (4/3) = 3 (a fourth), and ¢ (4/3) ≈ 498¢
> If R = 3/2 and S = 4/3, 1 / R = 2/3, 1 / S = 3/4, R · S = 2/1, R / S = 9/8, R^2 = 9/4 and S^2 = 16/9

AXIOMS:

1. k (R) + k (S) = k (R · S)	Keyspans add up as expected
2. s (R) + s (S) = s (R · S)	Stepspans add up as expected
3. ¢ (R) + ¢ (S) = ¢ (R · S)	Cents add up as expected (follows from the definition of cents)
4. k (1/1) = 0	A single note has only one key
5. s (1/1) = 0	A single note has only one name
6. ¢ (1/1) = 0	A single note has only one pitch (follows from the definition of cents)

> For example:
> k (3/2) + k (4/3) = k (2/1) = 12 semitones
> s (3/2) + s (4/3) = s (2/1) = 7 steps = an 8ve
> ¢ (3/2) + ¢ (4/3) = ¢ (2/1) = 1200¢

THEOREMS:

The larger the ratio's decimal equivalent, the more cents it has (the decimal equivalent of 3/2 is 1.5).

If R > S, ¢ (R) > ¢ (S), and vice versa	Follows from the definition of cents
If R > 1, ¢ (R) > 0, and vice versa	Follows from the previous theorem

> For example, 3/2 > 4/3 (because 1.5 > 1.333...), thus ¢ (3/2) > ¢ (4/3).

These two theorems do <u>not</u> generalize to keyspan or stepspan. See the inevitability of paradoxes below.

A descending ratio has the opposite keyspan, stepspan and cents of the corresponding ascending ratio.

From axiom 1, k (R) + k (1/R) = k (R · 1/R) = k (1/1). From axiom 4, k (1/1) = 0. Therefore:

k (1/R) = - k (R)	From axioms 1 and 4
s (1/R) = - s (R)	From axioms 2 and 5
¢ (1/R) = - ¢ (R)	From axioms 3 and 6

> For example:
> k (2/3) = - k (3/2) = -7 = descend by 7 semitones
> s (2/3) = - s (3/2) = -4 = descending fifth
> ¢ (2/3) = - ¢ (3/2) ≈ -702¢ = descending by 702¢

Stacking an interval twice doubles its keyspan, stepspan and cents, stacking it three times triples them, etc.

For any integer N:

$k(R^N) = N \cdot k(R)$ From axiom 1

$s(R^N) = N \cdot s(R)$ From axiom 2

$¢(R^N) = N \cdot ¢(R)$ From axiom 3

For example, $9/4 = (3/2)^2$, therefore $k(9/4) = 2 \cdot k(3/2)$, $s(9/4) = 2 \cdot s(3/2)$, and $¢(9/4) = 2 \cdot ¢(3/2)$.

From the previous 3 theorems and from axioms 1-3, we can derive a method of assigning keyspan, stepspan and cents based on the prime factors of any ratio:

For any ratio $R = 2^a \cdot 3^b \cdot 5^c \cdot 7^d$ ...

$k(R) = a \cdot k(2/1) + b \cdot k(3/1) + c \cdot k(5/1) + d \cdot k(7/1)$ ...

$s(R) = a \cdot s(2/1) + b \cdot s(3/1) + c \cdot s(5/1) + d \cdot s(7/1)$ ...

$¢(R) = a \cdot ¢(2/1) + b \cdot ¢(3/1) + c \cdot ¢(5/1) + d \cdot ¢(7/1)$ …

The keyspan and the stepspan depend on the sizing and naming frameworks.

For example, let $R = 36/35 = 2^2 \cdot 3^2 \cdot 5^{-1} \cdot 7^{-1}$

Assume a 12-tone sizing framework and a 7-note naming framework

$k(36/35) = 2 \cdot k(2/1) + 2 \cdot k(3/1) - k(5/1) - k(7/1) = 2 \cdot 12 + 2 \cdot 19 - 28 - 34 = 0$

$s(36/35) = 2 \cdot s(2/1) + 2 \cdot s(3/1) - s(5/1) - s(7/1) = 2 \cdot 7 + 2 \cdot 11 - 16 - 20 = 0$

$¢(36/35) = 2 \cdot ¢(2/1) + 2 \cdot ¢(3/1) - ¢(5/1) - ¢(7/1) \approx 2 \cdot 1200¢ + 2 \cdot 1902¢ - 2786¢ - 3369¢ = 49¢$

A more musician-friendly method uses octave-reduced rungs: 3/2, 5/4, 7/4, etc. For 12-tone, $k(2/1) = 12$, $k(3/2) = 7$, $k(5/4) = 4$, and $k(7/4) = 10$. For 7-note, $s(2/1) = 7$, $s(3/2) = 4$, $s(5/4) = 2$, and $s(7/4) = 6$.

For any ratio $R = (2/1)^a \cdot (3/2)^b \cdot (5/4)^c \cdot (7/4)^d$ ...

$k(R) = a \cdot k(2/1) + b \cdot k(3/2) + c \cdot k(5/4) + d \cdot k(7/4)$ ...

$s(R) = a \cdot s(2/1) + b \cdot s(3/2) + c \cdot s(5/4) + d \cdot s(7/4)$ ...

$¢(R) = a \cdot ¢(2/1) + b \cdot ¢(3/2) + c \cdot ¢(5/4) + d \cdot ¢(7/4)$ ...

For example, let $R = 36/35 = (2/1)^0 \cdot (3/2)^2 \cdot (5/4)^{-1} \cdot (7/4)^{-1}$

(Find the octave exponent last, after determining the other exponents.)

Assume a 12-tone sizing framework and a 7-note naming framework

$k(36/35) = 0 \cdot k(2/1) + 2 \cdot k(3/2) - k(5/4) - k(7/4) = 0 \cdot 12 + 2 \cdot 7 - 4 - 10 = 0$

$s(36/35) = 0 \cdot s(2/1) + 2 \cdot s(3/2) - s(5/4) - s(7/4) = 0 \cdot 7 + 2 \cdot 4 - 2 - 6 = 0$

$¢(36/35) = 0 \cdot ¢(2/1) + 2 \cdot ¢(3/2) - ¢(5/4) - ¢(7/4) \approx 0 \cdot 1200¢ + 2 \cdot 702¢ - 386¢ - 969¢ = 49¢$

The cents can of course be calculated directly from the formula $¢(R) = 1200 \cdot \log(R) / \log(2)$. However, if the rungs are tempered, this method is better.

A mathematical proof of the inevitability of paradoxical ratios:

To prove inevitability, first assume the opposite, that for some framework there are no paradoxical intervals:

POSTULATES:

1. If $\text{¢}(R) > 0$, $k(R) >= 0$, and vice versa	Assume there are no upside-down intervals
2. If $\text{¢}(R) > 0$, $s(R) >= 0$, and vice versa	Assume there are no negative intervals

PROOF:

First we'll disprove postulate 1:

Let R and S be two unique ratios with the same nonzero keyspan, with $R > S$ and thus $\text{¢}(R) > \text{¢}(S)$.

$k(R/S) = k(R) + k(1/S) = k(R) - k(S) = 0$	The keyspan of R/S will be zero
$\text{¢}(R/S) = \text{¢}(R) + \text{¢}(1/S) = \text{¢}(R) - \text{¢}(S) > 0$	The cents of R/S will be greater than zero

Let $x = \text{¢}(R) - \text{¢}(S)$, the size in cents of R/S.
Let N be the smallest integer such that $N \cdot x > \text{¢}(S)$.
Let Z be the ratio $(R/S)^N$, the musical sum of N intervals R/S.
The ratio Z/S will be an upside-down ratio:

$\text{¢}(Z) = \text{¢}((R/S)^N) = N \cdot \text{¢}(R/S) = N \cdot x$	The cents of Z is greater than the cents of S
$\text{¢}(Z/S) = \text{¢}(Z) - \text{¢}(S) = N \cdot x - \text{¢}(S) > 0$	The cents of Z/S is positive
$k(Z) = k((R/S)^N) = N \cdot k(R/S) = 0$	The keyspan of Z is zero
$k(Z/S) = k(Z) - k(S) = 0 - k(S) < 0$	The keyspan of Z/S is negative
$\text{¢}(Z/S) > 0$, but $k(Z/S) < 0$	Z/S is an upside-down ratio
$k(S/Z) > 0$, but $\text{¢}(S/Z) < 0$	S/Z disproves the "vice versa" part of postulate 1

> There are many pairs of ratios R and S that can be used to find an upside-down ratio.
> For example, in the 12-tone framework, 6/5 and 7/6 are both min 3rds, 10/9 and 9/8 are both maj 2nds, etc.
> For these pairs, Z/S will be a very complex ratio with 10-digit numbers or higher.
> For simplicity, choose R and S to have both a small keyspan and a large cents difference between them.
> Let $R = 16/15$ and $S = 28/27$
> $k(R) = k(S) = 1$ (both are minor 2nds)
> $\text{¢}(R) \approx 112\text{¢}$
> $\text{¢}(S) \approx 63\text{¢}$
> $R/S = (16/15) \cdot (27/28) = 36/35$
> $x = \text{¢}(R/S) \approx 112\text{¢} - 63\text{¢} = 49\text{¢}$
> $2 \cdot 49\text{¢} = 98\text{¢} > 63\text{¢}$, therefore $N = 2$
> $Z = (36/35)^2 = 1296/1225$
> $Z/S = (36/35)^2 \cdot (27/28) = 8748/8575$
> $\text{¢}(Z/S) = 98\text{¢} - 63\text{¢} = 35\text{¢}$
> $k(Z/S) = -1$
> Z/S is upside-down, therefore postulate 1 is false.

In any given sizing framework, is there always a pair of unique ratios R and S such that their keyspans are equal?
Yes. (But see "Tempered ratios" below.)
Proof: For there to be no such pair, each ratio less than 2/1 must have its own unique keyspan.
In other words, the number of unique ratios less than 2/1 would have to be less than $k(2/1)$, the keyspan of an octave.
But the octave-reduced lattice is infinite. Therefore one octave would have to span an infinite number of keys!
(If the period is not an octave, just substitute "period" for "octave", and substitute the period's ratio for 2/1.)

Note that this proof always finds an upside-down interval, but usually not the least remote upside-down interval.

The proof of the inevitability of negative ratios is done similarly:

Let R and S be two unique ratios with the same nonzero stepspan, with R > S and thus ¢ (R) > ¢ (S).

s (R/S) = s (R) + s (1/S) = s (R) - s (S) = 0	The stepspan of R/S will be zero, and it will be a unison
¢ (R/S) = ¢ (R) + ¢ (1/S) = ¢ (R) - ¢ (S) > 0	The cents of R/S will be greater than zero

Let x = ¢ (R) - ¢ (S), the size in cents of R/S.
Let N be the smallest integer such that N · x > ¢ (S).
Let Z be the ratio $(R/S)^N$, the musical sum of N intervals R/S.
The ratio Z/S will be a negative ratio:

¢ (Z) = ¢ $((R/S)^N)$ = N · ¢ (R/S) = N · x	The cents of Z is greater than the cents of S
¢ (Z/S) = ¢ (Z) - ¢ (S) = N · x - ¢ (S) > 0	The cents of Z/S is positive
s (Z) = s $((R/S)^N)$ = N · s (R/S) = 0	The stepspan of Z is zero
s (Z/S) = s (Z) - s (S) = 0 - s (S) < 0	The stepspan of Z/S is negative
¢ (Z/S) > 0, but s (Z/S) < 0	Z/S is a negative ratio
s (S/Z) > 0, but ¢ (S/Z) < 0	S/Z disproves the "vice versa" part of postulate 2.

> There are many pairs of R & S that can be used, for example 5/4 and 6/5 are both 3rds in the 7-note framework.
> For simplicity, choose R and S to have both a small stepspan and a large cents difference between them.
> For example, let R = 9/8 and S = 21/20
> s (R) = s (S) = 1 (both are 2nds)
> ¢ (R) ≈ 204¢
> ¢ (S) ≈ 84¢
> R/S = (9/8) · (20/21) = 15/14
> x = ¢ (R/S) ≈ 119¢
> 119¢ > 84¢, therefore N = 1
> Z = $(15/14)^1$ = 15/14
> Z/S = (15/14) · (20/21) = 50/49
> ¢ (Z/S) ≈ 119¢ - 84¢ = 35¢
> s (Z/S) = -1
> Z/S is negative, therefore postulate 2 is false.

In any given naming framework, is there always a pair of unique ratios R and S such that their stepspans are equal?
Yes. (But see "Tempered ratios" below.)
Proof as above: otherwise one octave (or period) would have to span an infinite number of steps!

Tempered ratios:

People often refer to a "tempered 3/2", even though strictly speaking a ratio can't be tempered, only an interval can. Even a slight amount of tempering will produce similar paradoxes for cents. For example, an equal-tempered 3/2 is 700¢, not 702¢. If the 5/4 is an untempered 386¢, the yo minicomma Ly-2 = $2^{-15} \cdot 3^{8} \cdot 5$ = 32805/32768, which is normally 2¢, becomes -14¢, a descending interval. This contradicts the theorem "if R > 1, ¢ (R) > 0". How can this be?

Mathematically, "tempering a ratio" means slightly altering the primes 2, 3, 5, etc. to nearby decimal numbers. In other words, what we call "3" might really have a value of around 3.001. For example, assuming untempered octaves, if the fifth 3/2 is slightly compressed to 700¢, "3" would actually equal 2.9966 (approximately). Using this value of "3" in the yo minicomma's ratio of $2^{-15} \cdot 3^{8} \cdot 5$ gives a value of R < 1 as expected, and the theorem holds.

In 12-ET, "2" remains 2, because octaves are still exactly in tune with the ratio 2/1. The ratio 3/2 becomes seven twelfths of an octave = 7\12. The twelfth 3/1 = 2/1 · 3/2 becomes 12\12 + 7\12 = 19\12, and "3" = $2^{19/12}$, which as noted comes to 2.9966. To find the value of "5", evaluate 5/1 similarly. The major third 5/4 becomes 4\12, 5/1 becomes 28\12, and "5" = $2^{28/12} = 2^{7/3}$ = 5.0397. The minor seventh 7/4 becomes 10\12 and 7/1 becomes 34\12 and "7" = $2^{34/12} = 2^{17/6}$ = 7.1272. The half-augmented fourth 11/8 becomes either 5\12 or 6\12, yielding either 10.6787 or 11.3137. Just as neither of these numbers is a good approximation of 11, neither the 500¢ perfect 4th nor the 600¢ augmented 4th are good approximations of 11/8.

Other edos alter primes similarly. For any prime P, simply calculate the value of P/1 as a fraction of the octave. For example, in 5-edo, 3/1 becomes 8\5, and "3" = 3.0314.

For EDONOIs, substitute "period" for octave. For example, for the Georgian tuning of 4 equal steps to a fifth, the octave 2/1 is 7 steps = 7\4 (seven fourths of a fifth), and "2" = $(3/2)^{7/4}$ = 2.033. 5/4 becomes 2 steps = 2\4, 5/1 = 2\4 + 7\4 + 7\4 = 16\4, and "5" is $(3/2)^{16/4} = (3/2)^4$ = 81/16 = 5.0625. 7/4 becomes either 5 or 6 steps, 7/1 becomes either 19 or 20 steps, and "7" becomes either $(3/2)^{19/4}$ = 6.86 or $(3/2)^{20/4} = (3/2)^5$ = 243/32 = 7.56.

Tempered ratios provide the sole escape from the inevitability of paradox. The proofs require two unique ratios R and S, with k (R) = k (S) > 0, and ¢ (R) > ¢ (S). In JI, ¢ (R) and ¢ (S) are never equal, because of the unique factorization theorem, so the 2nd condition is easily met.. Consider keyspan first. Let K_n = the keyspan of the nth lattice rung, with n = 1 for the octave. If all lattice rungs are tempered to K_1-edo to equal exactly $K_n \backslash K_1$ = 1200¢ · K_n / K_1, then every interval's keyspan will have a direct correlation to its cents. In other words, ¢ (R) / k (R) will be a constant. In this case, if k (R) = k(S), ¢ (R) = ¢ (S), and the proof will not work.

The stepspan proof can be treated the same way. If S_n is the stepspan, temper the nth rung to $S_n \backslash S_1$ = 1200¢ · S_n / S_1, so that if s (R) = s (S), ¢ (R) = ¢ (S). Note that in general, you can't temper the rungs to both $K_n \backslash K_1$, and $S_n \backslash S_1$ simultaneously, and you can only eliminate one type of paradox. You can eliminate either negative intervals or upside-down intervals, but not both.

In 12-edo, there are no upside-down intervals, but there are still negative intervals. For example, the pythagorean comma is a diminished negative 2nd.

Part VI – The Alt-tuner Manual – version 1.2

Chapter 6.1 – Overview & Setup....311
Chapter 6.2 – Quick Start: Basic Operation....313
Chapter 6.3 – Alt-keyswitcher....320
Chapter 6.4 – Basic Alt-tuner Preference Screens....325
Chapter 6.5 – Multi-keyboard Setups....340
Chapter 6.6 – Recording With Alt-tuner....347
Chapter 6.7 – Advanced Preference Screens....350
Chapter 6.8 – Advanced Topics....370
Chapter 6.9 – Advanced Examples....387
Chapter 6.10 – Hardware & Software Issues....399
Chapter 6.11 – Troubleshooting and Other Considerations....421
Chapter 6.12 – Customizing Alt-tuner....426
Appendix 6.1 – 3000 Ratios....430
Appendix 6.2 – Alt-tuner Flowchart....441
Appendix 6.3 – List of Clickable Items....442

Chapter 6.1 – Overview & Setup

Alt-tuner retunes MIDI to alternate tunings (just intonation or JI, tempered, etc.) via pitch bends, channel redirection, note transpositions, virtual keyswitches and/or sysex messages.

Alt-tuner is written in Jesusonic, a programming language that lets you create your own midi and audio effects. It's part of Reaper, a great $60 DAW that runs on macs, PCs and Linux. Get Reaper here: www.cockos.com/reaper/index.php. Reaper is not copy-protected and the free demo version is identical to the paid version. To use another DAW, you have to run both DAWs side-by-side, using Rewire and/or a free virtual midi cable. Rewire is free and is already included in your DAW. PC users also may have the option of running alt-tuner inside their DAW with ReaJS. See chapter 6.10.

Installation: download and install Reaper. Run Reaper, and with all effects windows closed, in the menu choose "Options/Show REAPER resource path in explorer/finder". Go to the Effects/MIDI subfolder and move all the alt-tuner files except the "sample project files" folder to this folder. To uninstall, delete the files.

If your keyboard has sounds: run the included Reaper project "solo play". Send the track's midi hardware output to your keyboard. On your keyboard, turn local control off and put the synth in multi-timbral mode. Set channels 1-12 of your tone generator to be all the same instrument, one for each of the 12 notes in the octave.

For softsynths: There are two methods, depending on your softsynth. Try the easier one first: load the "solo play" file and put your softsynth in the effects chain after alt-tuner. Test it by holding down one key and playing short random notes over it. Listen for pitch shifts in the drone note. This will be easier to hear with a sustaining sound. If there are no shifts, your softsynth is multi-midi-channel, and you're all set. Otherwise, each new note is retuning any previous notes still sounding. Set up 8 or so instances of the softsynth running in your DAW, each one on a separate track, each track receiving a different midi channel 1-8, with all audio sent to one single track for recording. The included Reaper project "solo play with ReaSynth" is all set up that way, using Reaper's built-in VSTi ReaSynth. If your computer has trouble running all 8 at once, try a different output mode, or see "Hardware & Software Issues". If your softsynth is one of the few that can be retuned via keyswitch or sysex, set alt-tuner's output accordingly and use the easier method.

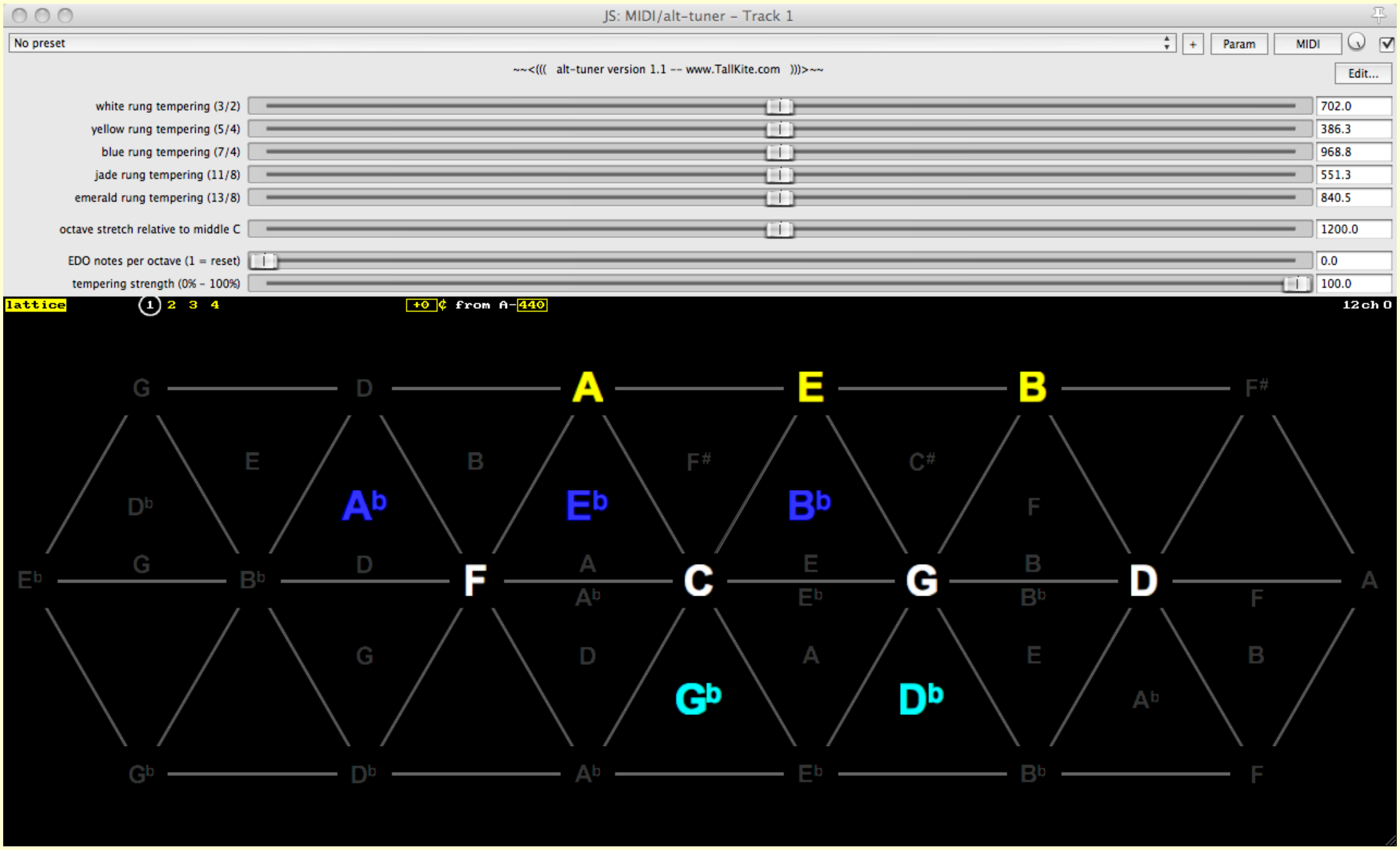
Check that the notes on the screen are being circled as you play them. If not: check that the track input is set to the proper midi input, the track is record-armed, and track record monitoring is on. If there's circles but no sound: check the audio device settings in Reaper preferences, and if you're using a hardware synth, check the "midi hardware output" setting in the track I/O box. Do some notes sound like cymbals? Sometimes midi channel 10 is hardwired to be drums. If so, see "first output channel" in "prefs/misc" in chapter 6.4.

The companion effect alt-keyswitcher sets up foot pedals and keyswitches; see chapter 6.3. Aftertouch-converter and other included files are covered at the end of chapter 6.8. The color notation is explained in parts I-V of this book. See the last two pages of this manual for two handy summaries: a flowchart and a list of what can be clicked on.

Chapter 6.2 – Quick Start: Basic Operation

Alt-tuner has four main screens: lattice, graph, table and preferences. You can see them all by clicking on the yellow rectangle in the upper left of the display. Let's start with the triangular lattice.

<u>The lattice</u>: The harmonic lattice reflects the current tuning. On a keyboard, the physical distance between notes corresponds to melodic distance. In the harmonic lattice, physical distance corresponds to <u>harmonic</u> distance; the lattice shows notes that sound good together close to each other. Horizontal lines are fourths and fifths, diagonal lines are thirds and sixths. Triangles are triads; major ones point up and minor ones point down. An upwards triangle plus a blue note makes a 3-D tetrahedral tetrad. See chapter 1.3 for a full explanation of lattices.



Directly above the lattice, alt-tuner displays the current custom tuning (1 through 4) and the current cents offset of the tonic from 12-ET tuned to A-440. The tonic is the note that appears exactly in the middle of the lattice. When you play notes, they are circled on the lattice, and the size in cents of the last interval played is shown. Play C & E, and you'll see "M3 -14¢ = 386¢ = y3 = 5/4". The interval is 14% of a semitone flatter than the usual 12-ET major 3rd. Its color and degree are a yellow 3rd and its frequency ratio is 5 to 4. The color notation is explained in part II of this book. For now, just be aware that each row of notes in the lattice has its own color.

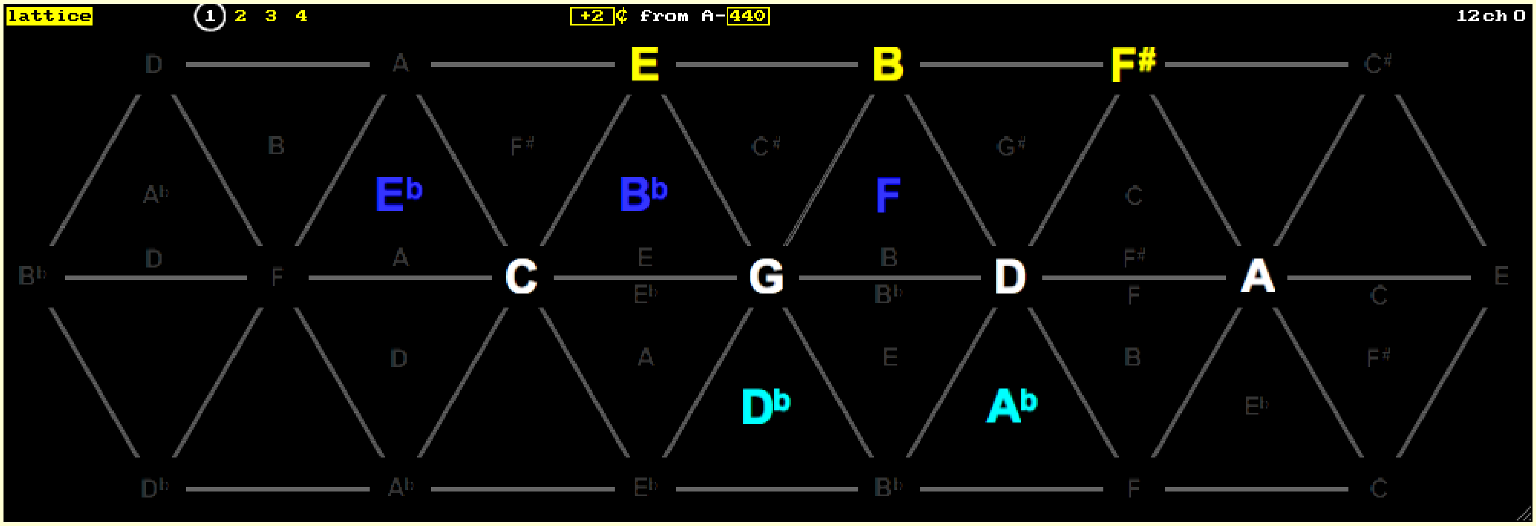
The pitchbend wheel normally affects all the notes at once. But in alt-tuner, it only affects the last note played, along with the same note in other octaves. Play any interval <u>except</u> an octave and you can bend it wider or narrower with the wheel. The interval display will show the change.

Alt-tuner has 4 main actions: cycling, tapping, modulating, and switching. You can do them 3 main ways: with pedals, keyswitches or mouse clicks. Speaking of mouse clicks, there are only two main kinds of clicking: regular clicking and what I'll call right-clicking, which includes alt-clicking, shift-clicking, control-clicking, etc. See also "speed-scrolling".

<u>Cycling taps / tuning taps</u>: These are called taps because they are usually done with the keyboard via tapzones, which are covered in the next chapter. For now, we'll tap with mouse clicks.

Clicking or right-clicking the center note cycles through four preset scales: two 7-limit just intonation scales, one 5-limit scale, and 12-ET (equal temperament, the standard tuning). 12-ET appears as a circle of fifths, the other three appear as lattices. These three presets are merely convenient starting points. Clicking on any note other than the center note will "tap" it. Clicking on the small gray D note on the yellow row will select it, turning it yellow, and unselect the white D, turning it gray. Clicking on the yellow D will make it "jump" back to the white row. Play the note being clicked to hear the effect. There are 4 options for the 11 non-center notes, making 44 intervals, plus the octave, for a total of 45 possible intervals. You are not limited to these 45 notes; you can change these intervals and add other ones.

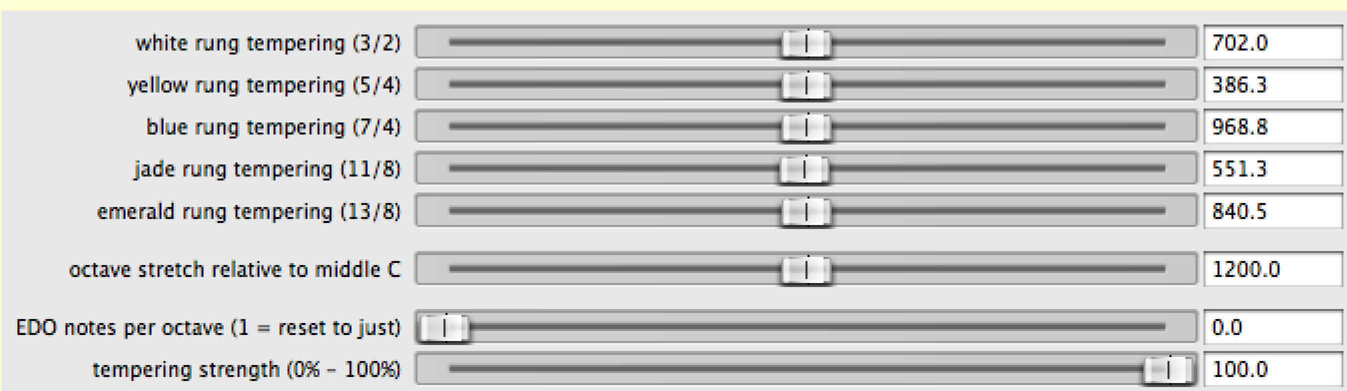
<u>Modulating</u>: Right-click on any note, selected or not, and that note becomes the new center note. A white arrow from the clicked note to the center note will briefly flash on the screen. The cents offset from A-440 adjusts automatically. This offset is the interval from the note in the center of the lattice to the nearest note in the standard 12-ET scale. Modulating fifthward from C to G will change the tonic by 702¢ and thus increase the cents offset by 2¢.



If you're in the 12-ET preset, you can also modulate by clicking or right-clicking on any note in the circle of fifths. That note becomes the new center note. See "tapzones" in the next chapter for more info.

In the next chapter you'll set up modulating pedals that will let you change keys as you play.

<u>The sliders</u>: There are five rung sliders, a stretch slider, an EDO slider and a strength slider. Double-click a slider to reset it. Control-drag (command-drag on a mac) for more fine control. Right-drag or right-click to temporarily move a slider; it will snap back to its former position when you release the mouse button. Each slider has a small number box on the right. You can type any number into this box, even one out of range, and the slider will take on that value.



<u>Tempering sliders</u>: The first five sliders let you alter the basic intervals that generate the tuning. Such an alteration is called a temperament. One reason for doing this is to avoid intervals mistuned by a comma ("wolf" intervals) and thus allow freedom to modulate. As you move these sliders, some of the gray unselected notes will "light up", changing to colored. Alt-tuner is indicating that the tempering has made that unselected note equivalent to a selected note.

The sliders range 100¢ sharp and flat of the default values. These ranges can be customized to be larger or smaller.

<u>Octave stretching</u>: The stretch slider stretches not just octaves but all intervals at once. The tempering sliders react to stretch slider movement: setting the stretch slider to 1212¢ increases them all by 1%. The octave stretch slider is "locked" at 1200¢ until you leave octave-equivalent mode; see "Midi output modes" in the prefs/misc screen.

<u>EDO slider</u>: EDO (pronounced "EE - doe") stands for equal division of the octave. For example, 6-EDO corresponds to the whole-tone scale, with 6 "EDO-steps" of 200¢. 10-EDO has 120¢ EDO-steps. Moving the EDO slider to 10 sets the five tempering sliders to the nearest multiple of 120. White becomes 720, yellow 360, blue 960, etc. When you move the sliders, they are restricted to multiples of 120; for example the white slider jumps from 600 to 720 to 840, etc. A temperament that conforms to an EDO creates an EDO-mapping. Moving the EDO slider produces the nearest EDO-mapping, which is an EDO-mapping in which each rung approximates just intonation as closely as possible. To access all the notes of an EDO higher than 12, either use EDOtap (see chapter 6.8) or increase the number of keys per octave on the keyboard screen. The EDO slider goes up to 72, and higher EDOs can be accessed by typing in the box to the right of the EDO slider. Alt-tuner distinguishes between 12-ET and 12-EDO, more on that in chapter 6.8.

Tempering strength: The tempering sliders affect both the intervals in the scale and the intervals one modulates by. Setting the tempering strength to 0% creates adaptive tuning, in which only the modulating intervals are tempered, and the scale intervals remain just. The fifth in the scale would be a different note than the fifth you modulate to. This allows in-tune chords but avoids comma pump problems. Settings between 0% and 100% produce partially tempered chords. As a general rule of thumb, high settings produce better melodies and low settings produce better harmonies. This slider does not reduce octave stretching; the octave will be stretched but the other rungs won't be.

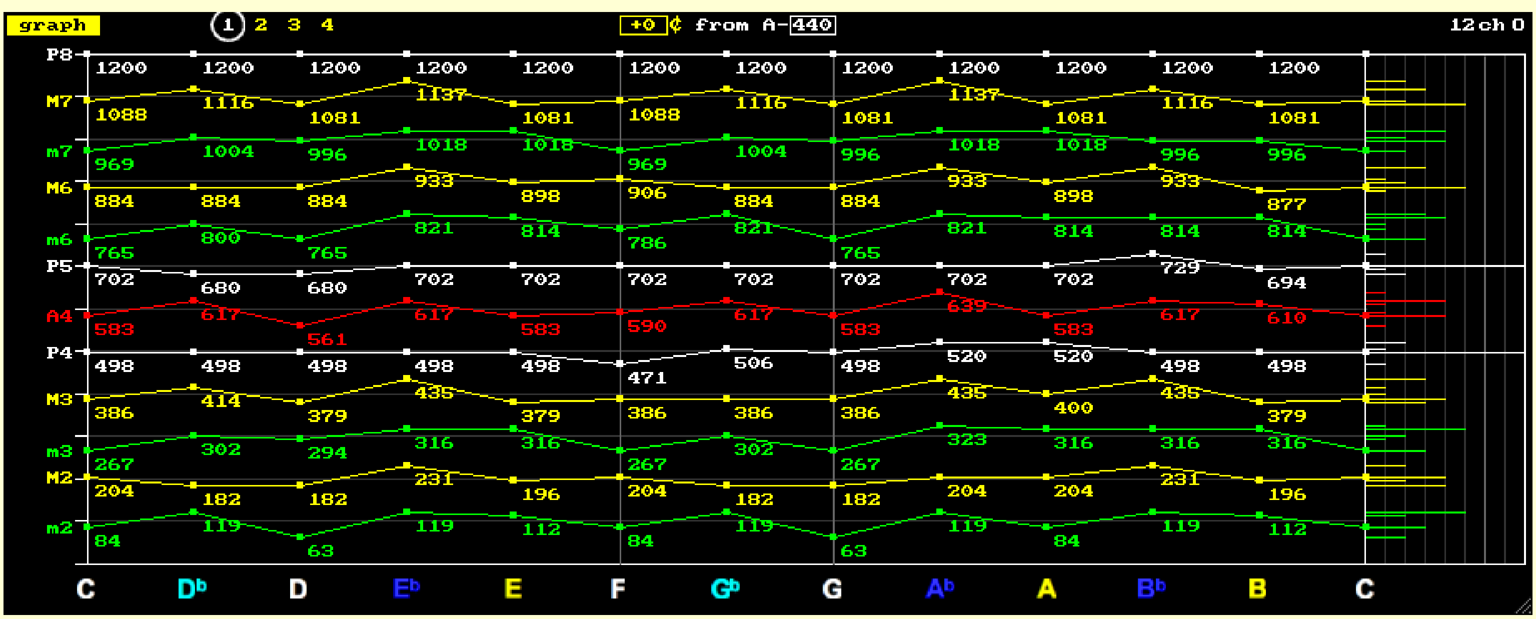
Switching: You can switch among 4 custom tunings instantly with switching pedals. We'll set up pedals later. For now, click on the yellow numbers in the upper left of the display to switch. Some of the tunings start you off in 12-ET. Click anywhere on the circle of fifths to cycle to a preset scale, then tap or modulate to get the scale you want. Or, you can right-click a yellow number to copy its tuning over to the current one. Each custom tuning "remembers" not only the scale and key, but also all the slider settings. You can switch between different temperaments or EDOs instantly. Later we'll see how to increase the number of custom tunings/temperaments from 4 to 30 or even beyond.

Cents offset: The "from A-440" doesn't literally refer to 440 hz. It refers to every note in the standard 12-ET tuning when calibrated to A-440, including C-262, G-392, etc. It's analogous to the calibration feature on a handheld electronic tuner.

Click or right-click the cents offset to increase or decrease it. Hold down the mouse button to autorepeat. Autorepeat will automatically stop when you reach 0¢. The "A-440" button functions like a panic button. Click it to reset the cents offset to zero, clear all red squares and dimmed notes, reset all controllers, and send all-sound-off and all-notes-off messages to every channel. Right-click "A-440" to do all that without resetting the cents offset.

Alt-tuner Presets: (not to be confused with preset scales.) All alt-tuner settings can be saved as presets. Each preset stores all 4 custom tuning/temperaments. Load them from the menu up top. Save a preset by clicking on the box up top with a "+". Each Reaper project file remembers alt-tuner's settings and in effect serves as a preset. To clear a Reaper project's memory, chose "reset to factory default" from the preset menu, and save the project. Alt-tuner presets are for individual songs. Alt-keyswitcher presets are for gear like keyboards, controllers or pedalboards.

Graph view: Click on the yellow rectangle that says "lattice" to get to the graph view. Right-clicking this rectangle takes you back to the lattice view.



The graph view shows you every possible interval between any two keys within an octave. The note names on the bottom start with the lattice's center note, assumed to be the tonic, and run in order. The horizontal lines show the size in cents of each type of interval. The green line at the bottom is the size of each semitone, the yellow line above it is the size of each major 2nd, etc. The number by each dot is the size in cents of that interval. The lines are color-coded for readability: white for perfect, yellow for major, green for minor, red for augmented, and blue for diminished. This use of color to indicate quality is different than alt-tuner's general use of color to indicate a ratio's prime factors.

The graph can also be read vertically. The far left column is the scale that starts on the tonic, the next column shows the scale that starts on the minor 2nd, etc. To find the interval from, say, D to F, find D on the bottom and look at the m3 (minor 3rd) line to get 294¢.

The notes along the bottom are clickable, just like the lattice notes. Clicking the leftmost note (the tonic) will cycle, clicking any other note will tap it up. Right-clicking a note will modulate to that note.

Clicking anywhere on the graph zooms in on each of the four corners in turn, zooming back out with the fifth click.

On the far right are miniature histograms. Cycle to any preset scale except 12-ET and look at the green semitone line. It will have 4 different sizes of semitone. The 4 short green lines to the right are at the appropriate height for those 4 sizes. The longer the line, the more common is that interval size. The box containing the histograms is just big enough to contain the lines. The vertical line representing 6 occurrences (or more generally, half the number of keys per period) is a slightly brighter gray.

As in the lattice view, the notes you play are circled and the latest interval is displayed up above. In addition, all the intervals in a chord are shown as black writing on a small colored rectangle. Playing a close-position triad will highlight a maj 3rd, a min 3rd and a 5th.

The graph lines react instantly to cycling, tuning taps, slider movement, etc. The stretching slider changes all the intervals at once, but the graph is automatically scaled to fill the screen, so the numbers will change but the lines won't move. If the tempering strength slider is less than 100%, the tempering sliders will affect the graph less or not at all.

Moving the EDO slider makes all the lines dance around dramatically. For lower EDOs, there are very few choices for interval size, and the histogram gets crowded. In 5-EDO, for example, the 240¢ area gets histogram lines from the min 2nd, maj 2nd, min 3rd and maj 3rd. Alt-tuner "stacks" the histogram lines one atop the other so they can all be seen.

Table view: Click on the yellow rectangle that says "graph" to get to the table view. This view displays the same basic information as the graph view, but in a different format.

table   1 2 3 4   0¢ from A-440

P8	1200	1200	1200	1200	1200	1200	1200	1200	1200	1200	1200	1200	12
M7	1088	1116	1081	1137	1081	1088	1116	1081	1137	1081	1116	1081	5 2 3 2
m7	969	1004	996	1018	1018	969	1004	996	1018	1018	996	996	2 4 2 4
M6	884	884	884	933	898	906	884	884	933	898	933	877	1 5 2 1 3
m6	765	800	765	821	814	786	821	765	821	814	814	814	3 1 1 4 3
P5	702	680	680	702	702	702	702	702	702	702	729	694	2 1 8 1
A4	583	617	561	617	583	590	617	583	639	583	617	610	1 4 1 1 4 1
P4	498	498	498	498	498	471	506	498	520	520	498	498	1 8 1 2
M3	386	414	379	435	379	386	386	386	435	400	435	379	3 4 1 1 3
m3	267	302	294	316	316	267	302	267	323	316	316	316	3 1 2 5 1
M2	204	182	182	231	196	204	182	182	204	204	231	196	4 2 4 2
m2	84	119	63	119	112	84	119	63	119	84	119	112	2 3 2 5
	C	D♭	D	E♭	E	F	G♭	G	A♭	A	B♭	B	

The table view is useful when the graph lines cross a lot. Again, the notes are clickable. The histograms on the right don't show cent size, just frequency. The intervals are sorted lowest to highest. For example, the lowest row shows "2 3 2 5", meaning the smallest semitone (63¢) occurs twice, the next largest (84¢) 3 times, etc.

If the table is too big to fit on the screen, you can click it to cycle through zoomed views, just like with the graph.

Click on the yellow "table" box to get to the preference screens, which are covered in chapter 6.4.

Suggested "course of study" for beginners: There are near-infinite tuning possibilities, and the newcomer to alt-tonal music may well feel overwhelmed. Here's a brief outline of a few of the more popular approaches.

Just intonation: Select a "normal" sound like piano or guitar, select the green and yellow preset, and play any major triad that produces an upward-pointing triangle. Cycle to 12-ET and back to green-and-yellow, and compare the sound. Train your ears to hear the waterfall we've been living next door to all our lives: the beating of tempered major thirds. Wide voicings with the third in the upper register will show this most clearly. Compare the minor chords, not as much difference.

Next cycle to the blue and yellow preset and play any dom7 chord that produces an upward-pointing tetrahedron. Cycle to 12-ET and compare the sound. Learn to hear the buzz saw we've been living next door to! Play a minor scale and acclimate your ears to the sound. Then cycle to the green and yellow preset. Does minor now sound oddly sharp?

Temperaments: Play a wolf fifth, which if you're in C would be D to A. This fifth is the main problem with just intonation. Move the white tempering slider to 696.6¢. Some of the gray notes will turn colored. All the fifths will now sound slightly worse, but the wolf fifth will sound much better. There are many possible temperaments. This one is the most common one, quarter-comma meantone. Tempering is covered in Part IV.

Equal divisions of the octave (EDOs): Go to the graph view and move the EDO slider. Play a familiar piece and listen to how the sound changes. Whereas just intonation favors harmonies, EDOs favor melodies. Try different sounds; EDOs work well with inharmonic timbres. As you move the EDO slider, the tempering sliders will jump around. The more off-center they are, the further from just intonation you are. High EDOs, 53-EDO to 72-EDO, will sound much like just intonation. Middling EDOs each have their own sound. I think of 11 and 13 as the "anti-just" EDOs, with greatly altered thirds, fourths, fifths and sixths. Low EDOs below 11 produce simple melodies with large steps. Popular EDOs are 5, 7, 10, 15, 16, 19, 22, and 31.

Talking about alt-tonal music is like the proverbial six blind men describing an elephant. Everyone hears it differently, and everyone is drawn to something different. This outline just reflects my own personal views.

Summary Click on the "edit" button in the upper right to see a handy summary:

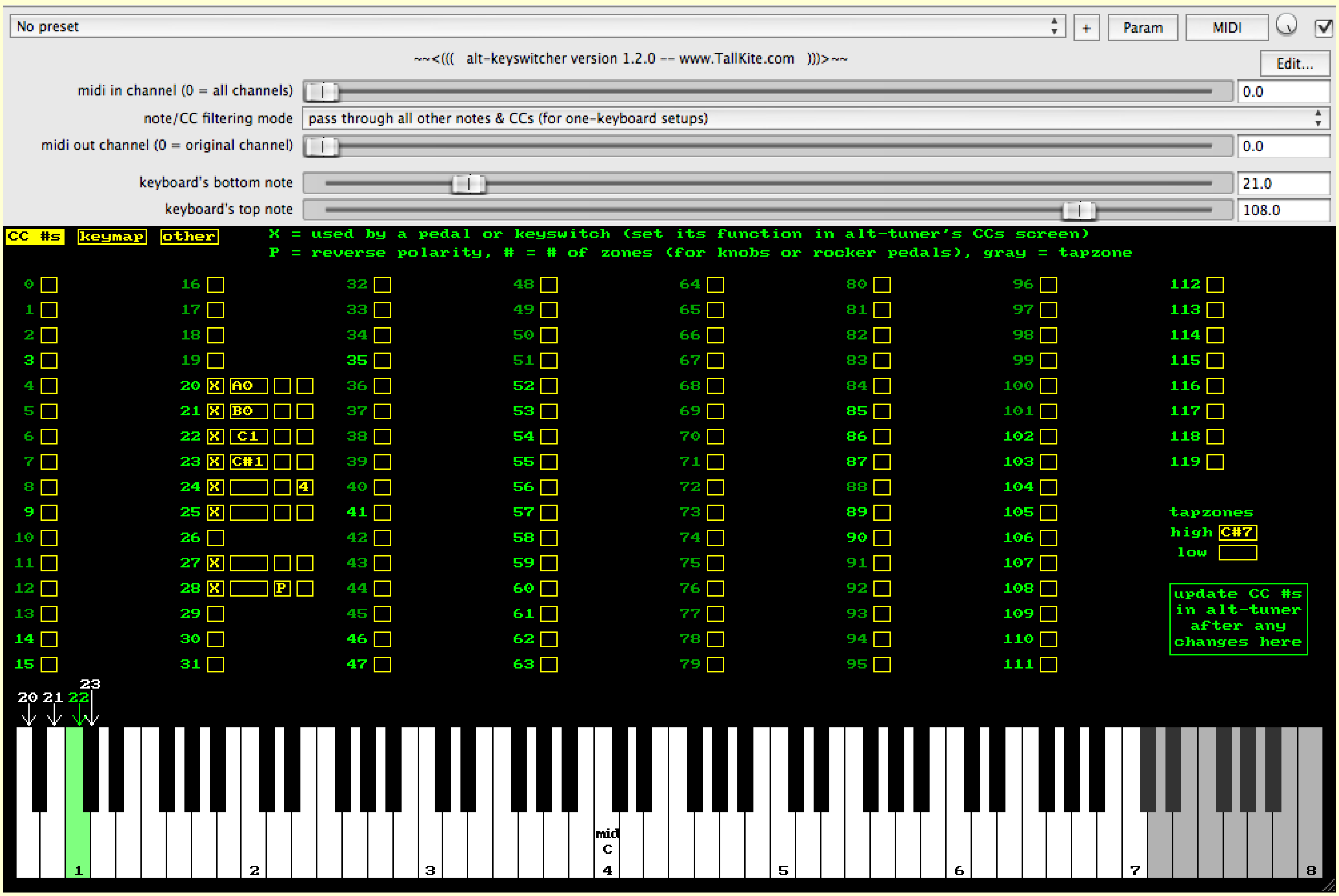
Use alt-keyswitcher before alt-tuner to set up tapzones, pedals, keyswitches, etc.
Right-clicking, shift-clicking, alt-clicking, control-clicking, etc. are all the same.
In the triangular lattice, click on the center note to cycle through the preset scales.
Click on a note to tap it, right-click it to modulate to it.
Click on a custom tuning (1 2 3 4) to select it, right-click it to clone it.
Click on "A-440" to reset/clear, right-click to only clear.
Click or right-click on the yellow rectangle to change screens.
In the "prefs" screens, click on yellow number boxes to increment, right-click to decrement.
Hold down the mouse button to increment or decrement number boxes quickly.
In the "prefs" screens, right-click a slider to reset it.
Control-Z will undo some operations.

w = white = perfect
y = yellow = major
g = green = minor
b = blue = subminor
r = red = supermajor
bg = bluish = diminished/minor
ry = reddish = augmented/major
j = jade = neutral
a = amber = neutral
e = emerald = neutral
o = ochre = neutral
W = wide = widened by an octave
T = tempered
L = large = relatively augmented
s = small = relatively diminished

Chapter 6.3 – Alt-keyswitcher

Alt-keyswitcher sets up tapzones, keyswitches and pedals. Tapzones and keyswitches make extra keys on your keyboard act like buttons or switches that can control alt-tuner. (These physical keyswitches are not to be confused with the virtual keyswitches sent from alt-tuner to your synth.) Without alt-keyswitcher, you would have to use mouse clicks for all tuning changes. With it, you can control alt-tuner while playing music, without having to touch the computer. To get the most out of alt-tuner, you should use either a second keyboard or a midi pedalboard (either the kind with stomp-able buttons made for electric guitarists or the kind with notes made for organists), or even both.

Put alt-keyswitcher immediately before alt-tuner in the effects chain. The first three sliders are useful in multiple keyboard setups, see chapter 6.5. For now, set the 1st and 3rd sliders to "0" and set the 2nd slider to "pass through". Then use the next two sliders to make the keyboard picture match yours (middle-C is marked "mid C"). It's best to do this before you set up your keyswitches. When you play your keyboard, the corresponding key in the keyboard picture should turn green. In both alt-tuner and alt-keyswitcher, yellow items are generally clickable and green ones are unclickable.



Pedals and keyswitches send CC (control change) messages to alt-tuner. Setting up your CCs is a multi-step process. First you tell alt-keyswitcher what pedals and keyswitches you have. Then you tell alt-tuner what those pedals and keyswitches do. The reason for this division of labor is so that when you buy new midi gear and sell old gear, you only have to change one alt-keyswitcher preset instead of hundreds of alt-tuner presets.

CC numbers: Click on the "CC #s" box. You'll see a list of all 120 midi Control Change numbers. Many numbers are already "taken", for example the mod wheel sends CC #1 messages, and the sustain pedal sends CC #64 messages. Verify this by moving these controls; the green CC numbers should flash white. These reserved CC #s are dark green and the ones more likely to be available are bright green. CCs #120-127 are reserved for all sound off, local control, etc. and are not suitable for keyswitches. Experiment with all your knobs and pedals and note any bright green CC #s that might be taken. The pitch bend wheel is not a CC, but can be converted into one. See "Jesusonic" in chapter 6.8.

If alt-keyswitcher uses a CC # that your gear already uses for something else, moving that control can cause tuning changes instead of its intended effect. This is usually but not necessarily to be avoided. For example, if you're an organ player who never uses your sustain pedal, you can choose to use it as an alt-tuner control instead. This choice is made in alt-tuner, so you won't lose the option of using it as a sustain pedal whenever you want. For this reason, you may want to select all the "taken" CCs, for possible later use. For example, your piano sound may not react to the mod wheel. For piano songs, you could set alt-tuner to use the mod wheel to switch between tunings.

Pedals: To set up a pedal, first figure out what CC # it's sending out. Work the pedal and watch for white CC #s. Sometimes your keyboard will let you set the pedal's CC #.

Next select that CC # by clicking the empty yellow box to its right. An "X" will appear in the box, and three more boxes will appear. The first one is a note box; see "Keyswitches" below. The second one is the polarity box and the third one is the pedal zone box; see "CC values" below.

Any time you make changes in alt-keyswitcher's CCs screen, you must tell alt-tuner what you've done by going to prefs/CCs and OKing "update CC #s?". Do this now, and your pedal's CC will appear on this screen. Set the pedal's function (switch, modulate, etc.) and test your pedal. The exact workings of modulators and switchers are controlled by alt-tuner's "modulate" and "switch" pref screens.

Most keyboards have various buttons, knobs and sliders on them. If you don't normally use such a control, then you might want to use it to control alt-tuner. Follow the same procedure as for a pedal to harness it. Not all such controls will output midi; some of them will only control the keyboard itself. Sometimes they will only output midi when the keyboard is put in "remote" mode, which is designed to let the keyboard control a DAW via a different midi port. To determine the output, put Reaper's (or ReaPlug's) included ReaControlMIDI effect in the first slot of your track's effect chain and click "Show Log". Work the control and watch the log. A control may send out a midi message that isn't a CC message. If so, a Jesusonic utility may be able to convert that message into a CC message that alt-tuner can use. For example, the pitch bend wheel's messages can be converted into CC messages.

Keyswitches: If you have a 2nd keyboard hooked up, or you have 88 keys and octaves to spare, you can set up keyswitches that will simulate pedals. Alt-keyswitcher will transform the key presses into CC messages which will appear to alt-tuner exactly like a pedal. These physical keyswitches are not to be confused with the virtual keyswitches sent from alt-tuner to your synth, when using the keyswitch midi mode in the prefs/misc screen.

When you select a CC #, three boxes appear, but only the first one applies to keyswitches. It's a note box which moves a white arrow along the keyboard picture. The arrow is labeled with the CC #. Hold the mouse button down to speed the arrow across the keyboard. Right-clicking sends it backwards. A key can only send one CC #, so the arrow will "jump over" other arrows. To delete a keyswitch, you can send the arrow all the way off the keyboard. Or you can simply unselect a CC # and then reselect it.

Typically keyswitches are assigned to unused CCs. However, a keyswitch can send the same CC # as a pedal; it will function as an alternate pedal. The polarity and zone settings will not affect the keyswitch.

Certain CCs aren't appropriate for keyswitches. For example, suppose you assign the D2 keyswitch to CC #7. That one happens to be the volume CC which affects the loudness of most synths. If that CC is assigned a function in alt-tuner, it will not be passed on, and all is well. But if no function is selected for CC #7, or if the CCs haven't been updated since assigning D2, alt-tuner will pass CC #7 on to the synth. Whenever you play the D2 key, on release it will send a volume CC of 0 to your synth, silencing it! To avoid such problems, use the bright green CCs for keyswitches.

Keyswitches have a higher priority than tapzones. If a keyswitch is in a tapping zone, the keyswitch CC is sent but the tapping CC is not. Remember to update your CC #s before you test it!

Two different keyswitches cannot send the same CC #. For example if you want to set up a 5-key "zone" on your keyboard that does anything other than tap or modulate, you'll have to set up 5 different CC #s with the same usage.

Tapzones: Notes played in the "tap zone" in the upper and/or lower octave of your keyboard will modify the tuning. If you can't spare an octave, tuning taps can come from a 2nd midi controller. Tapzones are shaded in the keyboard picture. You can have a high one or a low one, or both, or neither. Because you can't tap with pedals, you'll probably want at least one tapzone, even if your keyboard is short. If you're only using one keyboard, you'll probably want the bare minimum of 12 keys in a tapzone. If you're using 2 keyboards, you have room for a multi-octave tapzone.

Click the empty box labeled "high tapzone" to define the size of the tapzone. The high end of the keyboard picture will turn gray and the box will show the name of the lowest note in this zone. Click this note name to make the zone bigger, right-click it to make it smaller. To set up the lower tapzone, click the "low tapzone" box.

Update the CC #s in alt-tuner/prefs/CCs, and also make sure the high tapzone usage is set to "tap up". Test your tapzone in the lattice view. Tuning taps are always silent, but you can immediately hear their effect by alternately "playing" tuning taps and then playing the same note in the middle of the keyboard. Tuning taps are inspired by the mandals on the Qanun or Kanun, a Turkish zither.

In alt-tuner, there are 4 tuning options for each note; tapping a note cycles through those 4 options. These 4 intervals can be altered, and others can be added, in the prefs/rows screen. Tap the center note to cycle through the preset scales. Once in 12-ET, tapping any key makes that note the new center note. It also resets the cents offset to zero, providing a quick way to start over in a new key. This method only lets you access 12 of the possible 17 keys. To play in, say, C♯ minor, go to D♭, modulate fourthward (leftward) to B, then modulate fifthward to C♯.

Tapzones can be set to tap up, tap down or modulate. The high tapzone defaults to tapping up, ascending from flattest to sharpest option, wrapping around to the flattest. The low tapzone defaults to modulating; the center note becomes the tapped key. The low tapzone can optionally tap down, sharpest to flattest. The ability to tap both up and down can be handy when tapping through a large number of options, like when EDO-tapping in 72-EDO. Either tapzone can also be set to have no usage, by clicking on the currently selected usage. This provides a handy way to temporarily bypass a tapzone, so that these keys will function as normal keys.

CC values: Some pedals are footswitches, sending a midi CC value of either 127 (all the way on) or 0 (all the way off), like a light switch. Other pedals are footpedals, sending a midi CC value somewhere between 0 and 127, depending on their exact position, like a light dimmer. Use ReaControlMIDI to determine whether your pedal is a footswitch or a footpedal. Any button or switch on your keyboard will usually operate like a footswitch, and any knob, dial or slider will usually operate like a footpedal. Keyswitches send a value of 127 for note-ons and a value of 0 for note-offs, and thus act as footswitches.

Any footpedal, knob or slider can be defined in alt-keyswitcher as a multi-zone control. This is especially useful for rocker pedals. The control's range of motion is divided evenly into a number of zones. As the control travels through these zones, it will output its CC message every time it crosses over into a new zone. As it returns back to the previous zones, it will output a CC message with the next higher CC #. For example, a pedal with 4 zones and a CC # of 24 will output 3 "on" #24 messages as you press it down and 3 "on" #25 messages as it returns up. All 6 "on" messages will have a value of 127. It will also output an "off" message of value 0 immediately after each "on" message. To set up a multi-zone pedal, click on the last box and specify the number of zones. Be sure to also select the following CC # (#25 in this example). CC #119 can't be a rocker pedal because CC #120 is unavailable.

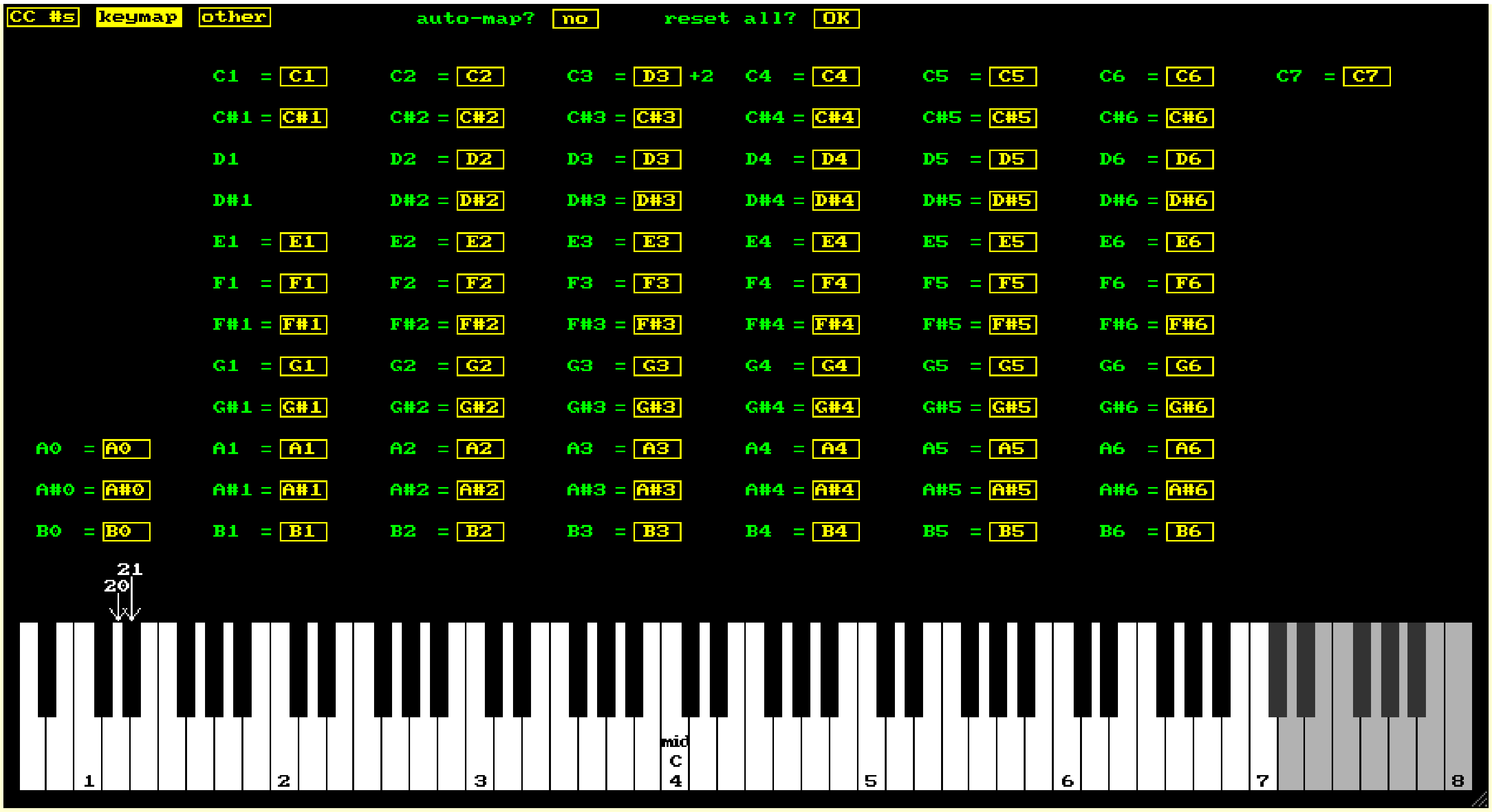
Sometimes pedals don't behave right, sending a value of 0 when down and 127 when up. If this happens, reverse the polarity by clicking on the second to last box, the polarity box, to get "P". If you reverse the polarity of the rocker pedal in the last example, it sends CC #25 messages on the way down and CC #24 messages on the way up.

See also the midi threshold on the "other" screen.

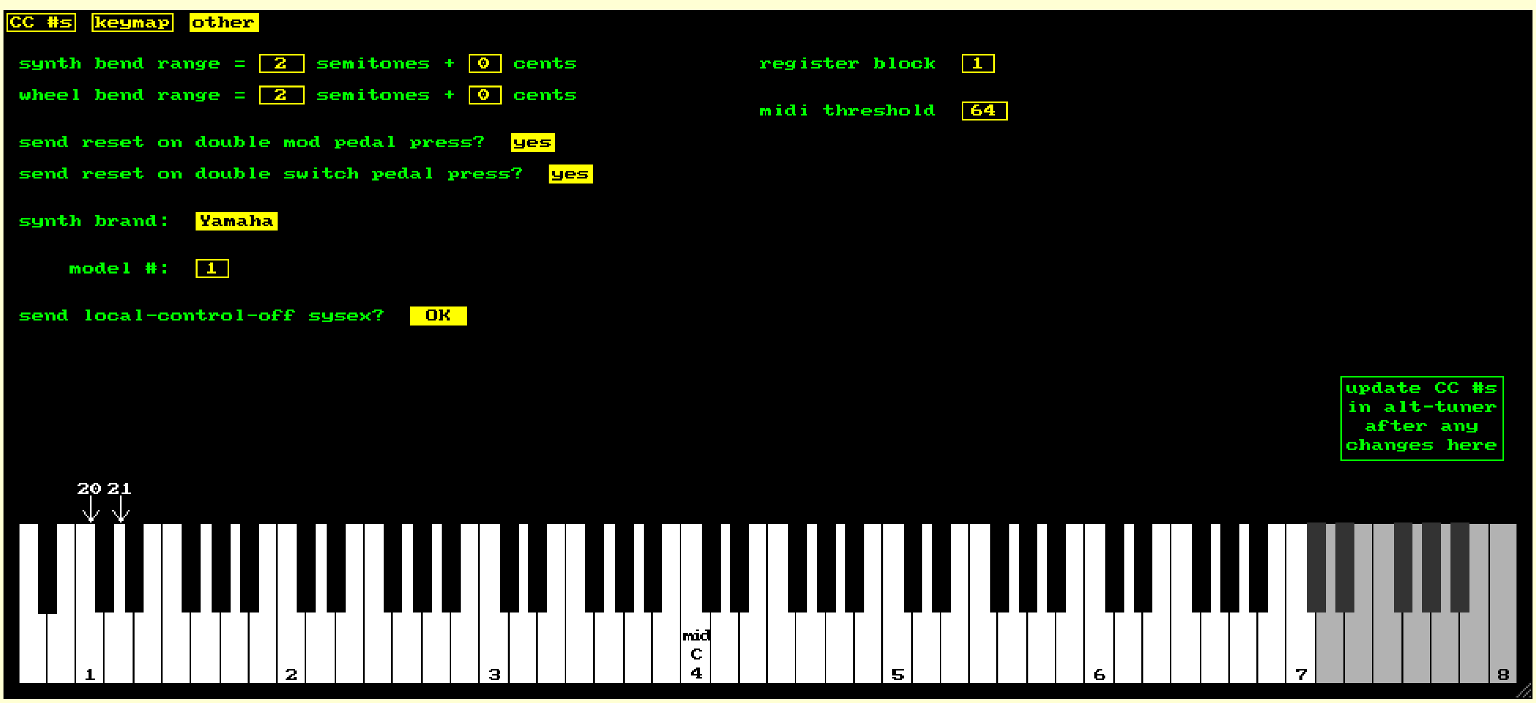
Keymap (advanced): Click on the "keymap" box in the upper left to get here. This screen lets you easily redirect keys to other keys. You can click on the note boxes, or use auto-map. When auto-map is on, play any two notes simultaneously, and the first one played will be mapped to the second one. The keymap won't affect the sound until auto-map is turned off, so that you can auto-map by ear. Don't leave auto-map on when you play, or you'll completely scramble your keyboard! To help avoid this, auto-map is automatically turned off whenever you leave the keymap screen. The main use of keymaps is with fully retunable tunings, see the advanced examples.

You can redirect the keys to any key, even ones not on your keyboard. For example, a 2-octave keyboard could be mapped to a 3.5 octave diatonic scale. Or the white keys of an 88-key keyboard could be mapped to a 4-octave chromatic scale.

In the picture below, C3 is mapped 2 semitones higher to D3. Keyswitches and tapzones have priority over keymaps. Because of the keyswitches for CC #20 and #21, D1 and D♯1 are not mappable. Likewise for C♯7 to C8, because of the tapzone.



Other: This screen has a few other gear-specific parameters to set. After changing anything here, you must update the CC #s in alt-tuner/prefs/CCs for the changes to take effect.



Bend range: Alt-tuner mostly uses midi pitch bend messages to retune. These messages don't contain any actual retuning information; they only say how far the wheel has been moved off-center. Your synth interprets these messages based on its pitch bend range, which specifies how much the pitch is bent when you move the pitch bend wheel all the way up or down. The standard pitch bend range is +/- 2 semitones, which means that G can be bent all the way up to A or all the way down to F. The range can be set as high as 127 semitones. In order for alt-tuner to work right, it is <u>essential</u> that alt-keyswitcher's pitch bend range match your synth's pitch bend range. With alt-tuner and alt-keyswitcher bypassed, use your pitch bend wheel to determine the bend range of your synth. If your synth uses a different range, either adjust its range or adjust alt-tuner's range here. Cents are for fine-tuning the range, 1 cent is 1/100 of a semitone. Wheel bend range sets the physical pitch bend wheel's range, which is usually equal to the synth bend range. You might want to set this range lower for finer control. See also "How retunable is your synth?" in chapter 6.11.

Register block: If you're using multiple instances of alt-keyswitcher and alt-tuner, you can use the register block to control which instance of alt-keyswitcher talks to which instance of alt-tuner. Otherwise, leave this set to block #1. See the end of chapter 6.5 for more info. Alt-keyswitcher automatically writes to this block every time you move a slider in the upper half of alt-keyswitcher or you click anywhere in the lower half. The only exception to this rule is when your clicks are changing the block, in order to avoid overwriting multiple blocks. After changing the block, you can ensure the block is written to by clicking on any blank area of the screen.

Send reset: If "Send reset on double pedal press" is on, simultaneously pressing the first two modulating pedals or the first two switching pedals will also reset alt-tuner's cents offset.

Midi threshold: is the dividing line between "on" and "off" CC values. Changing this will change the trigger point in a pedal's travel.

Send sysex: Alt-keyswitcher will attempt to turn off local control on your synth when you first launch it. It does this via midi CC message #122. Not all synths respond to this message. If you select your synth make, alt-keyswitcher will send a custom sysex message for that make. If you see a model number, click to your model and OK it. See chapter 6.10 for model numbers. (More makes will be added as requested.)

Alt-keyswitcher Presets: Once you get alt-keyswitcher set up, save your settings as a preset (see alt-tuner presets in chapter 6.2). The name of your preset should list all the gear you're using. You may want to save it as the default. Alt-tuner will automatically load all your alt-keyswitcher settings when you first open your Reaper project.

Midi learn: A final way of controlling alt-tuner with your keyboard has nothing to do with alt-keyswitcher. Most DAWs have a "midi learn" feature which lets you easily associate any midi control with any effect slider. For example, you can set up a dial or a pedal to directly temper the fifth while you are playing. Consult your DAW's manual for details.

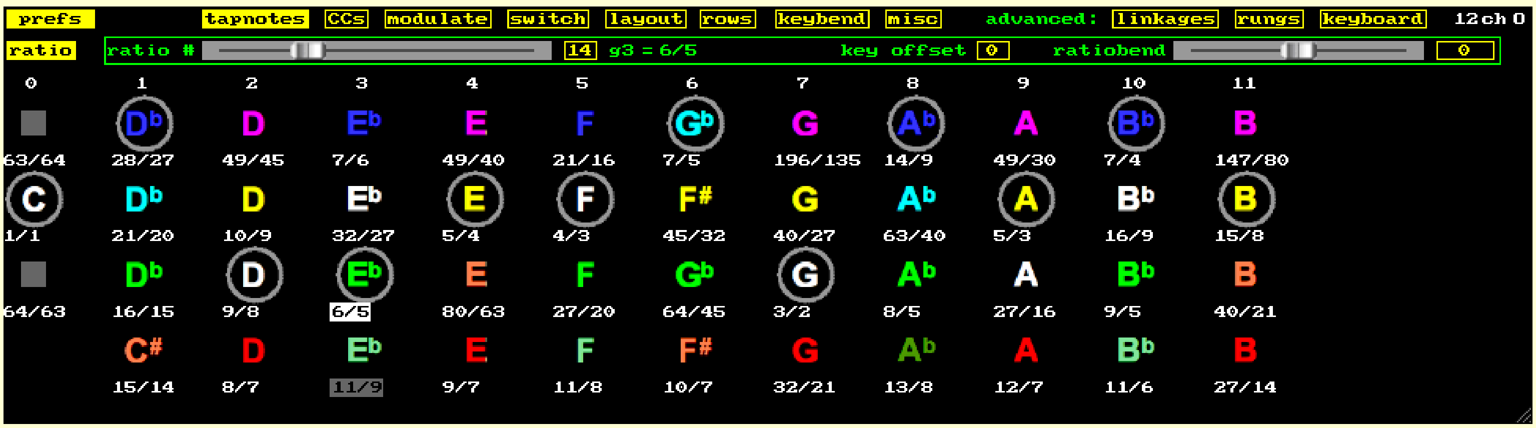
Chapter 6.4 – Basic Alt-tuner Preference Screens

General operation: Any yellow words or numbers in a box are clickable. Yellow numbers can be clicked to increase and right-clicked to decrease. Hold the mouse button down to increase or decrease quickly. They often wrap around, so that clicking up beyond the maximum takes you down to the minimum. Sliders can be dragged like the tempering sliders. You can fine-tune a slider by clicking or right-clicking the yellow number box to its immediate right. Right-clicking on the slider, not the yellow box, will reset most sliders to their default values.

The yellow boxes at the top of the screen (tapnotes, CCs, etc.) are for preference submenus, or screens. Click on a yellow box to select its screen. Click the selected yellow box to return to the previous screen.

There are 8 basic screens and 3 advanced ones. The advanced screens require some knowledge of tuning theory to use. Otherwise one can easily render one's keyboard unplayable (dead keys, duplicate keys, out-of-order scales, etc.)

Tapnotes screen: This screen shows all 45 ratios available via tapping. The currently selected ones are circled in gray. They will be circled in white as you play them. These notes are clickable just like the lattice notes. Clicking an unselected note will select it. Clicking an already selected note will tap it up, selecting the next higher note.

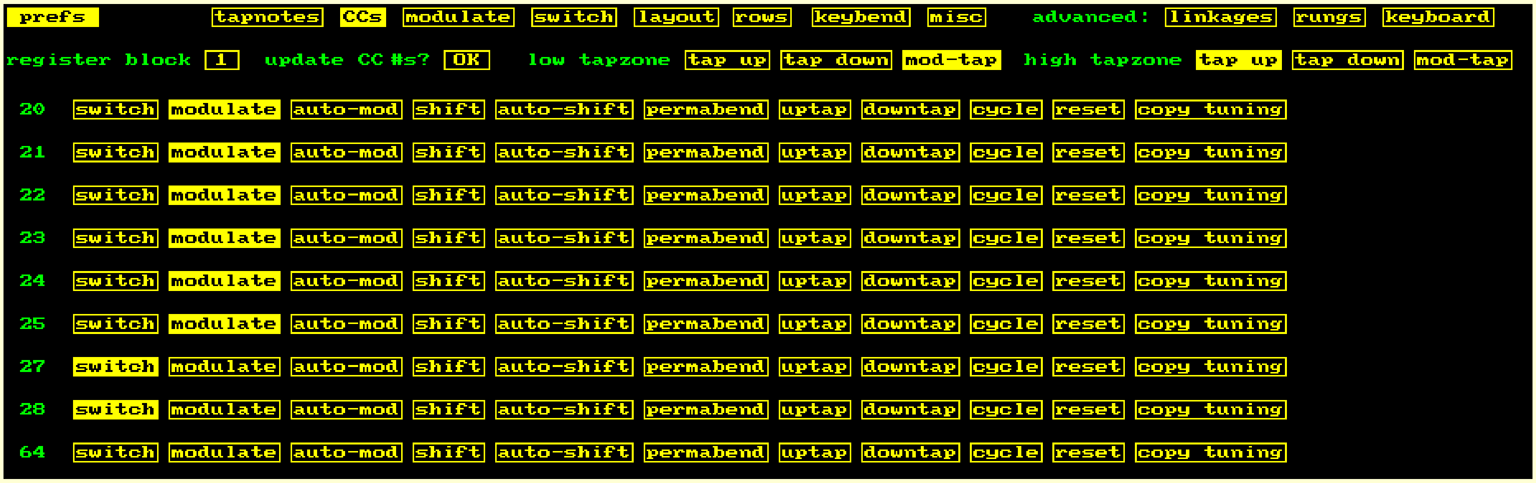


The columns are numbered by keyspan, 0 to 11. Each column contains all the notes with that keyspan. Each note is displayed with its ratio below it. For example, the yellow "E" in the picture above has a ratio of 5/4. Click on the yellow "ratio" box in the upper left to view other information instead: the color & degree of each note (e.g. "y3"), the interval from the tonic in cents (e.g. "386¢"), and the steps from one note to the next, in cents (e.g. "27¢" from the purple "E" up to the yellow "E").

Advanced: The first gray slider allows you to focus on any of the 45 ratios. Clicking on a colored note will also focus on it. The note focused on (the green E♭ in the picture above) has its data highlighted in black on white. Information on that ratio is shown to the right of the slider. You can change that ratio's keyspan offset or ratiobend. The keyspan determines where a ratio "lands" on the keyboard. Changing this will move the ratio from column to column. For example, if you want to be able to play both 9/8 and 10/9 without tapping, you can redefine 10/9 to be a min 2nd. The center note's keyspan can't be changed; its offset is an unclickable green. See also "keyspan offset" in the rows screen. Ratiobending is covered in chapter 6.8. Right-click the ratiobend slider to reset it to zero. If the ratio that is focused on is bent, an asterisk appears next to its data. If any other ratio is bent, its data is highlighted in black on gray, so that you can easily find all bent ratios. For example the jade E♭ in the picture above has been bent.

Advanced: The gray slider starts at the 2nd note, b2, but the ratio number is 4. That's because center note tapping is not allowed (see prefs/misc) and there are several alternate ratios for w1 = 1/1 that are hidden. Alt-tuner indicates these ratios with gray squares in the first column. You can access the hidden ratios with the slider. Changing a hidden ratio's keyspan will unhide it. Turning on center note tapping will unhide all the ratios in the first column.

CCs screen: If you haven't already, set up some CC #s (control change numbers) in alt-keyswitcher, and then update the CC #s here. The CCs screen determines the function of these CC #s.



You should update the CC #s here whenever you change anything in alt-keyswitcher. Those CC #s will appear here in green. When you press or release a pedal or a physical keyswitch, the green CC # will briefly flash white. Note: pitch bend messages are not CCs, but can be converted into them. See "Jesusonic" in chapter 6.8.

When you first access this screen, no functions are selected, and alt-tuner will pass these CCs through unchanged. Once you select a function for a CC, alt-tuner will not pass the CC through, and will instead use it to modify the tuning. You can return a CC to its unselected state by clicking on the selected function. For example, suppose you rarely use the sustain pedal (CC #64), and you have assigned it a switch function. If at times you want to use the sustain pedal as a sustain pedal, just click on "switch" to unselect it. All boxes on that row will be black, as shown above. If you unselect a tapzone's function, alt-tuner will pass through the tapzone notes as normal keys.

If you're using multiple instances of alt-keyswitcher and alt-tuner, you can use the register block to control which instance of alt-keyswitcher talks to which instance of alt-tuner. Otherwise, leave this set to block #1. See the end of chapter 6.5 for more information. Changing the register block automatically updates the CC #s. Changing to a block that alt-keyswitcher hasn't used yet produces a "can't read from alt-keyswitcher" error message. Set alt-keyswitcher's block to match and update here to get rid of the error.

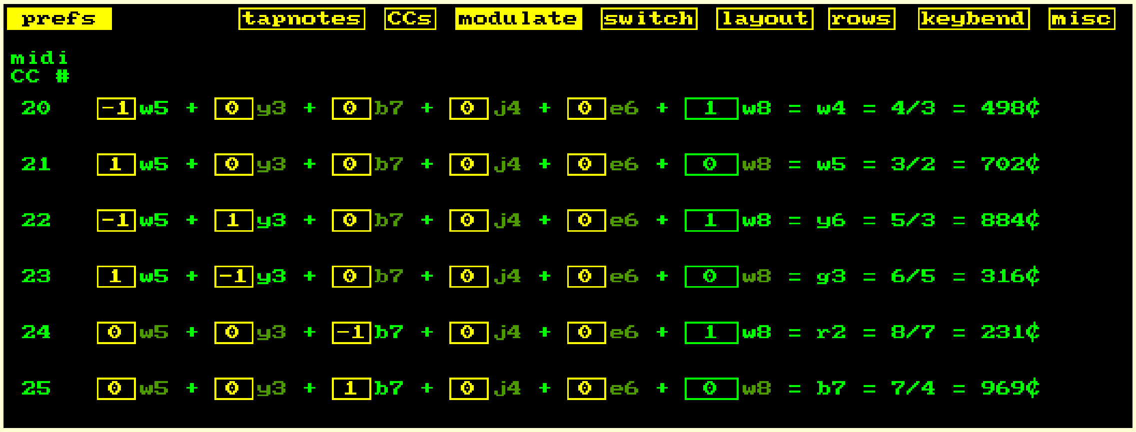
The high and low tapzones are shown on the top line, along with their CC #s and functions. The options are: tap up, tap down, or mod-tap. Tuning taps are covered in the previous chapter under "tapzones". Modtaps modulate directly to the note tapped. Modtapping the D note is exactly like right-clicking the D note in the graph or table views. If you are already in D, modtapping D has no effect.

For non-tapping CCs, the options are: switch, modulate, auto-modulate, shift, auto-shift, permabend, uptap, downtap, cycle, reset and copy previous tuning. Don't be overwhelmed by the possibilities. Many of them are simply alternatives to mouse clicks. Switching and modulating are the most essential functions. They're covered in the next two sections. Auto-modulate, shift, auto-shift and permabend are covered in chapter 6.8, "Advanced Topics". When an uptap or downtap pedal is pressed, every key press becomes a tuning tap. This allows you to make tuning taps without having to move your hand to the far end of the keyboard. Pressing either one also reverses the direction of regular tuning taps in the tap zone, as does pressing a permabend pedal. Cycling does what clicking on the lattice's center note does: it cycles you through the 4 preset scales. Resetting resets the cents offset from A-440 to 0¢, as if you had clicked on "A-440". Both cycling and resetting can alternatively be done by tapping the center note in a tapzone. Tuning-copying makes your tuning the same as the previous tuning you were on, as if you had right-clicked the previous tuning's yellow number. This function only works if you have switched tunings since launching alt-tuner, because otherwise there is no previous tuning to copy.

Inevitably over the years, you'll buy new midi hardware and sell your old hardware. This will often require you to change your CCs around in alt-keyswitcher. When you do, alt-tuner will reassign the functions you've selected to the new CC #s. In other words, alt-keyswitcher keeps a list of what CC #s are in use, and alt-tuner keeps an independent list of what functions you want to assign to those CC #s. The first list is saved in alt-keyswitcher's presets, and the second in alt-tuner's presets. Reaper projects function as an "uber-preset" that save all choices for all effects. See

"Customizing alt-tuner" for an easy way to make your pedal function choices permanent.

Modulate screen: This screen determines what the modulating pedals and keyswitches do. First set up some modulators in the CCs screen; otherwise this screen will be blank.



Every CC that functions as a modulator is listed on the left. When you use a modulating pedal or keyswitch, this green CC # will briefly flash white. Click on the yellow boxes to specify what interval that pedal or keyswitch will modulate by. The lattice's center note will change by the modulating interval, and the relative scale will be unchanged. The interval is expressed as a sum of rung factors. The green rung names will be brighter for nonzero entries. Because the interval is automatically octave-reduced, the octave rungs are an unclickable green. The modulating interval is shown on the right as a color/degree combination, as a ratio and as cents.

The intervals usually come in complimentary pairs, one undoing the other, with the more fourthward one coming first. The default intervals work along the 3 axes of the 7-limit lattice. Alt-tuner defaults to y6 & g3 instead of g6 & y3, to make it easier to modulate through a g1 comma pump (w5 – w5 – w5 – g3 vs. w5 – w5 – w5 – w5 – g6). Likewise, r2 & b7 is used instead of z3 & r6, for modulating through r1.

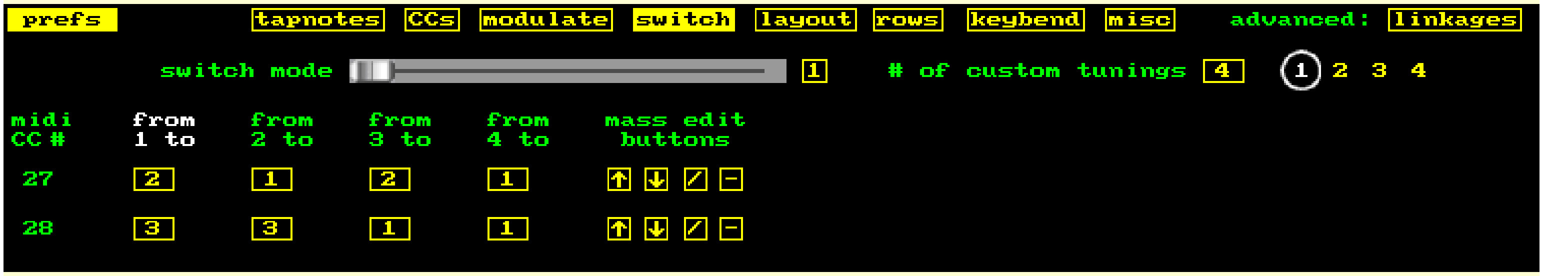
This screen can double as a handy ratio calculator and interval finder. Expressing intervals in rung-factor format is an important concept in JI. If this concept is at all unclear, you may want to spend some time on this screen.

Tempering a rung will also temper the modulating interval. The interval will appear with a "T" and the cents will have decimal places.

Advanced: When you modulate while on the lattice screen, alt-tuner draws an arrow on the screen from the ratio you're modulating by to the center note. If the modulating ratio isn't in the lattice, alt-tuner will use the nearest ratio instead. For example, suppose you set up CC #24 to modulate by an emerald third e3 = 39/32 = 342¢. Because there is no emerald 3rd in the lattice, whenever you use CC #24, alt-tuner will draw an arrow from the jade third j3 = 11/9 = 347¢.

Advanced: Because of the deep purple microcomma 2401/2400 = 0.4¢, there's essentially only 4 septimal planes in 7-limit JI. So if you use a rocker pedal to modulate septimally, you might want to set it up with 4 zones.

Switch screen: Here you can set up how the switching pedals (and/or keyswitches) move you from one tuning to another. If you haven't already, set up 2 switchers in the prefs/CCs screen.



When you press or release a switching pedal, the green CC # on this screen will briefly flash white. Move the slider on the upper left, and notice the effect on the 2 x 4 array of yellow boxes below. They show you where each pedal takes you to when you're in a particular tuning. Five example modes are already set up for you. Here's what the two switching pedals do in each mode:

switch mode	first pedal's action	second pedal's action	matrix rows
#1	go to tuning #1 or #2	go to tuning #3 or #1	2 1 2 1 and 3 3 1 1
#2	cycle through the first 3 tunings forwards	cycle through them backwards	2 3 1 1 and 3 1 2 1
#3	go to tuning #1 or #2	go to tuning #3 or #4	2 1 1 1 and 3 3 4 3
#4	cycle through the first 3 tunings	go to tuning #4	2 3 1 1 and 4 4 4 1
#5	cycle through all 4 tunings forwards	cycle through them backwards	2 3 4 1 and 4 1 2 3

Switch modes are related to song structure. In these examples, the first two modes are for a song with three sections that requires a different tuning for each section. Suppose the A section is the verse, the B section is the chorus, and the C section is the bridge. The first mode would be best for a song that goes AB-AB-C-AB. For a song that goes A-B-C-B-A-B, the second mode works better. Switchmode #3 would be for a song with 4 sections that goes AB-AB-CD-CD-AB. Switchmode #4 is for a song that goes ABC-ABC-D-ABC. Switchmode #5 is for a song that goes ABCDCBA.

There are 8 switch modes. Set the switch mode to 6 and click on the 2 x 4 switching array to create your own pattern.

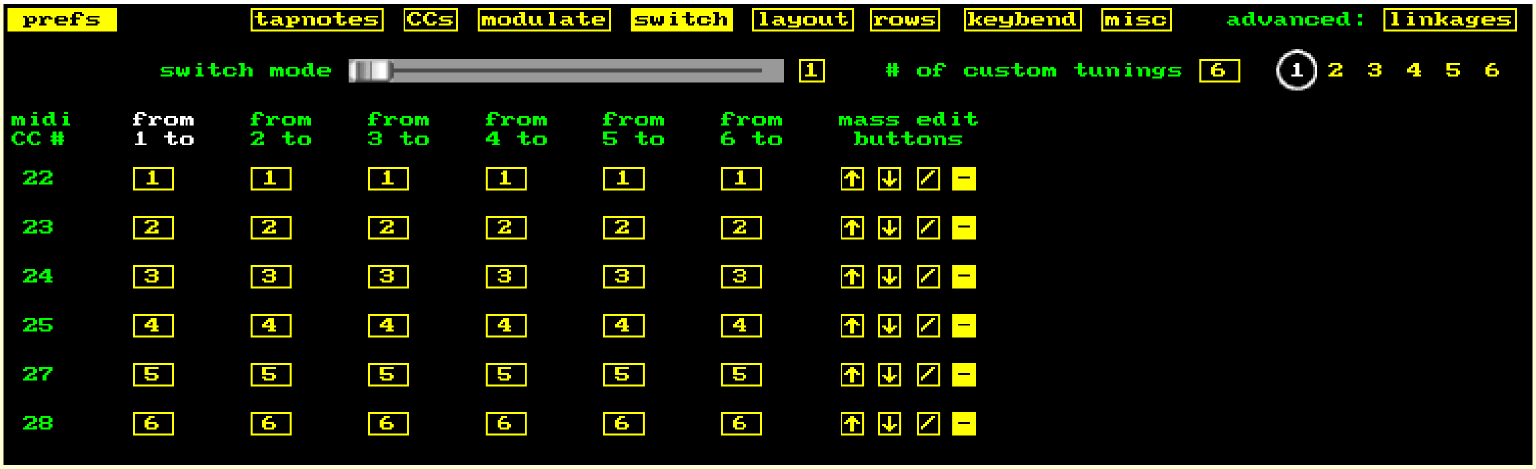
You can have up to 30 custom tunings/temperaments to switch among. Click on the "# of custom tunings" box to add tunings, right-click it to remove tunings. Warning: if you remove tunings and then add them back, you'll probably alter the current switch mode. You can set up as many switchers as you want in alt-keyswitcher, but alt-tuner will only recognize the first 30.

To get more than 8 switch modes or more than 30 custom tunings or switchers, see "Customizing alt-tuner".

Switching pedals are great for performing complex pieces with ease. I use tuning taps and modulating pedals while composing. Once a song is complete, I set up an alt-tuner preset and/or a Reaper project for that song with the appropriate custom tunings and switching mode. Then I can play the song using only the switching pedals.

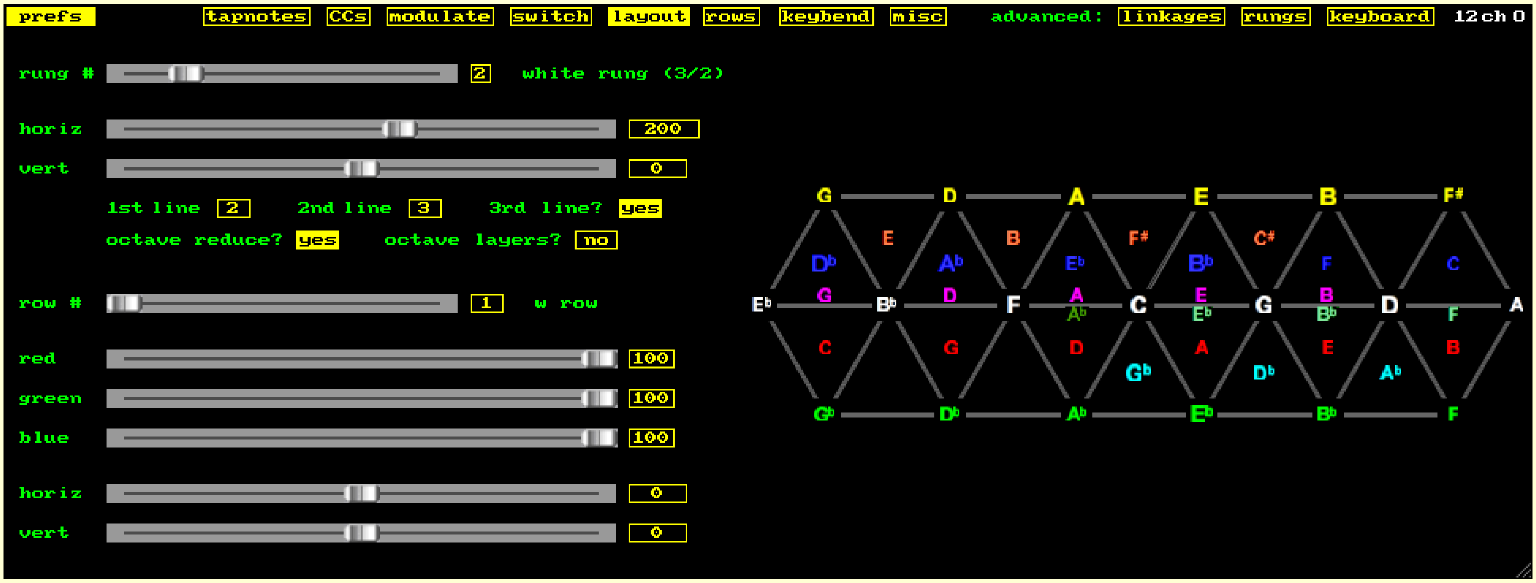
Some things in alt-tuner are switchable, like the key or the scale. Some things like the lattice are not switchable; they are the same for all custom tunings. Everything controlled by the preference screens is unswitchable, except for what's on the linkages screen. The flowchart on the last page of this manual shows what's switchable and what's not.

Advanced: With lots of switchers and tunings, clicking on the yellow boxes can get tedious. The mass edit buttons are designed to speed up the process. Click on the up-arrow button to increase all entries in that row by 1, wrapping around as needed. Click on the down-arrow to decrease them. The other buttons' symbols are like a miniature graph of the row values. Click the button containing a "/" symbol to make the numbers in that row run 1-2-3-4. Such a switcher would actually be useless, because it will never switch tunings. However, if you now click the up-arrow button, you'll get 2-3-4-1, which cycles forward. Click the down-arrow button to create a switcher that cycles backwards. Click the button containing a "-" symbol to make the numbers in the row all the same. The first row will become 1-1-1-1, the second row will become 2-2-2-2, etc. When the row conforms to either of the last two buttons, that button will light up. To use many keyswitches to switch directly to many custom tunings, use the "-" mass edit button to set up your switching array like this:



Advanced: When you switch from one tuning to the other, the center note and/or the cents offset may change. Alt-tuner interprets this as a form of modulating. For example, switching from a tuning centered on C +0¢ to one centered on G +2¢ is interpreted as modulating by 702¢, which happens to be exactly a white fifth. If you're on the lattice screen when you switch, alt-tuner will draw an arrow on the screen from the white fifth to the center note. Often the difference between the old tonic and the new tonic won't be exactly equal to any of the ratios in the lattice. In this case, alt-tuner will instead use the nearest ratio to draw the arrow. For example, switching from C +0¢ to G -4¢ would also be interpreted as a white-fifth modulation, but from C +0¢ to G -15¢ would be a yellow-fifth mod. Because the smallest interval in the default lattice is r1 = 27¢, switching between C +0¢ and C +27¢ (or even C +14¢) is interpreted as modulating by an r1. This is true even if center tap is off and r1 is invisible (see prefs/misc for more on center tap).

<u>Layout screen</u>: This screen lets you modify the lattice's appearance.



Start by selecting a rung by moving the "rung #" slider. Each rung type corresponds to a direction in the lattice; for example white is sideways, yellow/green is NE-SW diagonal, etc. You can control the rung's length and direction with the "horiz" and "vert" sliders. The scaled-down lattice on the right shows the result.

This lattice is the same as the big one on the lattice screen, with a few minor differences. Unselected notes are not gray, and they are almost as big as the selected ones. If center tap is not allowed, alternate center notes are displayed anyway, but they are unclickable. The center note is also unclickable, except for right-clicking it to cycle. Cycling to 12-ET makes not a circle of fifths but an all-white lattice. The lattice is fully clickable in every other way. The lattice reflects the current custom tuning and may change when you switch tunings.

The "1st line" and "2nd line" options determine which rungs are used in the "base plane" of the lattice. The default is white and yellow/green, with blue floating above this plane and red beneath it. For a septimal-centric lattice, set yellow/green's vertical to 67 and blue/red's vertical to 173, and set "2nd line" to 4.

The "3rd line" option determines whether the lattice has triangles or parallelograms. For a rectangular lattice, turn this off and set the yellow/green horizontal to zero by right-clicking it.

Octaves are by default invisible because the clear rung has zero length. If you make octaves visible, you get an "octave lattice". There will be multiple "C" notes, one for each "C" on the keyboard. The exact number depends on how you have set up your keyboard in alt-keyswitcher. The "octave reduce?" and "octave layers?" options are only useful for octave lattices. For more on these options, see "octave lattices" in the "Advanced Topics" chapter.

Each rung adds a dimension to the lattice. The white, yellow and blue rungs create a 3-D lattice of tetrahedrons. Higher rungs create 4-D and 5-D lattices. See chapter 3.6 for advice on keeping the lattice readable.

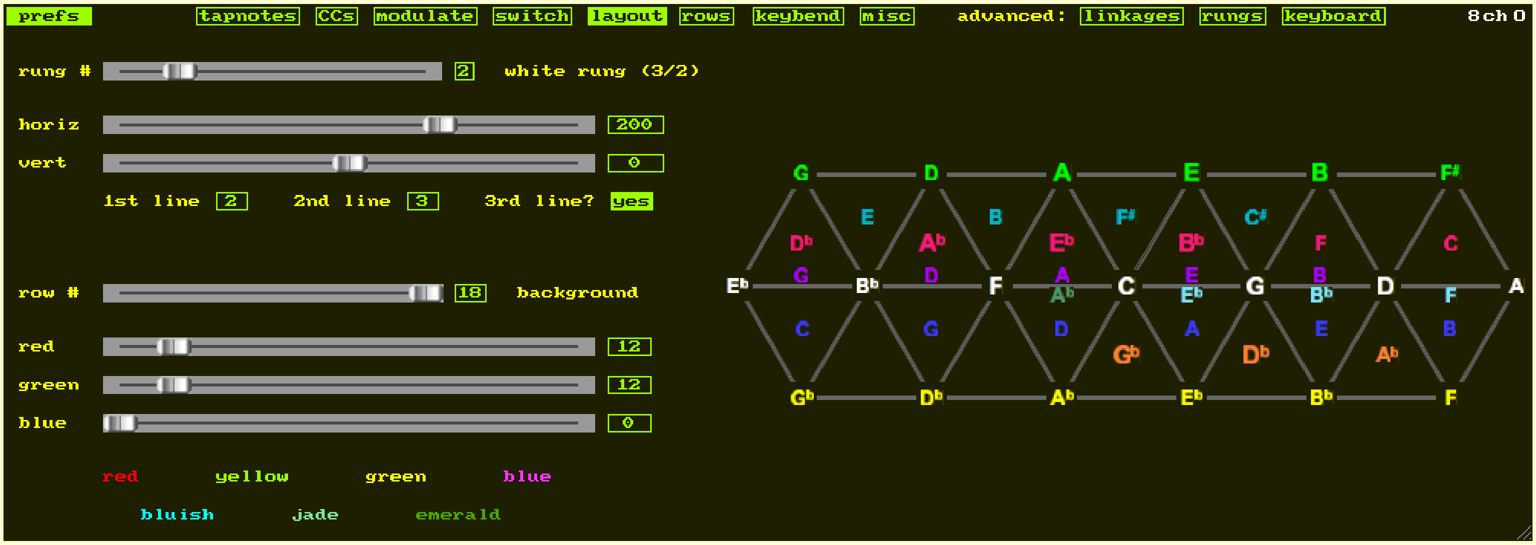
Select a lattice row with the "row #" slider. Use "horiz" and "vert" to shift an entire row around. The purple row (which is actually the bbg row) is shifted this way. See chapter 3.4 for more about purple. The three color sliders control the color of the letters in the lattice. Right-clicking these sliders will set them to the midpoint, 50%.

Moving the row # slider further right accesses the standard on-screen colors. Adjusting yellow will have an immediate effect on the menu buttons. It will also affect the graph lines and table rows for major intervals. It won't affect the yellow row, but it will be the default color of future yellow rows. Here's what the colors are used for, besides future rows:

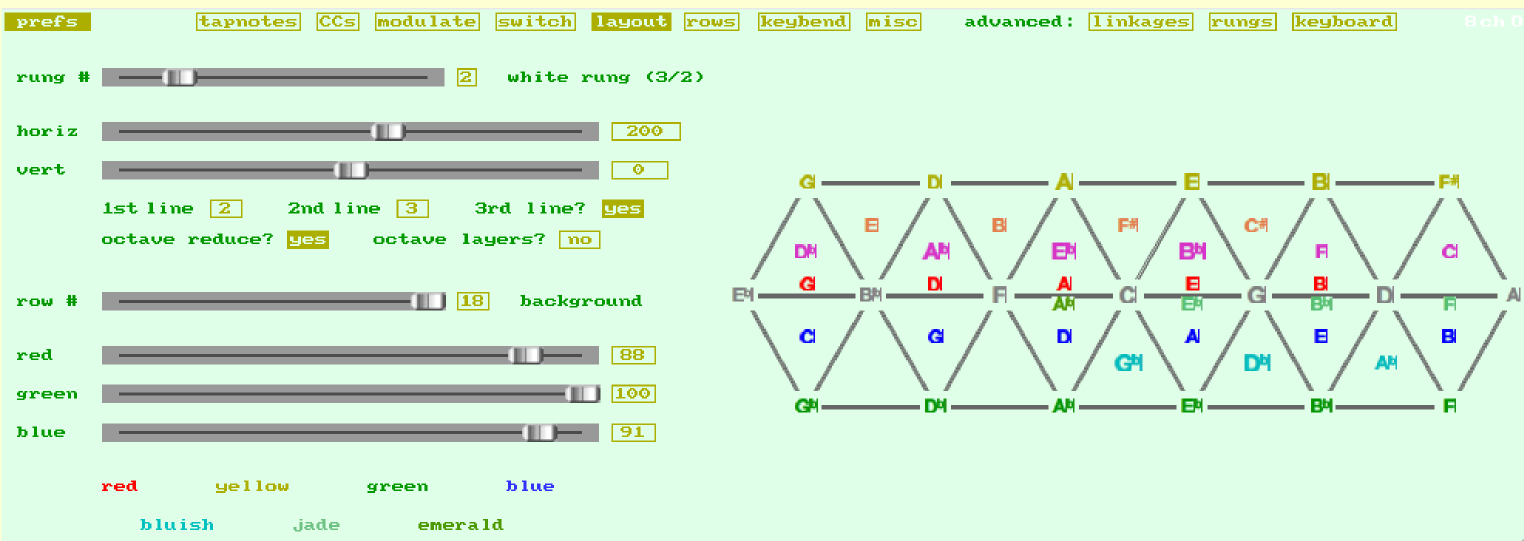
standard red: augmented graph lines and table rows, red square for dropped notes
standard yellow: major graph lines and table rows, clickable items
standard green: minor graph lines and table rows, unclickable items
standard blue: diminished graph lines and table rows
standard bluish: interval readout, linkage warnings
standard jade: interval readout

standard emerald: unclickable items in the modulate and linkage screens
background: the background color of the screen for the entire program

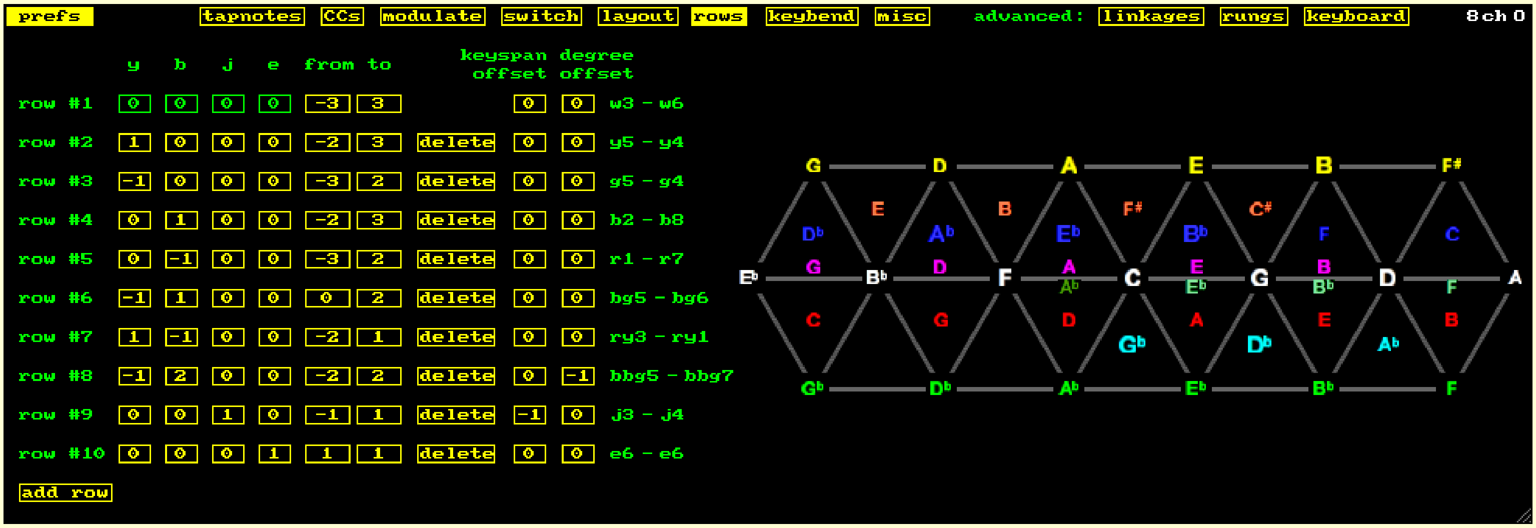
Those with color vision deficiencies can adjust the colors for easier discrimination. See the customizing chapter for an easy way to make these changes permanent. Here's a screenshot using customized colors:



You can change the background color too:



Rows screen: This screen controls the contents of the lattice. Lattice rows can be lengthened, shortened, added or deleted, creating new alternatives to tap to. The scaled-down lattice on the right shows the result. See the previous section on the layout screen for differences between this lattice and the big one.



There's a column for each rung except the white rung and the octave (clear) rung. The numbers in these columns are rung factors indicating which colors are present in each row. The first row is always the white row, which has all zeros in an unclickable green. The yellow row has a "1" in the "y" column, and the green row has a "-1". The rung factors can range from -9 to 9.

The "from" and "to" columns control the length of the row. The first and last ratios are shown on the far right. The white row runs from the white 3rd to the white 6th, 7 notes connected by 6 rungs. The "from" and "to" are measured from the row's midpoint. For a midpoint ratio, the sum of the rung factors is zero, not including the octave rung. The yellow row's midpoint is y6 because y6 is 1 yellow rung minus 1 white rung. Example midpoints are w1, y6, g3, b3, r6, bg5, ry4, j6, a3, e2, o6, etc. The sum of any two midpoints is another midpoint, so the deep yellow midpoint is y6 + y6 = yy4. If the "from" is less than -3, the first ratio (shown on the far right) will be small, written with a "s". If the "to" is more than 3, the last ratio will be large, written with an "L". If the "to" is more than 10, the last ratio will be double large, written LL. Triple large is L3. The "from" and "to" can range from -99 to 99.

Rows can be added or deleted with the appropriate buttons. When you add a row or change a row's rung factors, alt-tuner automatically assigns it a color based on the standard colors. You can change both the standard colors and the actual row color in the layout screen.

If you delete a row by mistake, click the "add row" button and it will reappear at the end of the list. If you haven't deleted any rows, adding a row creates a white row. You'll see a redundant ratio warning, because it overlaps the existing white row. Redundant ratios are not a problem because any duplicate ratios only appear once. Set the new row's color to whatever color you want and the warning will disappear.

The 47 default alt-tuner ratios are merely those that I personally consider to be useful ratios. You may prefer different ones, for example Harry Partch's 43-note scale. You may prefer a higher or lower prime limit. You may prefer more or fewer tapnotes, or different neutral intervals. You can use the rows screen to get exactly what you want. For example, if you want only 3 tapnotes per key, just delete the bbg, j and e rows, and set the ry row's "to" to 0. You can have up to 1000 ratios in 100 rows, and even more if you customize alt-tuner.

The keyspan offset affects where a row's notes "land" on the keyboard. The jade row has a keyspan offset of -1 in order to keep the tapnotes to 4 per key. As a result, every jade ratio in the tapnotes screen has a key offset of -1. The "key offset" number in the tapnotes screen is the sum of both the row's keyspan offset and the individual ratio's key offset.

The degree offset will affect the degree of every note on the row, changing for example a C♯ to a D♭. The purple (bbg) row's degree offset is -1 because otherwise 49/40 = 351¢ would be a fourth, not a third. (Purple is covered in chapter 3.4.) Again, the jade and purple offsets are just my personal preferences; feel free to change them.

Once enough rows are added to fill up the screen, adding another row will create a new page. This page will have the new row and a "nextpage" button. This button will cycle you through the pages. The row numbers on the far left will help you keep track of the pages. Speaking of row numbers, the rows can come in any order, except that the first one is always white. The first row can't be deleted and always includes the w1 = 1/1. Because w1's keyspan can't be changed, the first keyspan offset box only changes the other white ratios.

Ratios with a keyspan of 0, like b8 or r1 in the picture above, will automatically be filtered out from the lattice and will appear as gray squares in the tapnotes screen. Go to the prefs/misc screen and allow center note taps to change this.

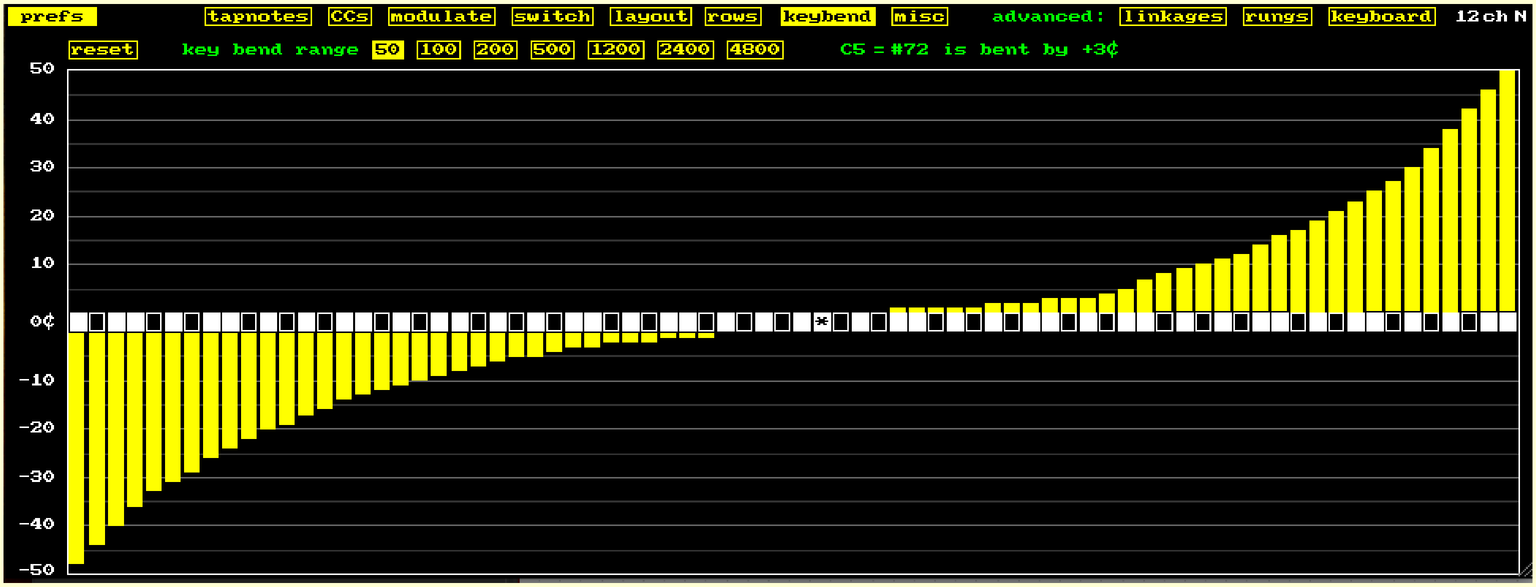
Advanced: Adding a row makes an exact copy of the last row deleted. You can use this fact to move a row to the end of the list by deleting it and then immediately adding a row. If you delete several rows and then add several rows, the first new row will be a copy of the last deleted row, but the other new rows will default to white rows of length 5. Note that the row's horizontal/vertical offset and color are also preserved. If this isn't what you intended, go to the prefs/layout screen after adding a row to reset the row offset and color. Because the color will be automatically altered if you change any rung factors, it's usually only the row offset that needs correcting. The only default row with an offset is purple. So as a general rule, if deleting and adding rows, don't delete purple last.

Advanced: There can be more than one row on the screen of a given color. Lattice rows with a gap require two rows on the screen: A yellow row from -2 to -1 and another yellow row from 1 to 3 would make a lattice row that runs y5-y2 and y3-y4, skipping y6. If two rows overlap, any duplicate ratios only appear once. Alt-tuner defines duplicate ratios as ratios that have the same keyspan and are within 1¢ of each other when untempered. Note that two completely different ratios differing by a microcomma (see chapter 3.2) would be considered duplicates. In this case the sharper one is filtered out. For example, create a reddish-red row from 0 to 0, which creates rry2 = 60/49 = 350.6¢. That ratio will appear in the lattice alongside bbg3 = 49/40 = 351.3¢ because they have different keyspans (rry2 = aug2 and bbg3 = maj3). But if you set the rry row's keyspan offset to 1, both rry2 and bbg3 will have the same keyspan, and only rry2 will appear in the lattice. Alt-tuner will display the message "1 redundant ratio" to indicate that a ratio has been filtered out.

Advanced: Deleting a row deletes its horizontal/vertical offset and its color (both set on the layout screen), as well as its keyspan and degree offset. However, setting the "from" greater than the "to" removes the row without deleting it. Such an "empty row" keeps its horiz/vert offset, color, keyspan & degree for possible later use.

Advanced: Removing rungs on the rungs screen doesn't delete a row that use that rung, it just removes the rung from the row. For example, go to the rungs screen and set the # of rungs to 5. Row #10 changes from emerald to white. Set the # of rungs back to 6 and row #10 changes back to emerald. There can be a slight issue when removing rungs, so it's usually best to have your rows run in order from low prime limit to high prime limit.

Keybend screen: This screen lets you bend each key individually. Keybends allow you to create stretched tunings to accommodate the natural inharmonicity of strings. Keybends also provide the freedom to create virtually any tuning you can imagine, including the non-octave-consistent ones found in gamelan or mbira music.



You create the keybends by drawing a curve on the graph with the mouse. The curve appears as a bar graph, one bar per key. The bend is rounded to the nearest cent. If you mouse over the graph, the bend for the key you're hovering over is shown on top in green. The key's name is the standard name, and is not affected by the misc/keyboard screen. The key's midi note number is also shown.

In certain midi output modes, octaves are "locked" to 1200¢. There will be a warning in the upper right: "octaves locked by ____ mode". Only the central octave of the graph will be used. Its bars will be yellow and the other octaves' bars will be green. The yellow bars will affect all octaves and the green bars won't affect any octaves. Midi output modes are covered in the next section, prefs/misc. The central octave is defined in the prefs/keyboard screen. No matter what the output mode is, the graph and table views are only affected by the keybends in the central octave.

The keyboard is represented by the central row of black and white squares. Middle-C is marked with a "*". Clicking on one of the black and white squares increments the bend, right-clicking decrements it. Hold to autorepeat. See "minimum/maximum decimal" in prefs/misc for how to bend keys by fractions of a cent, and how the keybend's cents is rounded off.

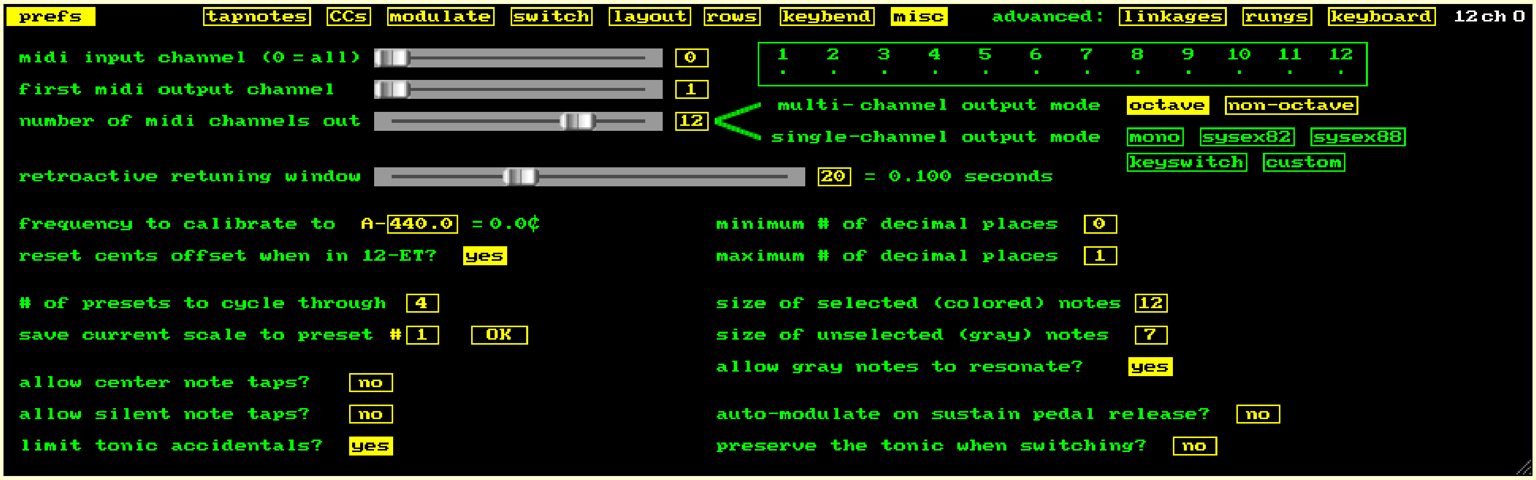
The range of the keyboard is determined by alt-keyswitcher. See "octave lattices" in the "Advanced Topics" chapter. As you play the keyboard, the keys you play will turn red, as will their bars in the bar graph. The "reset" button removes all the keybends. Right-clicking anywhere in a key's column other than the black and white squares resets that key's bend to zero. Right-click-and-drag to quickly reset part of the keyboard.

You can set the vertical range of the graph to be 50¢, 100¢, 200¢, 500¢, 1200¢, 2400¢ or 4800¢. On the highest setting, you can bend any key a full four octaves. If you decrease the range, any keybends outside the new range will be automatically clipped to fit.

While ratiobend bends one of the key's tapnotes in all octaves, keybend bends all the key's tapnotes in one octave. You can also create keybends with the pitch bend wheel and a permabend pedal. This lets you fine-tune each key by ear. Permabending is covered in "Advanced Topics".

Keybend is applied "on top of" all the other bends – tempering, ratiobend, EDOtapping, etc. If you want to retune solely with keybends, set the EDO slider to 12-EDO. Alt-tuner distinguishes between 12-EDO and 12-ET. If you cycle or switch to 12-ET, keybends are not applied. This lets you compare keybent 12-EDO with non-keybent 12-ET. Keybends are not switchable; there is only one keybend graph for all custom tunings. To compare two keybend graphs, see "combining radically different tunings" in the "Advanced Examples" chapter.

Misc prefs screen: This screen sets miscellaneous preferences.



midi channels: If you're unfamiliar with midi, see the "Basic Midi Guide" in chapter 6.10.

The "midi input channel" controls which channel of the incoming midi stream alt-tuner "listens to". If it's set to 0, all 16 midi channels are listened to. The "number of midi channels out" slider controls how many channels on your synth alt-tuner "talks to". This slider determines which output modes are available, see below. The "first channel" slider controls exactly which channels alt-tuner will talk to. The channels start with the first channel and wrap around to 1 if necessary. For example, sometimes midi channel 10 is hardwired to be drums. If you avoid this channel by setting the first channel to 11, alt-tuner will send your 12-channel output to channels 11-16 and 1-6.

To the right of the "midi input channel" slider is the midi channel monitor. Its format depends on the output mode. It shows which channels are being talked to and which notes or how many notes are currently being sent to that channel.

midi output modes: There are seven output modes:

multi-channel polyphonic via pitchbends, octave-equivalent (octaves must be exactly 1200¢)
multi-channel polyphonic via pitchbends, non-octave-equivalent (octaves can be stretched)
single-channel monophonic via pitchbends
single-channel polyphonic via sysex82 (used by Xen-Arts synths)
single-channel polyphonic via sysex88 (used by many Roland keyboards)
single-channel polyphonic via virtual keyswitches (used by Kontakt)
single-channel polyphonic via custom sysex

The simplest and safest polyphonic mode is the default 12-channel octave-equivalent output. In this mode, each midi channel handles an entire pitch class. An example of a pitch class is all the "D" notes in every octave. You have the option of "compressing" the output to fewer channels, since it's rare that all 12 pitch classes are played at once. For soft synth users, this allows you to minimize CPU usage. For hardware synth users, this allows you to share one synth's tone generator between two players, or to convert several midi tracks into audio at once.

In octave-equivalent mode, if there are enough channels, each pitch class is always sent to a specific channel. "A" is always channel 1, "B♭" is always channel 2, etc. This prevents sounds with a lengthy decay, like synth pads, from "getting their tails bent" by the notes that follow. But if there aren't enough channels, the output is compressed, and the channels are allocated on the fly according to what notes are played, first come first served. If more pitch classes are played than there are channels, the least-recently played pitch class is dropped. As a warning, that pitch class will be outlined with a red square in the lattice, graph and table views. It'll also appear on the far right of the midi monitor as a red note name. To see this warning in action, set the "number of midi channels out" slider to 2 and play a triad. This warning is different from overbent notes, which are dimmed. Red squares disappear when that note's key is no longer held down. Occasionally one may persist. To clear the red squares from the lattice or graph, right-click the "A-440". This will also clear the white circles and dimmed letters. Only one red square or dimmed note is displayed at a time, no matter how many notes you drop or overbend.

In octave-equivalent mode, alt-tuner can't stretch octaves via pitch bending, since the two notes of an octave are going to the same channel. As a result, the stretch slider is restricted to 1200¢. To freely stretch octaves, use non-octave-equivalent mode. In this mode, each channel handles only one note, not an entire pitch class. The channels are allocated on the fly. You can only play as many notes as you have channels, so the maximum polyphony is 16. If too many notes are played, notes will be outlined in red and dropped.

In multi-channel modes, alt-tuner uses round robin channel allocation. The next channel to be used is always the channel that was freed up the longest time ago. This helps prevent older notes from getting their release tails bent by newer notes. As a side effect, playing one note repeatedly will send that note out to each channel in turn. This is useful for checking by ear that every instance of your synth is set identically.

The current number of channels and mode are shown in the upper right corner of the screen as "12 ch O", "8 ch N", "1 ch M", etc. You can right-click the O in "12ch O" to change it to N and back to O. This is a shortcut intended for advanced users who mostly work with non-octave scales.

Setting "# of midi channels out" to 1 replaces the octave/non-8ve choice with the mono/sysex82/sysex88/keyswitch choice. In these modes, notes will never be dropped. An additional choice, "custom", is for specific synthesizer models. These sysexes are add-ons sold separately. See the hardware section of the "Hardware & Software Issues" chapter for details.

When using mono mode, be sure to set your synth to monophonic. If you hold down one key and repeatedly press and release a neighboring key, you should hear a realistic trill. Mono mode is especially useful for guitar-to-midi converters. Set your converter to 6 channels of mono output (one for each string). Set alt-tuner to mono mode and set the midi input channel to 0 = all.

In all single-channel modes, all midi is output to its original input channel, offset by the first channel. In other words, if the input channel is set to 0 = all, 1-channel output is actually 1-channel output per input channel. If you set the first channel to 3, midi from channel 6 will be sent to channel 8. To send output to a specific channel, no matter what the input channel, set the channel in your DAW track. In Reaper, use the MIDI hardware output menu in the I/O section.

Sysex retuning is not supported by many synths. If yours does read sysex messages, use this mode to retune it directly. No need for multiple instances or multi-timbral mode. Certain DAWs block sysexes, see chapter 6.10. You have the choice of MTS Universal Sysex #82 (real-time single-note tuning change) or #88 (non-real-time scale/octave dump). Alt-tuner assumes that sysex82 messages will retune the synth's current notes as well as future ones, and that sysex88 messages will only affect future notes. As a result, <u>retroactive retuning is not possible in sysex88 mode</u>.

Sysex88 mode is limited to 12-note scales within 64¢ of 12-ET. In this mode, the octave stretch slider will be locked to 1200¢, same as in octave mode. In addition, the "# of keys" slider on the prefs/keyboard screen will be locked to 12. In most modes, the pitchbend wheel affects only the last note played. In octave-equivalent and sysex88 modes, it affects an entire pitch class; that is, it also affects that same note in other octaves.

Keyswitch mode uses the upper five midi notes, D# to G five octaves above middle-C, to encode tuning information in a new format. This mode is used with Kontakt when a custom script is loaded into a Kontakt instrument. See chapter 6.10 for more information. These <u>virtual</u> keyswitches are sent from alt-tuner to your synth, as opposed to the <u>physical</u> keyswitches which alt-keyswitcher converts to CC messages.

The midi channel monitor's format depends on the output mode. In non-octave mode, the monitor's note names include an octave number. Middle C is C4. In octave mode, there are no octave numbers, because each channel is used by an entire pitch class. See prefs/keyboard for an exception to this. In mono, sysex and keyswitch modes, the monitor shows how many notes are currently being sent to each active channel. If the "midi input channel" slider is set to 0 = all, all 16 channels are potentially being talked to and all 16 are shown. Otherwise, only the channel being talked to is shown.

<u>Retroactive retuning</u> will retune a note after it has sounded. Tuning changes (pedal presses, tuning taps or certain slider moves) affect future notes, not current ones. However, retroactive retuning will, upon tuning changes, look back into the past and instantly retune any recently played notes. If the window is set to 100 milliseconds, this allows you to pedal on the chord change, not before it, and lets you be up to a 1/10 second late on your pedaling. If the window is something long like 5 seconds, you can retune already sounding chords.

In sysex88 mode, retroactive retuning is not possible, and the slider will be replaced with a warning: "disabled by sysex88 mode".

The time scale is logarithmic, but setting the slider to zero completely turns off retroactive retuning. The possible window times range from 1 millisecond to 1000 seconds, which is about 16 minutes. Normally, retroactive retuning only affects notes still being played. Recently released notes that have a lengthy decay will not have their release tails retuned, even with a long window of 16 minutes. However, if you move the slider all the way to the right, the window will be set to "infinite". This allows you to retune the tails of notes you've already released. In addition, whenever you change the tuning, if possible alt-tuner will immediately output a complete "tuning dump" for your new tuning. This output is midi, in accordance with the current midi output mode. It contains tuning information for every single note or pitch class, not just the ones that have been played. A tuning dump is only possible in these midi output modes: octave (when the # of channels is set to 12 or higher), sysex82, and keyswitch. In sysex88 mode, because of the sysex format, the complete 12-note tuning is always dumped with every tuning change, regardless of the window size.

Frequency to calibrate to lets you calibrate your tuning to A-440, A-441, etc. Changing this will automatically retune your entire keyboard. This is very useful when playing along with other instruments or recordings far from A-440. The cents difference between A-440 and your new A is shown. To return to 440, hold down the mouse button to autorepeat, and autorepeat will stop when you reach 440. See also "maximum # of decimals" below. If you're working with non-12-note keyboards, see the prefs/keyboard screen section for more info.

Reset cents offset: When you cycle through 12-ET to reach a new key, alt-tuner will automatically reset the cents offset, unless you deselect "reset cent offset when in 12-ET".

# of presets to cycle through: You can have up to 8 preset scales (more with customization). The last preset is always the standard tuning, 12-ET. When you first start alt-tuner, there are 4 preset scales. Preset #1 is a 7-limit scale, preset #2 is a 5-limit scale, and preset #3 is a utonal 7-limit scale.

Save current scale to preset: Pick a preset and click OK, and it will be replaced by the current scale. Preset scales are cycled to, and custom tunings/temperaments are switched to. Unlike custom tunings, presets only save the scale, not the tempering sliders, the linkage, the EDO-slider, etc. Presets are only meant to be a convenient starting point to tap from; it's not good to have too many of them.

Center note taps let you shift the center note to other nearby ratios like r1 or g1. These nearby ratios all have a keyspan of zero. They are automatically filtered out of the lattice if center note tapping is off. On the tapnotes screen they'll appear as gray squares. Turn center tapping on and these ratios will appear. When center tapping is off, tapping or clicking the center note will cycle instead. The advantage to turning off center tapping is that you get a built-in cycling keyswitch in your tapzone.

Silent taps give you an additional choice to tap to: silence. Silent taps remove notes from the lattice, graph and table views. The center note can't be silenced. When silent taps are allowed, tapping or clicking on a note that is already tapped to its sharpest ratio (like the red second) will silence it and the note will disappear from the lattice. To get it back, tap the appropriate key or click on any similar note (e.g. any major second) on the lattice. To silence a note while in the tapnotes screen, click the sharpest note in the column to select it, and click it again to silence it. See the "Advanced Topics" chapter for more.

Limit tonic accidentals prevents you from modulating to keys like B♯ and D, by converting these keys to C. If this option is set to "no", modulating fifthward seven steps will sharpen C to C♯, then to C^{x}, then to C$\sharp^{3}$, C$\sharp^{4}$, etc.

Minimum/maximum # of decimals is the number of decimal places used for cents displays. It only affects the visual displays, not the actual midi output, which is always as accurate as the midi standard allows. The maximum decimal has an immediate effect on the calibration frequency number box. If you reduce the maximum decimals and then click on the number box, the number will be rounded off to the new number of decimal places. For example, set the frequency to 441.7, and set the maximum decimals to 0. The frequency will appear as 442, but it is actually still 441.7. You can confirm this by setting the maximum decimals back to 1. Now set it back to 0 and click on the freqeuncy number box to change 442 to 443. Set maximum decimals to 1 to confirm that the frequency is now rounded off to exactly 443. In other words, changing the # of decimal places affects the appearance of numbers immediately, but won't affect the value of numbers until the numbers are actually changed. The rung cents number box in the prefs/rungs screen behaves similarly. The keybend screen behaves similarly with respect to the minimum # of decimals.

Note sizes: These two options only affect the lattice screen, not the graph or table or tapnotes or keyboard screens. You can control the size of the selected (colored) or unselected (gray) letters. If you add or lengthen rows in the rows screen, the lattice will grow, and the notes may be scaled down so small that you'll need to increase the note sizes here to make them legible.

Allow resonating: Sometimes gray notes will turn colored. Unselected notes in the lattice will "resonate" when tempering brings them to within 1¢ of a selected ratio. You can turn off this effect here. The selected ratio need not be the same note. For example, if you set the jade slider to 590¢, the jade E♭ will resonate with the yellow E. Resonating notes are circled when the corresponding selected note is played, so you can tell what's resonating with what by playing one note at a time.

Preserve the tonic when switching: When this is on, switching among custom tunings won't affect the center note or the cents offset. This is useful when using different tunings to access different scales. For example, tuning #1 defaults to Centaur in C and tuning #2 defaults to a more fifthward tuning, also in C. If you're in C, switching between #1 and #2 provides a simple method to avoid wolf intervals like the red fifth and the yellow fifth. If while in #1 you modulate to D, this method won't work because tuning #2 will still be in C. You would have to go to #2 and modulate to D there too. If you change key again, you'll have to perform the same key change again in the other tuning. But if you opt to preserve the tonic, you can go to any key and the other tuning will in effect "follow" you to the new key. This feature is also useful for adaptive tuning setups in which different custom tunings temper out different commas, see "Auto-modulate" in the "Advanced Topics" chapter.

Auto-modulate on sustain pedal release is used to navigate comma pumps. See "Auto-modulate" in the "Advanced Topics" chapter.

Chapter 6.5 – Multi-keyboard Setups

Adding a second keyboard for physical keyswitches is very low-cost. Just go to the thrift store and get the cheapest keyboard that has either midi out or usb. It needn't be velocity-sensitive. A midi-to-usb-cable can be bought online for US$5. You can even use your computer's keyboard for keyswitches. Most DAWs support "musical typing". Just substitute your computer keyboard for keyboard B in examples 6.5.2 through 6.5.5.

Alt-tuner supports multiple musicians each playing their own keyboard(s). Start a band! In general, use one instance of alt-keyswitcher per keyboard and one instance of alt-tuner per person. Reaper users, but not ReaJS users, can instead use midi busses, which only require one alt-tuner instance. This is generally preferable, see below for details.

For one-person setups, it's simplest to have one DAW track that receives all channels of all midi inputs. Make sure your keyboards are transmitting on different channels. Each instance of alt-keyswitcher is set to input one of those channels. Guitarist-style pedalboards usually don't need alt-keyswitcher and can share a channel. If you can't set the transmitting channel on your keyboards, put them on separate tracks, as in the third example below, Table 6.5.3.

Multi-person setups usually require that each keyboard's midi is sent to different Reaper tracks.

Alt-keyswitcher will optionally filter out all other notes and CC messages. There are two situations in which to do this. One situation is when playing with another keyboardist, and you each have your own instance of alt-tuner running. Filtering lets you send your retuning actions, but not the actual music you play, to the other player's alt-tuner. This allows you to control both alt-tuners at once, and retune both synths at once. The other filtering situation is when using a 2nd keyboard just for keyswitches. Black keys work well for keyswitches because they're easy to see and hit. If you miss a keyswitch and accidentally hit a nearby white key, it makes a random sound. To avoid this, turn filtering on.

Here's some examples. Output to a softsynth means the synth is the last effect. In Reaper, output to a hardware synth is done through the MIDI hardware output in the track's I/O box. Output to a track is done with a send in the I/O box.

Table 6.5.1 – 1 player, 1 keyboard (keyboard A)

track	player	keyboard input	effect 1	effect 2	output
1	1st	A: plays & retunes	alt-keyswitcher	alt-tuner	synth 1

Table 6.5.2 – 1 player, 2 keyboards (plays keyboard A, retunes with keyboard B)

track	player	keyboard input	effect 1	effect 2	output
1	1st	all midi input: A: plays only B: retunes only	alt-keyswitcher on B's channel (set filtering on)	alt-tuner	synth 1

Table 6.5.3 – 2-track alternative to the above, use this if keyboards A & B must use the same channel

track	player	keyboard input	effect 1	effect 2	output
1	1st	A: plays only	alt-tuner		synth 1
2	1st	B: retunes only	alt-keyswitcher (set filtering on)		track 1

Table 6.5.4 – 1 player, 2 keyboards (plays and retunes with keyboard A, retunes with keyboard B)

track	player	keyboard input	effect 1	effect 2	effect 3	output
1	1st	all midi input: A: plays & retunes B: retunes only	alt-keyswitcher on A's channel	alt-keyswitcher on B's channel (set filtering on)	alt-tuner	synth 1

Table 6.5.5 – 2-track alternative to the above, use this if keyboards A & B must use the same channel

track	player	keyboard input	effect 1	effect 2	output
1	1st	A: plays & retunes	alt-keyswitcher	alt-tuner	synth 1
2	1st	B: retunes only	alt-keyswitcher (set filtering on)		track 1

Table 6.5.6 – 2 players, 2 keyboards, 1st player retunes synth1 only, 2nd player retunes synth2 only

track	player	keyboard input	effect 1	effect 2	output
1	1st	A: plays & retunes	alt-keyswitcher	alt-tuner	synth 1
2	2nd	B: plays & retunes	alt-keyswitcher	alt-tuner	synth 2

Table 6.5.7 – 2 players, 2 keyboards, 1st player retunes both synths, 2nd player retunes neither synth

track	player	keyboard input	effect 1	effect 2	output
1	1st	A: plays & retunes	alt-keyswitcher	alt-tuner	synth 1
2	1st	A: plays & retunes	alt-keyswitcher (set filtering on)		track 3
3	2nd	B: plays only	alt-tuner		synth 2

Table 6.5.8 – 2 players, 2 keyboards, 1st player retunes both synths, 2nd player retunes synth2 only

track	player	keyboard input	effect 1	effect 2	output
1	1st	A: plays & retunes	alt-keyswitcher	alt-tuner	synth 1
2	1st	A: plays & retunes	alt-keyswitcher (set filtering on)		track 3
3	2nd	B: plays & retunes	alt-keyswitcher	alt-tuner	synth 2

Table 6.5.9 – 2 players, 2 keyboards, both players retune both synths

track	player	keyboard input	effect 1	effect 2	output
1	1st	A: plays & retunes	alt-keyswitcher	alt-tuner	synth 1
2	1st	A: plays & retunes	alt-keyswitcher (set filtering on)		track 3
3	2nd	B: plays & retunes	alt-keyswitcher	alt-tuner	synth 2
4	2nd	B: plays & retunes	alt-keyswitcher (set filtering on)		track 1

This example can be converted into the previous three examples by muting tracks 2 and/or 4.

Table 6.5.10 – 2 players, 3 keyboards, 1st player retunes both synths, 2nd player retunes neither synth

track	player	keyboard input	effect 1	effect 2	output
1	1st	A: plays only	alt-tuner		synth 1
2	1st	B: retunes only	alt-keyswitcher (set filtering on)		tracks 1 & 3
3	2nd	C: plays only	alt-tuner		synth 2

Table 6.5.11 – 2 players, 3 keyboards, 1st player retunes both synths, 2nd player retunes neither synth, alternate gear

track	player	keyboard input	effect 1	effect 2	output
1	1st	A: plays & retunes	alt-keyswitcher	alt-tuner	synth 1
2	1st	A: plays & retunes	alt-keyswitcher (set filtering on)		track 4
3	1st	B: retunes only	alt-keyswitcher (set filtering on)		tracks 1 & 4
4	2nd	C: plays only	alt-tuner		synth 2

Table 6.5.12 – 2 players, 3 keyboards, 1st player retunes both synths, 2nd player retunes synth2 only

track	player	keyboard input	effect 1	effect 2	output
1	1st	A: plays only	alt-tuner		synth 1
2	1st	B: retunes only	alt-keyswitcher (set filtering on)		tracks 1 & 3
3	2nd	C: plays & retunes	alt-keyswitcher	alt-tuner	synth 2

Table 6.5.13 – 2 players, 3 keyboards, 1st player retunes both synths, 2nd player retunes synth2 only, alternate gear

track	player	keyboard input	effect 1	effect 2	output
1	1st	A: plays & retunes	alt-keyswitcher	alt-tuner	synth 1
2	1st	A: plays & retunes	alt-keyswitcher (set filtering on)		track 4
3	1st	B: retunes only	alt-keyswitcher (set filtering on)		tracks 1 & 4
4	2nd	C: plays & retunes	alt-keyswitcher	alt-tuner	synth 2

Table 6.5.14 – 2 players, 3 keyboards, both players retune both synths

track	player	keyboard input	effect 1	effect 2	output
1	1st	A: plays only	alt-tuner		synth 1
2	1st	B: retunes only	alt-keyswitcher (set filtering on)		tracks 1 & 3
3	2nd	C: plays & retunes	alt-keyswitcher	alt-tuner	synth 2
4	2nd	C: plays & retunes	alt-keyswitcher (set filtering on)		track 1

Table 6.5.15 – 2 players, 3 keyboards, both players retune both synths, alternate gear

track	player	keyboard input	effect 1	effect 2	output
1	1st	A: plays & retunes	alt-keyswitcher	alt-tuner	synth 1
2	1st	A: plays & retunes	alt-keyswitcher (set filtering on)		track 4
3	1st	B: retunes only	alt-keyswitcher (set filtering on)		tracks 1 & 4
4	2nd	C: plays & retunes	alt-keyswitcher	alt-tuner	synth 2
5	2nd	C: plays & retunes	alt-keyswitcher (set filtering on)		track 1

Table 6.5.16 – 2 players, 4 keyboards, both players retune both synths

track	player	keyboard input	effect 1	effect 2	output
1	1st	A: plays only	alt-tuner		synth 1
2	1st	B: retunes only	alt-keyswitcher (set filtering on)		tracks 1 & 3
3	2nd	C: plays only	alt-tuner		synth 2
4	2nd	D: retunes only	alt-keyswitcher (set filtering on)		tracks 1 & 3

Table 6.5.17 – 2 players, 4 keyboards, both players retune both synths, alternate gear

track	player	keyboard input	effect 1	effect 2	output
1	1st	A: plays & retunes	alt-keyswitcher	alt-tuner	synth 1
2	1st	A: plays & retunes	alt-keyswitcher (set filtering on)		track 4
3	1st	B: retunes only	alt-keyswitcher (set filtering on)		tracks 1 & 4
4	2nd	C: plays only	alt-tuner		synth 2
5	2nd	D: retunes only	alt-keyswitcher (set filtering on)		tracks 1 & 4

Table 6.5.18 – 2 players, 4 keyboards, both players retune both synths, alternate gear

track	player	keyboard input	effect 1	effect 2	output
1	1st	A: plays & retunes	alt-keyswitcher	alt-tuner	synth 1
2	1st	A: plays & retunes	alt-keyswitcher (set filtering on)		track 4
3	1st	B: retunes only	alt-keyswitcher (set filtering on)		tracks 1 & 4
4	2nd	C: plays & retunes	alt-keyswitcher	alt-tuner	synth 2
5	2nd	C: plays & retunes	alt-keyswitcher (set filtering on)		track 1
6	2nd	D: retunes only	alt-keyswitcher (set filtering on)		tracks 1 & 4

Table 6.5.19 – 3 players, 3 keyboards, all players retune all synths

track	player	keyboard input	effect 1	effect 2	output
1	1st	A: plays & retunes	alt-keyswitcher	alt-tuner	synth 1
2	1st	A: plays & retunes	alt-keyswitcher (set filtering on)		track 3 & 5
3	2nd	B: plays & retunes	alt-keyswitcher	alt-tuner	synth 2
4	2nd	B: plays & retunes	alt-keyswitcher (set filtering on)		tracks 1 & 5
5	3rd	C: plays & retunes	alt-keyswitcher	alt-tuner	synth 3
6	3rd	C: plays & retunes	alt-keyswitcher (set filtering on)		tracks 1 & 3

When you have multiple instances of alt-keyswitcher, be aware that alt-tuner only updates from one of them. When you OK "update CC #s now?" in alt-tuner, it reads CC #s from the most recently active instance of alt-keyswitcher. Even if that instance is on a different track or a different project tab. To activate an instance of alt-keyswitcher, change any parameter in it, or just click on a blank area of the black screen. If you inadvertently load unwanted CC #s, just click on the proper alt-keyswitcher instance and then re-update alt-tuner. To clear all CCs, create a new (blank) instance of alt-keyswitcher, click on it, and then update alt-tuner.

Updating affects not only CC #s but also most parameters from alt- keyswitcher's "other" screen, like the bend ranges, the midi threshold, and the pedal resets. Updating also reports the range of the keyboard, see "octave lattices" in advanced topics.

Alt-tuner only listens for tuning CC messages on the output channel that the last activated alt-keyswitcher is set to. If that output channel is 0 = original, alt-tuner instead listens to that alt-keyswitcher's input channel. If that input channel is 0 = all, alt-tuner listens to all 16 channels. This allows you to do things like use keyboard A's sustain pedal for sustaining and use keyboard B's sustain pedal for switching or modulating. To have alt-tuner respond to tuning CC messages from both keyboards, set alt-keyswitcher B's output channel to match keyboard A's channel. Alternatively, set both keyboards to the same channel and use two tracks, as in the third example above, table 6.5.3.

Register blocks: You can also control exactly which instance of alt-keyswitcher talks to exactly which instance of alt-tuner with the register blocks. In alt-keyswitcher, it's set on the "other" screen, and in alt-tuner, it's set on the "CCs" screen. With multiple players, set the first player's alt-tuner and alt-keyswitcher(s) to all have one register block, set the second player's alt-tuner and alt-keyswitcher(s) to all have a second register block, etc. There are 20 blocks, each using 5 registers. The 100 registers are shared by all the Jesusonic effects in your DAW, including those in other tracks and other Reaper project tabs. Other Jesusonic effects may use registers too. In particular, the included Transient-driven Auto-pan effect does. If two effects happen to use the same register for different purposes, data will be garbled. If this happens, use a different register block. Be sure to change it in both alt-keyswitcher and alt-tuner so that they match. Block 1 uses reg00-reg04, block 2 uses reg05-reg09, etc. See forum.cockos.com/showthread.php?t=78460 for more info about other Jesusonic effects.

Midi busses: In Reaper, but not in ReaJS, there is an additional option of using midi busses. This allows a single instance of alt-tuner to retune up to 16 independent midi streams from up to 16 different keyboards. Each player uses his/her own bus. Midi busses are only needed when alt-tuner's "number of midi channels out" is more than 1.

One advantage of using busses is stability. With two alt-tuner instances, it's possible for them to get out of sync. Another advantage is that you can use mouse clicks to retune both synths. A disadvantage is that you lose the ability to have each player control just his/her own tuning. There can only be one tuning, because there is only one alt-tuner instance.

In Reaper, open "duo play with midi busses 1.RPP". It's in the "sample Reaper files" folder that came with alt-tuner. Use the view menu to display the VMK, the virtual midi keyboard. Here's the tracks:

track 1: "controller 1" inputs from VMK channel 1, contains alt-keyswitcher, sends to track 3, midi bus #1
track 2: "controller 2" inputs from VMK channel 2, contains alt-keyswitcher, sends to track 3, midi bus #2
track 3: "all midi" no input, contains alt-tuner & two ReaControlMIDI instances
track 4: "synth 1" receives from track 3, midi bus #1, all channels
track 5: "synth 2" receives from track 3, midi bus #2, all channels

The "all midi" track contains two instances of ReaControlMIDI, renamed "bus 1 ReaControlMIDI" and "bus 2 ReaControlMIDI". You can rename an instance by right-clicking its name on the effect list. You can use these two instances to monitor the midi coming out of tracks 1 and 2, but the main reason that they are in this project is to provide an example of midi bus routing. Open one of them and right-click on the "MIDI" button in the upper right. You can set the midi input to any of the 16 busses, or you can disable the input. You can't set it to read more than 1 bus at once. Be sure to set the midi output similarly. You have the additional choice of either replacing the input bus (overwriting it) or merging with the input bus (overdubbing onto it). You will usually want to replace, not merge. While midi-only effects have a "MIDI" button, softsynths instead have a button that says something like "2 out" or "2/32 out". This button will also let you route the audio. Right-click this button to assign the midi bus input and output. Softsynths by default listen to only midi bus #1. Both alt-keyswitcher and alt-tuner are permanently set to read from and write to all midi busses, outputting each midi message to the same bus it was input on. This allows them to retune all the track's busses at once.

The next step is to configure this project to work with your synths. First read chapter 6.10. Softsynth users: If your synth is multi-timbral, multi-midi-channel or sysex-tunable, put two instances in the "all midi" track, at the end of the effects chain. Set the midi input of one to bus 1 and the other to bus 2, by right-clicking the "2 out" button. Delete the two unused "synth" tracks. Alternatively, you can put the two softsynth instances on the two "synth" tracks. This method uses more tracks but allows you to record the output of the two softsynths independently. Either way, set the two synths to different sounds.

If your synth is multi-instance, you'll need to use multiple tracks, as in the "solo play with ReaSynth" project. Open "duo play with midi busses 2.RPP" and put instances in tracks 5-16 and tracks 18-29.

Hardware synth users: Set the midi hardware output of "from bus 1" to one keyboard and set the output of "from bus 2" to the other keyboard. Set the two keyboards to different sounds.

The two "to bus" tracks are set to input from channel 1 and 2 of the virtual midi keyboard. Play the VMK, setting its channel first to channel 1, then to channel 2. You should hear two different sounds. Set the two "to bus" tracks to input from your two keyboards or controllers. Each player should now produce their own sound, with both players having the same tuning.

A single instance of alt-tuner can only use one midi output mode. If you want to use different output modes simultaneously, you'll need multiple instances of alt-tuner. For example, using a multi-instance softsynth like Kontakt alongside a sysex-retunable synth like the xen-arts synths. However, multi-instance synths, multi-midi-channel synths and multi-timbral synths all use the same midi output mode.

Chapter 6.6 – Recording With Alt-tuner

This chapter assumes that Reaper is your DAW. Consult your DAW's manual to adapt this chapter to your DAW. Each Reaper track can contain both midi and audio takes. Other DAWs will require separate tracks.

The most flexible recording method uses three steps. First record the "raw" midi, including any retuning pedals and physical keyswitches, next convert it to tuned midi, and finally convert that to audio.

Setup: Create a Reaper track and name it "midi template". Right-click the input box, set the recording mode to midi overdub, and select a midi input (perhaps "all midi inputs, all channels"). Set record-monitoring to ON. Open the FX chain and load alt-keyswitcher and alt-tuner, with perhaps other midi effects before them and perhaps a softsynth after them. Save this track as a track template for future use.

First step (raw midi): Record-arm the track. You may want to create an empty midi item first, then overdub midi onto it. Hit the record button, and play, tap, modulate and/or switch away. The recording can be done in several passes if the pedal work becomes cumbersome. You can lay the pedals down first, then play the music, or vice versa. When you're done recording, un-record-arm the track.

Reaper has recorded the pre-FX input, not the post-FX output, so you will have a single-channel midi file with no pitch bends in it (unless you actually moved the pitch bend wheel yourself, of course). To check your take, hit play and watch alt-tuner as you listen. You can easily edit the midi at this point to correct any playing or pedaling mistakes. For example, if you pressed a pedal too late, you can move the CC it made over to the left.

Before replaying, be sure to initialize alt-tuner to the starting key, scale, custom tuning, etc. For example, suppose you plan to start in C and modulate halfway through to G. If alt-tuner is set to G instead of C when you hit play, the first half will be in G and the second half will be in D. This is very easy to do by mistake, because simply playing the song will leave you in G. For this reason, you may want to modulate back to C at the end of the piece. Remember, the raw midi file only contains information about when to modulate or switch. It doesn't contain any information about what to modulate or switch to.

For simple recordings with only a few tuning changes, the easiest way to initialize alt-tuner is to set up a Reaper preset for your song and select it before you play. If you're editing a section in the middle of the song, and you don't want to have to replay the song from the beginning, you'll need to set alt-tuner to the appropriate state for that section of the song. You may want to set up another preset for that section.

For complex recordings with many tuning changes, the best way to avoid this problem is to configure alt-tuner to switch to a specific custom tuning or modulate to a specific key upon receiving a specific keyswitch or CC, regardless of context. For switching, set up as many keyswitches or CCs as you have custom tunings. For modulation, always use a tapzone set to mod-tap on the CCs screen, and never use the relative modulation set up on the modulation screen. This is only possible if you're modulating to a note selected in your lattice. If not, use switching instead of modulating.

When performing, the number of CCs is limited to your available foot pedals, and the number of keyswitches is limited to the size of your keyboard. But when recording, you can use any CC, and any key. You can record CCs at the appropriate times in your song with any foot pedal, then edit the CC #. Or just draw in the CC message in the midi editor. You can record keyswitches using your keyboard, then use the midi editor to transpose them to midi notes 0-20 or 109-127, outside of normal piano range. Or you can record keyswitches in the normal range, then set them to a different channel, as if you had a 2nd keyboard dedicated to keyswitches.

In the prefs/switch screen, use the final mass edit button ("-") to set each row to only one number, so that each keyswitch or CC always switches to a specific custom tuning. You may want to add redundant keyswitches or CCs at the start of every section. For example, suppose your verse ends in custom tuning #1 and the chorus that follows also starts in #1 but ends in #2. You may want to put a CC that switches to #1 at the start of the chorus, even though it wouldn't be needed in normal playback, in case you want to replay the chorus repeatedly.

Second step (tuned midi): Initialize alt-tuner to the starting key, scale, custom tuning, etc. Right-click the raw midi item and select "Apply track FX to item as new take (MIDI output)". A new midi item will quickly appear below the original one. A four-minute song might take 20-30 seconds to process. This is the tuned midi. It won't have any pedal CCs or physical keyswitches. Instead there will be multi-channel midi with pitch bends, or single-channel midi with sysexes or virtual keyswitches, depending on your output mode. Once this file is created, disable (bypass) alt-keyswitcher and alt-tuner and all other midi effects, to prevent double-processing. This midi file can also be edited, but it's much trickier. If you move notes, you must keep the pitch bends or sysexes aligned with the notes. If you add new notes, they must be on the proper channel or have the proper pitch bends or sysexes.

Third step (tuned audio): Select the tuned midi take by clicking on it. Disable (bypass) alt-keyswitcher and alt-tuner.

Softsynth users: Right-click the tuned midi item and select "Apply track FX to item as new take". A third item will quickly appear below the other two, your audio take. If you want to tweak the sound, you can change the settings on your synth, and recreate the audio take. You can have several audio takes stacked up on the track for comparison.

Hardware synth users: Send the output of this track to your synth. Set the volume of your synth to a good level so that you don't clip. Set up a new track that receives audio from your synth. Record-arm this audio track and turn record-monitoring on. Put the play cursor at the start of your song and hit record. Let the whole song play, and stop recording when it's done.

Shortcuts: You can skip either or both of the first two steps and make fewer midi files. To skip the first step, right-click the input selection and set the track to "Record output (MIDI)".

Softsynth users: To skip the second step, record the raw midi, then with all effects enabled, "Apply track FX" to create audio. To skip the first two steps, set the track to "Record: output" for any of the non-MIDI options.

Hardware synth users: To skip the second step, record the raw midi, then with all effects enabled, record on your audio track to create audio. To skip the first two steps, record on both tracks at once.

Automation envelopes (advanced): If you need additional control over alt-tuner, you can automate the tuning changes over the course of the song using envelopes.

Click on the ^-shaped button in the lower left of the track. You'll see a list of all the sliders for every effect on the track. In addition to alt-tuner's 8 visible sliders, there are many other sliders that are hidden. The first 14 sliders control the key and the scale, and the EDOtap slider controls the EDOtap button. The other sliders won't be used until later. These 15 sliders plus the 8 visible sliders, 23 in all, determine the exact tuning. When recording with envelopes, you record the changing values of these sliders directly. This is done by recording automation data to some or all of the 23 corresponding envelopes in Reaper.

Each slider has small white boxes for checkmarks. To use a slider's envelope, check the first 3 boxes: the one by the slider name, Visible, and Arm. "Visible" here refers to the envelope, not the slider. A hidden slider can have a visible envelope, and vice versa. You only need to use an envelope if the corresponding slider changes over the course of the song.

To write automation, start with the raw midi file. Put the play cursor at the start of the song. Initialize alt-tuner to the starting state. Record-arm your envelopes and set the track's automation mode to write. Hit the record button, but don't play anything. Hit the stop button when the song ends. Reaper automatically sets the track's automation mode back to trim/read. Your envelopes should now reflect your tuning changes. For example, suppose in your chorus w2 becomes y2 (Maj2 slider), and your key changes from D to G (center note slider and perhaps the cents offset slider). You should see the envelope jump up or down where the chorus starts.

You can now "Apply track FX to item as new take (MIDI output)" as before. Don't play this file through alt-tuner!

Mostly you'll be letting alt-tuner "write" the envelope. But you can also click on the envelope and make your own envelope points. This is the main advantage of using envelopes. For example, you can make the cents offset envelope ramp up or down to deal with comma pumps. Reaper will let you enter numbers with decimals; for whole-number sliders like the center note and the 12 scale sliders, alt-tuner rounds off the number.

When you replay your song, alt-tuner will read the envelopes. You won't need to initialize alt-tuner to the appropriate state, because the envelopes will do that for you. This is the other advantage of using envelopes.

If an envelope is visible and being read from, the corresponding slider is locked. So for example if the Maj2 slider is locked, you won't be able to tap that key. If the center note slider is locked, you won't be able to modulate. Furthermore, the slider will be limited to the range and resolution of the envelope. For example if you have more than 4 tapnotes for the min 2nd key, you'll have to customize alt-tuner to increase the range of the min2 slider.

More about alt-tuner's sliders: If you click on the "Edit..." button on the right just above the white slider, a box will appear containing the values of all the variables. The sliders are near the bottom. Watch these as you use alt-tuner to better understand alt-tuner's sliders. You can also unhide any slider by customizing alt-tuner, so that you can watch it on the main alt-tuner screen.

Slider 1 is the key, the note in the center of the lattice. The value for this slider runs 1-12 for A through G♯, as well as 0, which corresponds to 12-ET, in which all other sliders except slider 2 are ignored. If you hover over the center note envelope, Reaper will show you the actual key note in the envelope. Slider 2 is the key note's offset from A-440, in cents. Sliders 3-14 are scale sliders for the 12 notes of your tuning. The 4 possibilities are numbered 1 to 4, or 0 if the key is silent. For example, if slider 1 is 6 for D and slider 5 is 2, we have wD and yE, but if slider 5 = 3, we have wD and wE. The scale slider format will change if EDOtap is on. In this format, the scale sliders show the number of EDO-steps the key spans, plus one; or else 0 if the key is silent. Slider 61 is 1 if EDOtap is on and 0 if it's off. This slider can only be set to 1 if the EDO slider is set to 2 or higher. The other sliders' ranges and values should be familiar to you from on-screen use. The range and resolution of the envelope is limited to the range and resolution of the slider. See the customization section for how to change the latter.

Additional tracks: The easiest way to set up the next track is to duplicate the first track, rename it, and delete its recorded midi and audio. This creates an empty track with the appropriate alt-tuner settings (and envelope data, if you're using envelopes). You may want to keep the raw midi file, but delete the notes, leaving only the tuning CCs and physical keyswitches. You can then overdub new notes onto this. Usually you'll want the same tuning in all your tracks. But if you want the 2nd track to use a different tuning, you can now set the 2nd track's alt-tuner independently (or edit the 2nd track's envelopes independently). If you later change the tuning in the 1st track and want to update track 2, duplicate track 1 again, delete its midi, copy the midi from track 2 and paste it into the new track, and delete track 2. Obviously, you'll have to regenerate the tuned midi and the audio. Alternatively, if you have only changed one or two things, it may be easier to just make the same changes on the 2nd track's alt-tuner. (To replicate minor changes to envelopes, you can copy and paste the automation data directly. In Reaper, right-click on an envelope in the source track and choose "select all points". Either choose "copy points" or just type control-C. Then select an envelope on the target track by clicking on the envelope panel on the far left. Click the "rewind" symbol on the play controls to go to the beginning of the track. Type control-V, and the points will be inserted at the play position.)

For projects with lots of tracks, it may be preferable to have one instance of alt-tuner retuning many different tracks. One way to do this uses sends and receives. Set up track A with alt-keyswitcher and alt-tuner. Record just the tuning pedals and physical keyswitches. Next record the raw midi on track B. Set up track C with your softsynth. Set this track's input to none and set it to record the output as either midi or audio. Now click track A's I/O box and set up a receive from track B and a send to track C. Finally, record-arm track C and hit the record button. You should get tuned midi or audio in track C. Now set up tracks D and E similarly to B and C. Set track A to receive from D and send to E. Hit record and track D's midi will make tuned midi or audio in track E. Repeat for all tracks.

These are just some of the possibilities. Reaper is extremely flexible and there are many other ways of routing midi and audio.

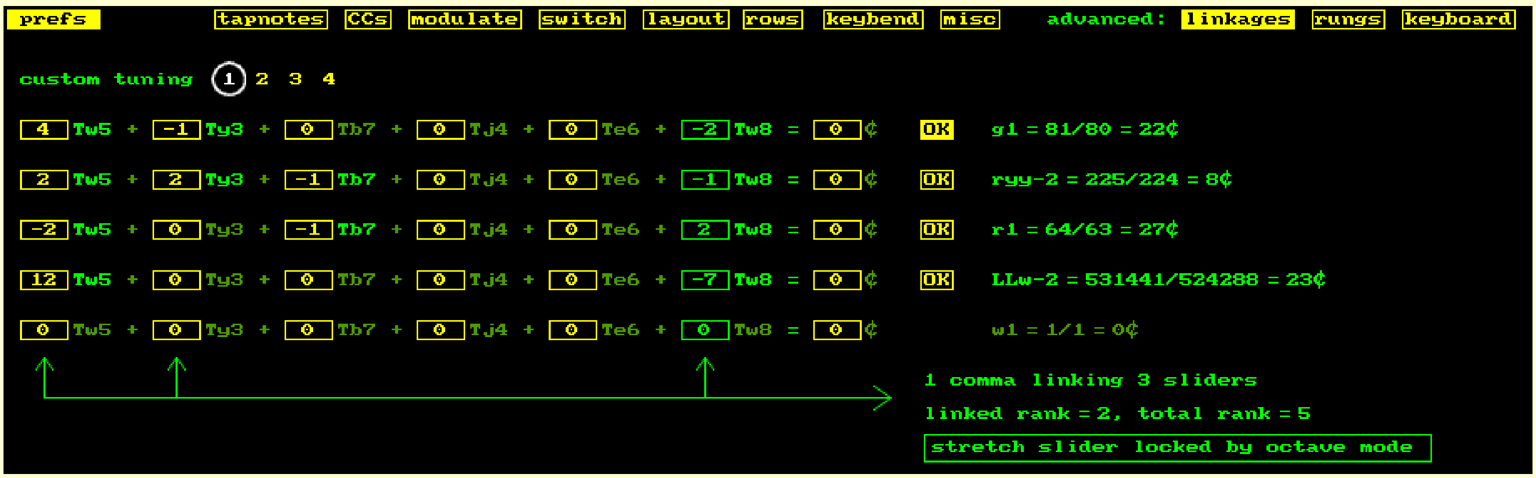
Other considerations: If you get midi latency, try turning on "Preserve PDC delayed monitoring in recorded items" on the track by right-clicking the record-arm button. It can also be set to be always on for new tracks in Reaper/prefs/project/track defaults.

If you're recording with many tracks and midi channels are tight, you can determine the minimum number of channels needed by setting the number of output channels to a low number, replaying your recorded raw midi, and watching the midi monitor on the prefs/misc screen for red squares.

Earlier I said the tuning was controlled by only 23 sliders. In the next chapter we'll see how to increase the number of rungs and the number of keys per octave, and we'll potentially have 63 sliders to automate!

Chapter 6.7 – Advanced Preference Screens

Linkages screen: We've seen how moving the tempering sliders causes unselected notes to "sympathetically resonate" and light up, as they become equivalent to selected notes. Linkages are a way to constrain the tempering sliders so that certain equivalences are always created. If you set the white slider to 696.6¢, the difference between w2 and y2 will disappear, and one will resonate with the other. Same with w5 and y5, with w6 and y6, and many other pairs. All such pairs are separated by the green comma g1 = 81/80. When the white slider is at 696.6¢, g1 becomes 0¢ and the green comma is said to be tempered out. This will also happen if white = 700¢ and yellow = 400¢, or if white = 702¢ and yellow = 408¢. There are actually many settings of white and yellow that will create this equivalence. To see all of them, go to the linkages screen, select custom tuning #1, and "OK" the first comma. Move the white and yellow sliders. They react to each other in such a way as to keep g1 tempered out. The two sliders are said to be *linked* by the green comma. This particular linkage is known as meantone temperament.



The green comma is written out in rung factor format in the first of the five rows. The "T" in "Tw5", "Ty3", etc. stands for tempered, which means intentionally made slightly sharp or flat. Reading across, the equation says that 4 tempered fifths minus 1 tempered yellow (major) third minus two tempered octaves equals 0¢. Thus the tempered comma Tg1 = 0¢. To the right of the "OK" button is the name, ratio and the untempered (just) size of the comma: g1 = 81/80 = 22¢.

You can use more than one comma to link sliders. A second comma is ready to go, just click on the 2nd OK button for ryy-2. Because this comma uses the blue rung, the blue slider will now be linked to the white and yellow sliders. Moving any one slider moves the other two as well. There are two more commas set up for you, the red one and the white one.

By the way, this screen can double as a handy ratio calculator and comma finder, just like the modulate screen. But unlike the modulate screen, it can only access the first 10 rungs (see the next section on rungs).

The lines below the list of commas detail the linkage. The rank describes the freedom of movement of the sliders. If all 6 sliders are unlinked, they can all be moved independently, and the rank is 6. Un-OK the ryy-2 comma to return to a 1-comma linkage in which only white and yellow are linked. Now only 5 sliders can be moved independently: white, blue, jade, emerald and octave stretch. The yellow slider "comes along for the ride". The total rank is 5. The linked rank looks at the linked sliders only, the ones with green arrows, which in this case are the white, yellow, and stretch sliders. (The stretch slider is always part of any linkage because it always moves the other sliders. This is a "built-in" linkage.) Because the white and yellow sliders move as one, there are only two degrees of freedom, and the linked rank is 2.

In the default octave-equivalent output mode, the octave stretch slider is locked in at 1200¢. Alt-tuner indicates this with a "stretch slider locked by octave mode" message. This lock in effect creates an additional "comma" (using the word <u>very</u> loosely!) with the equation w8 = 1200¢. Mathematically speaking, this lessens the rank by one, so that the linked rank becomes 1. However, alt-tuner will still report the rank as 2. This is because the terms "rank 1 temperament", "rank 2 temperament", etc. have come to take on specific meanings, as we'll see below. So the g1 linkage in octave mode is called not "rank 1" but "locked rank 2". To unlock the octave and free up the stretch slider, right-click the "12ch O" in the upper right of the screen.

If the octave slider is locked and a comma uses only the clear rung and one other rung, that rung will be locked too. In octave mode, if you OK just the 4th comma, the white rung will be locked at 700¢. If you OK the 3rd comma too, the blue rung will also be locked. If you unlock the octave, moving either the white or the blue slider will move the stretch slider, which will in turn move all the sliders.

The rank equals the number of rungs (including clear) minus the number of commas in your linkage. The rank is also the maximum number of sliders you can get centered on the JI default. In this example, you'll never get all 6 sliders centered, because the total rank is 5. You'll never get all 3 linked sliders centered, because the linked rank is 2.

You can modify the commas or create your own. You will have to re-OK any commas that you modify. You can reset a comma to zeroes by right-clicking on the OK button. Because alt-tuner automatically octave-reduces whatever interval you enter, the clear column is an unclickable green. If a comma is greater than half an octave, alt-tuner will automatically invert it when you OK it. If you enter 1 Tw5 + 1 Ty3 to make a y7, when you OK it, it becomes a g2.

Unlike all the other preference screens, the linkages screen is completely switchable. Each custom tuning has its own linkage. This lets you quickly compare different linkages or different variations of the same linkage. The first tuning is set up with four often-used commas, and the other three tunings are blank. If you add more tunings via the prefs/switch screen, they too will be blank. The yellow tuning numbers are the same as the ones on the lattice, graph and table screens; clicking them will switch tunings. You can copy the commas from one tuning to another by right-clicking on a yellow tuning number. This is the same as right-clicking a tuning number on the other screens; it also copies the tuning's center note, cents offset, scale, etc.

If the rank gets down to 1, an EDO is created. To see this, OK the first and fourth (green and white) commas. You will see "EDO = 12", the familiar standard tuning. If you go to the lattice view and cycle to the 5-limit preset scale, you will indeed have the standard tuning. But if you cycle to a septimal preset, the keyboard won't sound like 12-ET. All the blue and bluish intervals will be 31¢ flat of 12-ET. These intervals do however create their own 12-ET subset, because each of them is equally flat of the white & yellow subset. Because the <u>linked</u> rank is 1, any tuning based on only the linked sliders (in this case white, yellow and clear) will be an EDO. Because the <u>total</u> rank is 4, not 1, not all tunings are EDOs.

If the rank gets all the way down to 0, some sliders will be "stuck" at zero cents. To see this, set one comma to 5 white fifths, set another to 2 white fifths, then OK them. The white and stretch sliders will be stuck at 0¢. OK another comma of 3 yellow thirds, and the yellow slider will be stuck too. I'll leave it to others to find a musical use for this tuning!

You don't have to reduce the comma all the way down to 0¢. You can set the size of the tempered comma to any number of cents. Just click on the yellow box to the immediate left of the "OK" button. You can even increase it above its usual size! You can only set the tempered comma size to the nearest cent. In other words, you can set Tryy-2 to 1¢ or 2¢, but not 1.5¢.

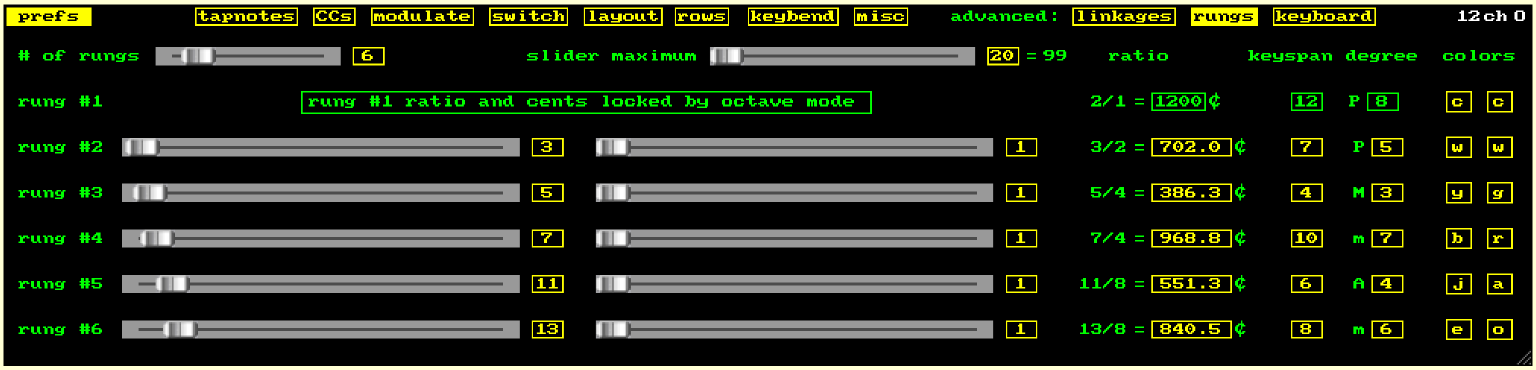
A comma needn't be tiny; it can be almost any interval at all. For example, you can temper out the blue 7th.

If you enter a "squared comma" aka a "doubled comma" like $(81/80)^2$ = 6561/6400 = 8 Tw5 - 2 Ty3, alt-tuner will treat it the same as an 81/80 = 4 Tw5 - 1 Ty3. There are exceptions to this rule. Because alt-tuner octave-reduces commas, $(10/7)^2$ is not 100/49 but 50/49. Tempering out 50/49 is very different than tempering out 10/7!

If you enter the same comma twice, alt-tuner will tell you that you have a redundant comma. The same thing happens if you enter a comma and its square, or if you enter three commas and one of them is the sum or difference of the other two.

Alt-tuner checks that all commas are indeed equal to their intended tempered size. If any are not, something like "warning: Tryy-2 = 23.45¢" in bluish writing will appear on the linkages screen. This warning occurs when you ask alt-tuner to do something impossible. To see this message, enter the same comma twice, but with different tempered sizes. Sometimes the EDO slider will "break" a linkage and create a warning. See the "Advanced Topics" chapter.

Rungs screen: A rung is a color, and a vector in the lattice, and a prime number, but it's essentially a melodic interval. The rungs screen lets you define that interval. In alt-tuner, rungs are a ratio made up of two numbers. Use the ratio sliders to specify these two numbers. The "ratio" column next to the sliders shows the actual rung ratio, which may be different because alt-tuner will automatically simplify and octave-reduce the ratio. Thus 3 and 1 become 3/2, not 3/1, and 15 and 12 become 5/4. The cents shown is for the untempered ratio; it doesn't change if the tempering sliders are moved. The cents are a clickable yellow, see "redefining rungs" below.



Certain midi output modes, for example octave and sysex88 mode, will restrict the first rung's ratio to 2/1 = 1200¢. Alt-tuner will indicate this by displaying a warning: "rung #1 ratio and cents locked by ____ mode". If you're in octave mode, right-click the "O" in the upper right corner of the screen to enter non-octave mode. Rung #1's two sliders will appear and the cents of rung #1 will become a clickable yellow.

The ratio sliders also double as a handy cents/keyspan/degree calculator. Unlike the modulate and linkage screens, instead of entering the component rungs, you enter the actual ratio.

The ratio sliders range from 1 to 99. To enter larger numbers, move the "slider maximum" slider. Because this slider is logarithmic, the yellow number merely reflects the slider's position, and the green number is the actual upper limit. Position 20 corresponds to a limit of 99, 30 corresponds to 999, etc. The maximum position is 70, for a limit of 9,999,999.

Rungs are the fundamental "building blocks" of scales and chords. The first rung is particularly important. Changing anything about it has many far-reaching affects. It defaults to an octave, but it can be any interval, although certain midi output modes restrict it to an octave. If it's anything other than an octave, for example 3/1 = 1902¢, alt-tuner displays "3/1" in the upper right of the screen, next to the output mode. The first rung's interval is called the **period**, because the scale repeats periodically at this interval. The other rungs' intervals are called **generators**, because combining these intervals generates the universe of all possible notes that are used to create scales and chords.

Rung ratios are usually a prime number over a power of two, and the rungs will default to this, but alt-tuner will allow you to specify any ratio at all. If you want a rung of a specific size, check the "3000 Ratios" table in appendix 6.1 for a nearby ratio, and adjust the cents as needed. For example the golden ratio phi = 833.09¢ is approximately 89/55 = 833.25¢. Alternatively, increasing the slider maximum will increase the range of the ratio sliders, making possible a closer approximation like 233/144. If you can't set it to exactly what you need, it's OK to have it be set a little higher. For example, position 24 creates a limit of 250, large enough for 233/144. The actual reduced ratio in green may be larger than this limit. For example, if you set rung #1 to 17/8, some rung ratios will have four-digit numbers, even if the slider maximum is only 99.

The slider maximum can be as high as 9,999,999, so rung ratios can become <u>really</u> large. Because computers only have 64-bit precision, it's impossible for them to accurately represent integers larger than $2^{53} \approx 9 \cdot 10^{15}$. Therefore rung ratios of more than 16 digits are not always accurate. Fear not, this represents an inaccuracy of less than a trillionth of a cent!

<u>Speed-scrolling</u>: For extremely large ratios, you can only get so close to your target with the slider, because moving the fader one pixel may increase the number by thousands. You'll have to click on the yellow number box, which can be tedious even with autorepeat. But if you click with a double-modified click (shift-right-click, alt-shift-click, etc.), the numbers will "speed-scroll" by 100 at a time. You can even "speed-autorepeat". This trick works with any yellow number box in alt-tuner.

The next two columns show each rung's keyspan and degree. Alt-tuner automatically calculates the quality of the interval (perfect, major, etc.) from these two columns. The first rung's keyspan and degree are always an unclickable green because they can only be set on the prefs/keyboard screen.

As with tapnotes, the keyspan column determines where the rung ratio "lands" on the keyboard, and the degree column determines which letter represents it on the screen. For example, the emerald rung is 841¢ and is an 8-semitone interval. The degree is 6, so 13/8 is a minor 6th. But the emerald rung could almost as easily be a 9-semitone interval, making it a major 6th. Changing the emerald keyspan will move all the emerald ratios up one key. Do this, and verify the change on the tapnotes screen. The lattice will change too, because the preset scale is based on the rung's old keyspan. Changing one rung can affect many ratios at once. For example, the blue rung has degree 7 (min 7th), but it could instead have a degree of 6 (aug 6th). Change it, and see the effect on the lattice. Not only blue and red, but also bluish, reddish and purple intervals are renamed. That's because on the rows screen, these are the rows with a nonzero entry in the blue column.

Where a ratio lands on the keyboard is affected not only by the keyspans of its rungs, but also by the keyspan offset in the rows screen and on the tapnotes screen. For example, the jade rung has keyspan 6, but the jade row has a keyspan offset of -1, so the 11/8 is a perfect, not augmented 4th. A ratio's name is affected not only by the stepspans of its rungs, but also by the stepspan offset in the rows screen.

When you change a rung's ratio, the keyspan is automatically recalculated to the "best fit" based on the ratio's cents, the first rung's cents and the first rung's keyspan. For example, the blue rung's keyspan is 12 · 969¢ / 1200¢ = 9.69, which is rounded off to 10. In effect, the rung is matched to the nearest 12-ET semitone, similar to what happens when you set the EDO slider to 12-EDO.

The degree is calculated similarly, using the <u>stepspan</u>, which is one less than the degree. There are 7 steps to an octave, so the first rung's stepspan is 7. The blue rung's stepspan is 7 · 969¢ / 1200¢ = 5.65, which rounds up to 6, to make a degree of a 7th. This is akin to setting the EDO slider to 7 and matching each rung to the nearest 7-EDO-step.

This matching can lead to some surprising results. The yellow rung is 5/4 which is a third, and the blue rung is 7/4 which is a seventh. The bluish fifth 7/5 is a seventh minus a third, making a diminished fifth. Suppose that instead of a blue rung we defined a bluish rung with a 7/5 ratio. You would expect the rung's degree to be a fifth, since 7/5 is a fifth. But 583¢ rounds down to 3/7 of an octave, for a stepspan of 3, which is a fourth. As a result, the 7/4 becomes a sixth, because it's the sum of a bluish rung and a yellow rung. If this is not what you want, you can adjust the bluish rung's stepspan to be a fifth. Similarly, the jade rung's keyspan is 6 and the white rung's keyspan is 7, so if we replaced the jade rung with an 11/9 rung, one would expect an 11/9 rung's keyspan to be 6 - 7 - 7 + 12 = 4 = maj 3rd, but instead it's 3 = min 3rd. (The 12 is added as part of the automatic octave-reducing.) The 11/8 becomes a perfect 4th because it equals an 11/9 rung plus two white rungs = 3 + 7 + 7 -12 = 5. You can easily fix this by adjusting the keyspan.

If you change the first rung's ratio, keyspan or degree, <u>all</u> the keyspans and degrees are recalculated like so:

> this rung's keyspan = roundoff (first rung's keyspan · this rung's cents / first rung's cents)
> this rung's stepspan = roundoff (first rung's stepspan · this rung's cents / first rung's cents)

Again, the rung is matched to an EDO-step as when moving the EDO slider. Changing the period may also affect all the rung ratios, as they are not actually octave-reduced but rather period-reduced. For example, if the period is set to 3/1, and the 3rd rung's sliders are at 5 and 1, 5/1 will be reduced to 5/3, not 5/4. Alt-tuner uses the word "octave" loosely to mean period, as in "EDO", which strictly speaking should be called "EDP".

The colors column contains the over and under color notation shorthand for each rung, which is used in the interval display and in the tapnotes, modulate, layout, rows and linkages screens. The octave slider is the clear rung. The first 6 rungs have default color names: clear for 2/1, white for 3/2, yellow and green for 5/4, blue and red for 7/4, jade and amber for 11/8, emerald and ochre for 13/8. Higher rungs default to over and under, or "o" and "u". (See chapter 3.6 for further explanation.) Rung ratios with two or more prime factors higher than 3 (7/5, 11/7) default to "x" and "z". Rungs can be renamed by clicking through the alphabet. In the color notation outlined in Parts II and III, the following letters are unused: H, J, K, N, O, Q, R, S, U, Y, Z, and z.

You can add and remove rungs with the "# of rungs" slider. A "nextpage" button will appear when the screen is full. If you remove a rung and then add it back, the rung will "remember" the ratio you set for it, but the keyspan and degree will be recalculated. There must be a minimum of two rungs. There can be up to 25 rungs, even more if you customize alt-tuner. Only the first 10 rungs are temperable and/or stretchable. This is a permanent limitation due to Jesusonic.

The sliders that temper these rungs are numbered 51 to 60, with 60 being the stretch slider, 51 the white slider, 52 yellow, etc. If you want to temper the rungs you've added, you'll need to unhide their sliders by customizing alt-tuner. If you never use the jade or emerald rungs, you can hide their sliders to reduce screen clutter.

If you remove a rung that is used in a row, that row remains, but without that color. For example, removing the emerald rung turns the emerald row into a white row. Because duplicate ratios are filtered out, this is generally not a problem, but you may want to delete that row. If you don't delete the row and you add the emerald rung back in, the row "remembers" and becomes an emerald row again.

We've seen how alt-tuner allows you to define a rung with a ratio and temper that rung with a tempering slider. You can also redefine the rung by clicking on the yellow box that contains the cents. Every rung can be redefined, even untemperable ones. If you change the white rung's cents from 702¢ to 709¢, alt-tuner will tune 3/2 to 709¢. Not just the interval but the ratio itself is redefined, as if the rules of physics had changed. (There is some real-world justification for this; see inharmonicity at the end of chapter 1.2.) All intervals based on the white rung are also changed. This redefining is overridden by the white slider, so it won't have any audible effect unless the tempering strength is less than 100%.

The redefining will persist until you change either that rung's ratio or the first rung's ratio. Either action will make alt-tuner recalculate the rung size from the actual ratio. To reset the yellow rung, you might change 5/1 to 6/1 and then back to 5/1. Alternatively, if you hold down the mouse button to autorepeat the rung cents, and you're headed back in the direction you came from, the autorepeat will stop at the rung ratio's mathematical value.

When tempering strength is set to 0%, alt-tuner will tune to the redefined rung ratio, not to the center mark on the tempering slider. This allows non-JI adaptive tuning. You can also change the center mark to match the redefined rung, by customizing alt-tuner.

The point of reducing the tempering strength is to make the sounding interval different than the modulating interval. The point of redefining rungs is to control what happens to the sounding interval when tempering strength is reduced. Tempering controls the modulating interval, and redefining plus tempering strength controls the sounding interval. You can't modulate by an octave (or more generally, by a period), but you still have the option of either redefining it or stretching it, as long as it isn't locked by the current midi mode.

Redefining a rung can change the nearest EDO-mapping. For example, moving the EDO slider to 12-EDO moves the jade slider from 551¢ to 600¢. But if the jade rung is redefined to 549¢, entering 12-EDO will move the jade slider to 500¢ instead. Redefining can also lead to ratios being equated and filtered out from the lattice.

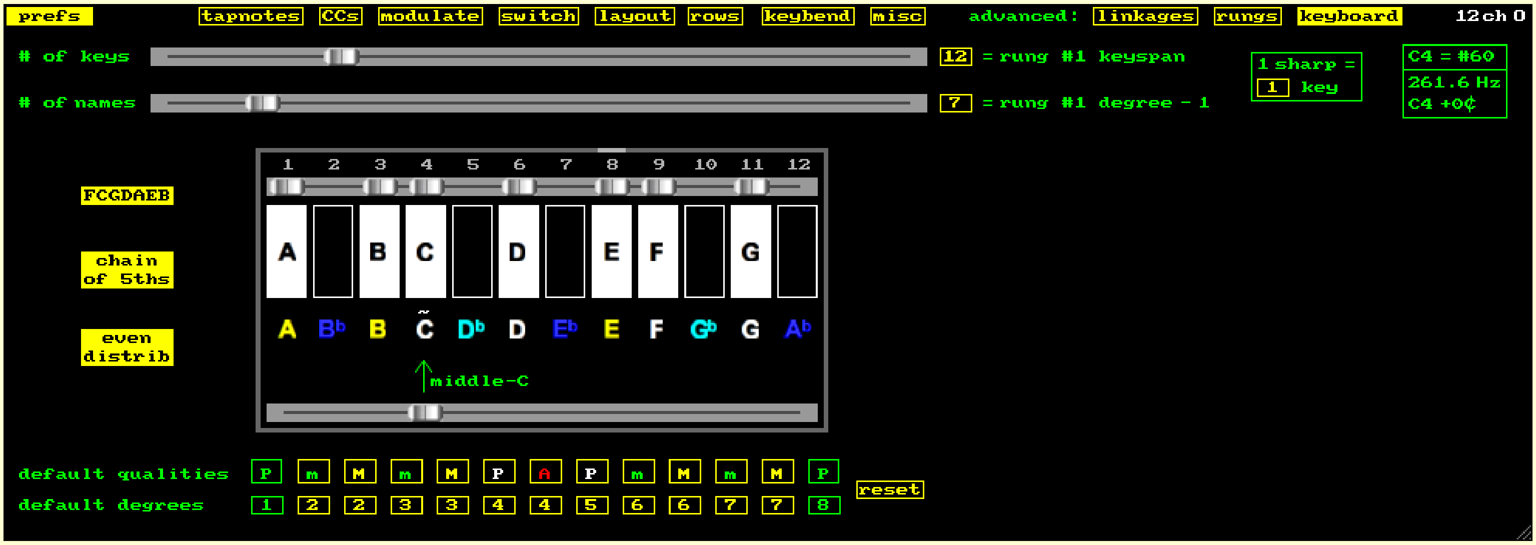
For an extreme example, redefine white to be 695¢. Go to the linkages page and set up g1 = 81/80. You'll see "g8 = 81//40 = 1194¢". When you OK it, you'll see "y1 = 80/81 = 6¢". This bizarre equation is a result of redefining the number 3 to be 2 ^ (1895 / 1200) = 2.988. The number 81 is the product of 4 "small" threes which makes it smaller than 80! It's as if the wizard Zarlino had surgically shortened Lady Tertia!

See "minimum/maximum decimal" in prefs/misc for how to redefine rungs with more accuracy, and how the rung's cents is rounded off.

It's possible to have redundant rungs, in which one rung is the sum of other ones. You can even have identical rungs.

Changes in the rungs screen and the keyboard screen will affect the tuning, the lattice, the tapnotes, the preset scales, the modulating intervals, almost everything in alt-tuner. Proceed with caution. For example, setting the white rung's keyspan to 8 and the jade row's keyspan offset to 0 results in all even-numbered keyspans, which makes half the keys go silent! Selecting "Reset to factory default" from the list of Reaper presets will usually restore normalcy. In extreme cases, select "Edit..." and select "Full recompile/reset", or else close the Reaper project and start over.

Keyboard screen: The "# of keys" slider controls the number of keys per octave, or more generally, per period. It directly controls rung #1's keyspan. Moving this slider changes almost everything, so it's a good idea to revisit the other prefs screens afterwards. Start with the rungs screen, because each rung's keyspan will be recalculated.

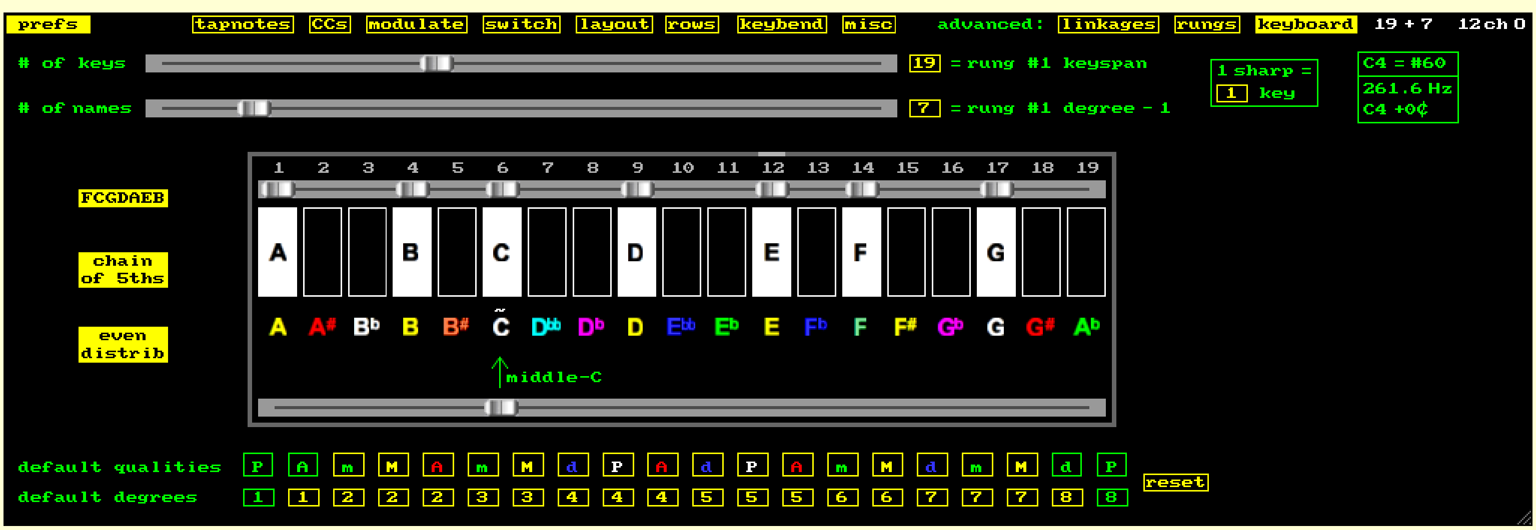


You can have up to 48 keys per octave (or period). This is a permanent limitation due to Jesusonic. The sliders for these 48 keys are numbered 3 to 50. To fully use keys past 12, add sliders for them. See "Customizing Alt-tuner". You can go beyond 48 keys by using two instances of alt-tuner; see tuning zones in chapter 6.9.

The "# of keys" slider and the "middle-C" slider are the only controls on the keyboard screen that affect the sound. All the other controls only affect the display. Alt-tuner automatically names every key and every interval based on these controls.

The "# of names" slider controls the degree of rung #1. Increasing it adds new letters of the alphabet. It causes all rung degrees to be recalculated. There are 92 note symbols (A-Z and 0-65). To add more symbols, see "Customizing Alt-tuner".

From the 15th to the 17th century, the cembalo cromatico, a "chromatic harpsichord" with 19 keys to the octave, was common in Italy. The five black keys were split and two more were added. See chapter 5.4 for pictures. To recreate the cembalo cromatico, just move the # of keys slider to 19. If the number of keys and number of names are anything other than the standard 12 + 7, alt-tuner displays a message like "19 + 7" in the upper right of the screen, next to the output mode, as seen here:



When the number of keys is other than 12, "cents offset from A-440" becomes a slippery concept. Alt-tuner attempts to supply a meaningful result, but repeated modulations may create contradictions. To be sure of your exact tuning, consult the reference pitch box in the upper right. The first line is the input note, complete with midi note number. It's the center note of the lattice, from the octave shown on the keyboard diagram below. The next two lines are the pitch produced when playing this key, both as a frequency and in relation to 12-ET calibrated to A-440. The last line has a standard note name with a cents offset of up to +/- 50¢. This output note describes the sound, not the keyboard. It's exactly what you would see if you played the center note and held an electronic tuner up to the speaker.

Sysex88 mode will restrict the # of keys to 12. Alt-tuner will display a warning: "# of keys locked by Sysex88 mode". In addition, the "12" near the "# of keys" slider will be green. Octave mode does not restrict the # of keys per octave.

The jade row defaults to a keyspan offset of minus one. This is useful when the number of keys is 12. For non-12 settings, it's usually better to set the jade row's offset to zero.

If the EDO slider is set to anything between 2 and the current # of keys, moving the # of keys slider will set the EDO slider to match. This lets you quickly create a tuning that accesses every note of an EDO. Turn EDOtap on to avoid duplicate pitches and/or silent keys. If you're curious why these problems arise: Duplicate pitches are caused by lattice rows with keyspan offsets. For example, set up a 13-EDO tuning with 13 keys per octave and with EDOtap off. Go to the tapnotes screen and select the "sizes" view. The notes in the first column are all 0¢, those in the second column are 1\13 = 92¢, the third 2\13 = 185¢, etc. But because the jade row has a nonzero keyspan offset, every jade note breaks this pattern. If a jade note happens to be selected, two keys on the keyboard will have the same pitch, and one of the 13-EDO pitches will be missing. To avoid this, turn on EDOtap. Duplicate pitches can also be caused by distant EDO-mappings. For example, set up a 15-EDO tuning on 15 keys. Move the yellow slider from 400¢ to 480¢, and every three keys will share the same pitch, and you get 5-EDO. Another problem is silent keys, or ratio-less keys, caused either by more keys per octave than tapnotes in the lattice (e.g. 48 keys per octave), or by all rung keyspans being even numbers, so that every other key is ratio-less (e.g. 20 keys per octave). Again, turn on EDOtap to avoid all these problems.

The EDO slider doesn't have to match the # of keys slider. To explore, say, 24-EDO and 19 keys per octave, set the # of keys slider before setting the EDO slider. And setting the EDO slider to a high number like 53 or 72 will ensure it is not affected by the # of keys slider. Moving the EDO slider automatically turns off EDOtap, but moving the # of keys slider doesn't.

The keyboard diagram has named keys which are white and unnamed keys which are black. The keyboard slider is immediately above the keyboard diagram. Unlike all the other sliders, it has many faders, one for each white key. You can use them to reposition any white key. If you move one up against a neighboring key, it will go on top of it, and one key will have two names. Right-clicking anywhere on the keyboard slider and dragging moves all the white keys en masse. There must be room to do this; it's not possible when both outermost keys are white.

You can skip over letters, so that a pentatonic layout could run A – C – D – E – G. Clicking directly on a white key skips a letter, and also "pushes" the following letters up as well. For example, clicking F up to G also pushes G up to H. Right-clicking a white key brings the key's letter down closer to the previous one, and if it bumps up against it, "pulls" it down too, if possible. Hold the mouse button down to autorepeat. Autorepeating the first white key up all the way reveals all the note symbols available.

The three buttons to the left of the keyboard diagram control the placement of the white keys. When clicked, the white keys will automatically be arranged using one of three methods:

The **FCGDAEB** method follows the traditional meaning of the 7 note names and uses the 6 perfect fifths F – C – G – D – A – E – B. In other words, the keyspans of F–C, C–G, G–D, D–A, A–E and E–B will all be equal to rung #2's keyspan as defined on the rungs screen. This method is not possible unless the note names consist of only the first seven letters. Thus changing "G" to "H" would make this button turn an unclickable green, as would setting the # of names to anything higher than 7. Not all seven letters need be used, this method works well with pentatonic scales like A – C – D – F – G.

In a **chain of 5ths** arrangement, almost every white key has another white key a perfect fifth above it. "Perfect fifth" is defined as an interval with the same keyspan and degree as rung #2, so it's actually a chain of generators. The standard keyboard has a chain of 5ths arrangement. The key 7 semitones above a white key is almost always another white key, with three other white keys between them. The exception is B – F#, because F# is black. One exception is allowed because 7 notes form a chain of only 6 perfect fifths.

An **even distribution** is one in which the gaps between any two white keys are all the same size or nearly so, and any unusually sized gaps are spread around the keyboard. For example, on the standard keyboard, adjacent white keys usually have a single black key between them. Those pairs that don't (B – C and E – F) are not right next to each other. Thus the standard keyboard layout A * B C * D * E F * G * is an even distribution, as is A * B * C D * E * F G *, but A * B C * D E * F * G * is not.

When you move either the # of keys slider or the # of names slider, alt-tuner will apply one of these three methods automatically. FCGDAEB has the highest priority and even distribution has the lowest. The first two methods use rung #1 as the period and rung #2 as the generator. The third method is independent of rung #2.

Sometimes a chain of 5ths or a FCGDAEB placement is impossible. For example, set the # of keys to 13. Alt-tuner will set rung #2's keyspan to the closest approximation of 702¢, which is 8. Because 8 is more than 3/5 of 13, the minor 2nd (defined as 3 octaves minus 5 fifths) has a negative keyspan, and the notes would run A–C–B–D–F–E–G. To prevent this, the first two buttons turn green and alt-tuner uses the even distribution method, which is always possible. The problem is caused by conflicting keyspans and degrees for rungs #1 and #2. If you go to the rungs screen and reduce rung #2's keyspan to 7, a chain of 5ths or a FCGDAEB placement becomes possible, and all the buttons become yellow. An alternative to reducing the fifth's keyspan is to reduce the # of names to 5.

Rung #2's keyspan is indicated in the keyboard diagram by a short light-colored line segment above one of the keys. This marks the key one generator above the leftmost key. In the previous 13 + 7 example, when rung #2's keyspan was 8 keys, the marker was above E#. Changing rung #2's keyspan to 7 moved the marker to E. When the lowest key is a white key, the marker should in general be above the white key that the second button indicates. If the second button says "4ths", it should be above the 4th white key.

A button will light up when the current arrangement follows its method. An arrangement often follows several methods, and several buttons may be lit up. Clicking on a button creates "the" arrangement for that method. The difference between "an" arrangement and "the" arrangement is usually just a matter of starting on a different key. In other words, all transpositions of "the" arrangement are "an" arrangement. For example, the chain of fifths arrangement of 16 + 7 is A * B * * C * D * E * * F * G *. Note the pattern: every adjacent pair of white keys is separated by either one or two black keys. There are two double-black-key sets, which are not very close to each other. Any other arrangement which follows this pattern will be both a chain of 5ths and evenly distributed. For example, A * * B * C * D * * E * F * G *. The FCGDAEB method is more exacting, requiring that the double black keys be between B–C and E–F. The only other possible FCGDAEB arrangement of 16 + 7 is * A * B * * C * D * E * * F * G.

Which method should you use? The FCGDAEB method has the advantage of more readable notation. For example, with 22 keys per octave, an equal distribution creates A * * B * * C * * D * * * E * * F * * G * *. Note that while some white-key-to-white-key fifths like A–E are the expected 13 keys wide, E–B and F–C and G–D are only 12 keys wide. However, on sheet music, these fifths look perfect, not diminished. These and other intervals sound different than they look. This is particularly disconcerting when playing in the keys of E, F or G. Now click the FCGDAEB button to get A * * * B C * * * D * * * E F * * * G * * *. Now every white-key-to-white-key fifth is 13 keys wide, except for B–F of course.

If the # of keys is a multiple of 5, the FCGDAEB method can cause one key to have two names. For example, 15 keys per octave makes A * * B/C * * D * * E/F * * G * *, permitting the use of standard diatonic notation.

Sometimes the chain of 5ths closes into a circle of 5ths, and the keyspan of the interval between the last note and the first one is the same as the others. For example, 12 + 7 has a chain of 5ths that doesn't close because B – F is diminished and has a smaller keyspan. However, B – F will be perfect when the # of keys is a multiple of 7. Thus 14 + 7 and 21 + 7 both close, because the # of keys is a multiple of the # of names. Sometimes the chain of 5ths closes before it reaches all the notes. For example, if the # of names is 10 and the fifth's stepspan is 6, after only 5 steps we return to the first note, and the circle only reaches half the notes. The other half form an additional unconnected circle. When this happens, alt-tuner places the other notes in its own circle, and the button will read "chains of 5ths" (plural chains) to indicate this. For example, the 15 + 10 system has two such circles, and the 25 + 15 system has three.

The period and the generator can be any ratios at all. The generator needn't be a fifth. If rung #2's degree isn't a fifth, the "chain of 5ths" button's label changes to read "chain of 4ths", "chain of 6ths", etc., even when it's green. If rung #2's stepspan isn't approximately 3/7 or 4/7 of rung #1's stepspan (and thus approximately a 4th or a 5th), the FCGDAEB button changes to read "FACEGBD" (approximate 3rds or 6ths) or "ABCDEFG" (approximate 2nds or 7ths). If rung #2's degree is 1, or is equal to rung #1's degree, this method is impossible, and the button will be a green "FCGDAEB".

Details: for FCGDAEB, A is always placed on the lowest key. B is two steps forwards on the FCGDAEB chain from A, so if there is a B, it's placed two fifths minus an octave above A (or more generally, two #2 rungs, reduced as needed by the #1 rung). If rung #1's keyspan is 17 and rung #2's keyspan is 10, B would be 20 - 17 = 3 keys from A. However, if the chain ran FACEGBD, B would be five steps forward from A, and would be placed five rungs away, octave-reduced of course. If the chain ran FGABCDE, B would be placed one rung away. C is three steps backwards on the FCGDAEB chain, so if present it is placed three fifths below, which is three fourths above, again octave-reduced. The notes are placed as if all 7 were present, even when some are missing. Some sets of note names will create a very uneven distribution. For example, A B C D E would run A * B C * D * E * * * *. To avoid this, when you reduce the # of names to 5, alt-tuner removes B and F, not F and G, producing A * * C * D * E * * G *.

The chain of fifths arrangement is derived logically from the stepspans and keyspans of rungs #1 & #2. For example, set the # of keys to 16 and the # of names to 9. Your notes will be A B C D E F G H I. The first note A is always placed on the lowest key in the diagram. Because rung #2's degree is 6, the next note in the chain is 5 steps from A, which is F. Because rung #2's keyspan is 9, F is placed 9 keys to the right of A. The next note in the chain is 5 steps from F. Counting 5 steps and wrapping around brings us to B. The fourth's keyspan is 16 - 9 = 7. Therefore B is placed 7 keys to the left of F. The next note in the chain is 5 notes after B, which is G. It's placed 9 keys to the right of B. And so on. The chain runs A – F – B – G – C – H – D – I – E and the notes run A * B * C * D * E F * G * H * I.

The even distribution is the simplest method. It starts off by placing the first note A on the lowest key, then using logic similar to Figures 5.2.3 and 5.2.4 to place the rest of the white keys.

If the # of names is 6, the first two methods are almost always impossible. One might conclude that the hexatonic framework isn't very useful. But if you set rung #2 to 9/8, it works better than the heptatonic framework.

Remember, you are not limited to the placements that these three buttons provide. You can reposition the white keys anywhere you want. You can even put several names on one key.

Alt-tuner supports the **<u>ups-and-downs notation</u>** discussed in Part V of this book. The sharp symbol ♯ represents a certain number of keys on the keyboard, as shown in the "1 sharp = [] keys" display. The flat symbol is of course the opposite of the sharp. Thus C – C♯, D – D♯ and E♭– E all have the same keyspan, which is the keyspan of the sharp symbol. With 7 note names, this keyspan is found by going up 7 fifths and octave-reducing. Sometimes this method would place C♯ just flat of C. If so, the meaning of sharp is reversed to mean 7 <u>fourths</u> up, octave-reduced. Normally, the extended chain of fifths has sharps on the right and flats on the left:

F♭ – C♭ – G♭ – D♭ – A♭ – E♭ – B♭ – F – C – G – D – A – E – B – F♯ – C♯ – G♯ – D♯ – A♯ – E♯ – B♯

However, if the meaning of sharp is reversed to mean 7 fourths, the sharps are on the fourthward side:

F♯ – C♯ – G♯ – D♯ – A♯ – E♯ – B♯ – F – C – G – D – A – E – B – F♭ – C♭ – G♭ – D♭ – A♭ – E♭ – B♭

Thus B♯ – F is a perfect fifth, as is B – F♭. This is called a fourthward system. This happens whenever the fifth's keyspan is less than 4/7 of the octave's keyspan, as with 9, 11, 16 or 23 keys per octave.

Most pentatonic systems are fourthward:

C♯ – G♯ – D♯ – A♯ – E♯ – C – G – D – A – E – C♭ – G♭ – D♭ – A♭ – E♭

The sharp symbol corresponds to 5 fourths up. However, if the fifth's keyspan is more than 3/5 of the octave's keyspan, the sharp corresponds to 5 fifths up. For example, 8, 13 or 18 keys per octave make fifthward pentatonic systems:

C♭ – G♭ – D♭ – A♭ – E♭ – C – G – D – A – E – C♯ – G♯ – D♯ – A♯ – E♯

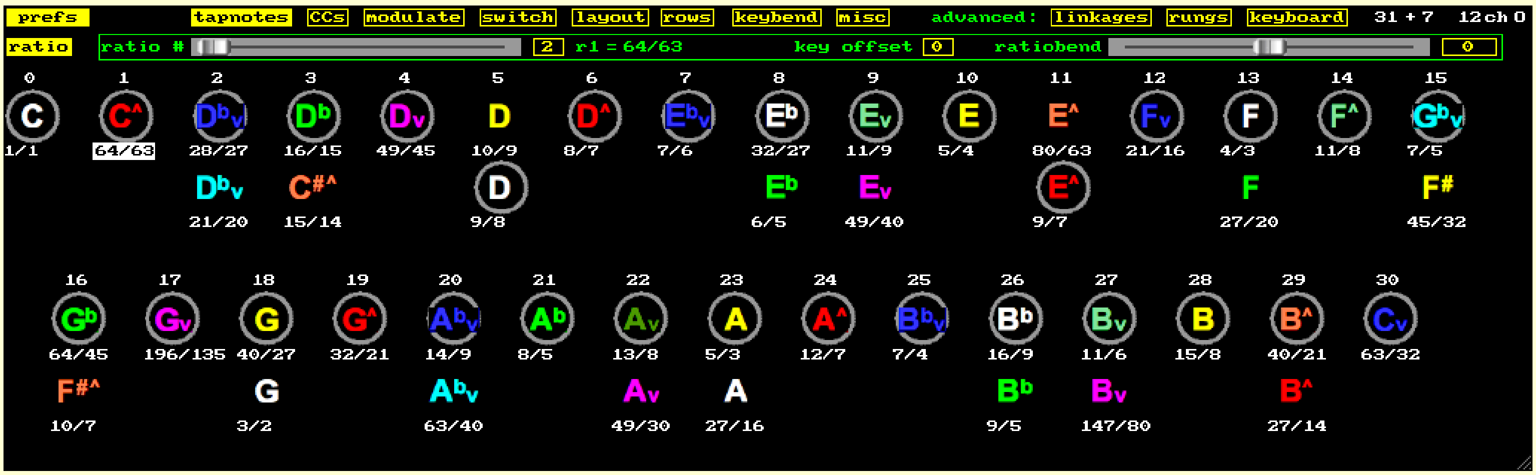
More generally, the sharp's keyspan is $(S_1 \cdot K_2) - (S_2 \cdot K_1)$, where S_1 is the # of names, K_1 is the # of keys, S_2 is rung #2's stepspan, and K_2 is rung #2's keyspan. Whenever you change K_1, K_2, S_1 or S_2, the sharp's keyspan is automatically recalculated by this formula. Changing the ratio for rungs #1 and #2 will affect their keyspans and stepspans, and thus will affect the sharp's keyspan too.

You can modify the sharp's default keyspan by clicking on the yellow number in the "1 sharp = [_] keys" display. The number can range from zero to half of the # of keys. Clicking on a placement button will not only reset the key positions but also reset the sharp's keyspan to the default.

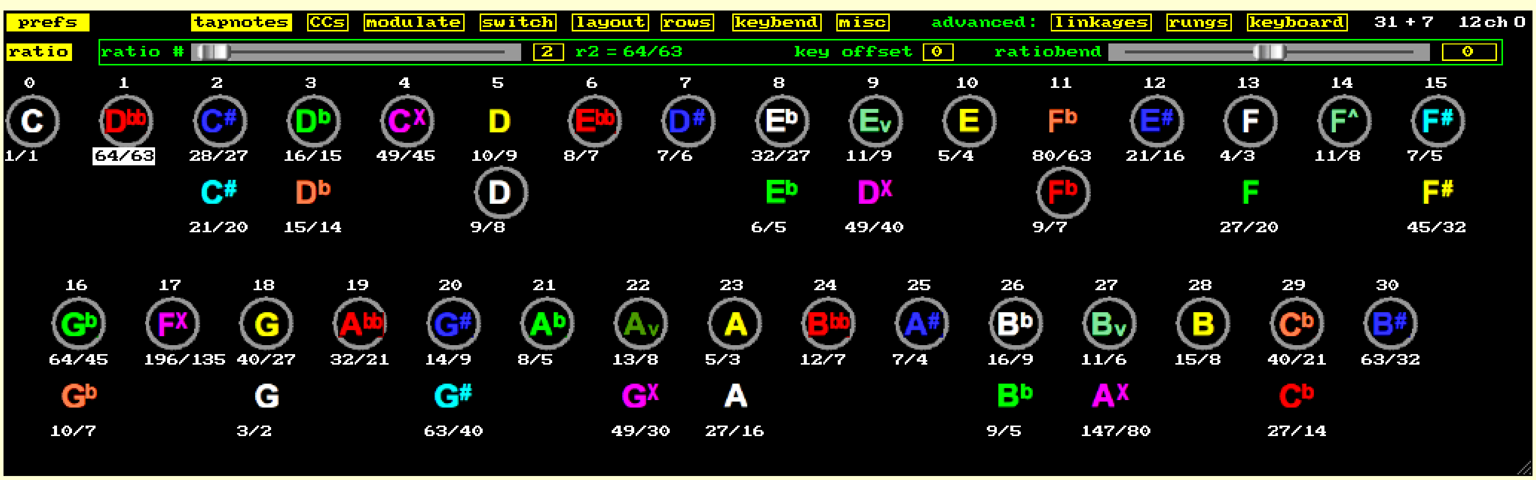
When the sharp is more than one key wide, the question arises, what is the key in between C and C♯ called? Alt-tuner uses the up and down symbols "^" and "v" to represent a change of just one key. Thus three adjacent keys might appear as C – C^ – C♯. If the sharp is four or more keys wide, alt-tuner uses special symbols for double-ups and double-downs that look like a sergeant's insignia. For triple-ups, etc., alt-tuner uses an exponent ("^3"). You can create your own custom up and down symbols as well as custom sharp and flat symbols. You can also create custom double-ups, double-sharps, triple-ups, quadruple-flats, etc. See chapter 6.12 for customizing info.

Dual accidentals are possible, for example C♯v or D♭^. If the # of keys is a multiple of the # of names (e.g. 14 + 7 or 15 + 5), the sharp's keyspan defaults to zero, the sharp and flat symbols are not used, and ups and downs are used in their place. If you prefer to use sharps and flats instead of ups and downs in these situations, simply set the sharp's keyspan to 1.

If the sharp's keyspan is greater than 1, ups and downs are usually needed. You can avoid them by reducing the sharp's keyspan to 1. Alternatively, you may be able to avoid them by instead altering the rung degree. For example, starting from alt-tuner's default state, set the # of keys per octave to 31. The sharp will be 2 keys wide and the arrangement will be FCGDAEB. Go to the rows screen and set the jade row's keyspan offset to zero. Go to the tapnotes screen. Many ratios will use ups and downs, and only the 5-limit ratios (white, yellow and green) will not.



Observe the 7/4 ratio. It is defined as a 7th, but because it is an odd number of keys from the white minor 7th, and the sharp's keyspan is 2, it requires a down. To remedy this, go to the rungs screen and set the blue rung's stepspan to 6. Because 7/4 is an even number of steps from the white major 6th, it no longer needs an up or a down. The purple ratios now have a rather unreasonable degree, remedy this by going to the rows screen and setting the purple row's degree offset to zero. Now all the 7-limit ratios will be without ups and downs.



To have the 11-limit ratios also without ups and downs, set the jade rung's degree to either 3 or 5. This will make 11/8 become either E𝄪 or G𝄫 . Note that while these changes simplify the notation by requiring fewer symbols, they inevitably complicate it by creating many more sequences such as E – F♭ – E♯ – F, in which the note names run out of order.

Below the keyboard diagram are colored notes which are fully clickable, just like the ones in the graph and table views. The center note of the lattice is marked with a tilde "~". The note colors and the location of the tilde, as well as the reference pitch box, all reflect the current custom tuning and may change when you switch tunings.

The "middle-C" slider specifies how the keyboard diagram is mapped onto the actual keyboard. Changing this slider does not shift the keyboard's overall pitch up or down. That's because in alt-tuner, middle-C refers not to a specific pitch but to a specific key on the keyboard. Midi note #60 corresponds to the middle-C arrow's key, note #61 corresponds to the next key up, and so forth. (If you do want to transpose the keyboard, use ReaControlMIDI.)

The middle-C slider is very useful if the # of keys is not 12, because it controls which physical keys get which note names. Otherwise it's somewhat pointless and confusing. It affects the naming of the keys, and hence for a given scale it will affect the tuning of the keyboard. For 12 keys and the 7-limit JI tuning of the first preset scale, moving the slider up two semitones to the D key will cause the physical C key to be labeled D on the screen, and the physical B♭ key to be labeled C. The default "centaur" scale in C will be transposed down 2 semitones and start on that B♭ key instead.

When "# of keys" is anything other than 12, think of the "frequency to calibrate to" number on the prefs/misc screen as actually setting the frequency of middle-C, not high A. The mid-C frequency is the high-A frequency divided by the fourth root of 8 = 1.6818. This corresponds to a difference of a 12-ET maj 6th = 900¢. When the key that the "middle-C" arrow points to is the center note of the lattice and the cents offset is zero, the middle-C key sends out this exact frequency. (Unless EDOtap is on, and the tonic has been tapped up.)

The midi channel monitor on the prefs/misc screen uses standard note names like C#. These note names won't reflect naming changes made on the keyboard screen. In octave mode, the monitor's note names usually don't have an octave number. But if the "# of keys" is other than 12, there will be an octave number. It indicates a key in the keyboard diagram, and represents the pitch class containing that key.

Each line on the graph and table screens is assigned a quality and a degree. Major, minor, perfect and augmented are all qualities. Third, fourth and fifth are degrees. These assignments are somewhat arbitrary. For example, a 6-semitone interval (the red line) is labeled as an augmented 4th, even though it's often a diminished 5th. When the number of keys or names is changed, alt-tuner automatically recalculates these qualities and degrees to fit the new framework. There are several possible ways to do this; alt-tuner uses the method described in chapter 5.2. This method depends only on the number of keys and the number of names to assign keyspans and qualities to each degree.

You can modify these default qualities and degrees by clicking on the yellow boxes under the keyboard diagram. The reset button will return you to the default qualities and degrees described above. The modifications you make here will mainly affect the color of the graph lines and table rows, and the placement of gray lines on the graph. They may also affect the top line interval display and the rung qualities on the rungs screen.

The unison and octave are an unclickable green because their quality is always perfect. The unison's degree is always 1, and the octave's degree is controlled by the "# of names" slider. Each key's degree is constrained by its neighboring values. For example, the tritone's degree can be changed from an aug 4th to a dim 5th, but not to a 6th. The qualities cycle from dim to aug. Alt-tuner distinguishes between augmented from major and augmented from perfect. Changing the quality of one key will affect all other keys with the same degree. For example, changing the maj 2nd to an aug 2nd pushes the min 2nd up to a perf 2nd.

Chapter 6.8 – Advanced Topics

So far, alt-tuner has been very lattice-centric, using the three "R"s: rungs, rows and ratios. Permabending and EDOtapping let you move beyond the lattice.

Permabending: Permabending allows you to "break the rules" and tune any note exactly how you want it. There are two kinds of permabend: ratiobend bends one tapnote in all octaves, and keybend bends all the tapnotes of one key in one octave. Permabending can be done with the mouse in the tapnotes screen or the keybend screen, or with a pedal.

First set up a permabender on the prefs/CCs screen. Permabenders are usually pedals, because physical keyswitches are more awkward to use. To permabend a ratio, press the permabend pedal, hold down a key, and move the pitchbend wheel. If you're in the graph view, you should see the lines move in response. Release the pitchbend wheel to return the ratio to normal. Now permabend the note again, but this time release the pedal or the key first. The permabend will be locked in. To remove it, permabend the note again and release the wheel first. Ratiobending affects only the currently selected tapnote. To permabend all the tapnotes of one key, permabend each one individually, or else use keybends.

On the tapnotes screen, you can permabend by dragging a slider and also clicking on a number, for more accuracy. Go to the tapnotes screen and select the "sizes" view. Click on the note you want to permabend. If the note isn't circled in gray, click on it again to select it. Play that note and permabend that ratio by switcher and wheel as before. The permabend box in the upper right will show you your exact bend. The white caption below the ratio's note will show you its exact size. You can fine-tune the permabend by clicking or right-clicking on the permabend box. The maximum permabend range is the same as the wheel bend range, which is set in alt-keyswitcher.

If you're on the prefs/keybend screen, the permabender can be used with the pitch bend wheel to create keybends, not ratiobends. The bending range is limited by both the wheel bend range and the screen's key bend range.

Permabends are not stretchable. Permabends are not switchable; if you permabend a ratio or a key and switch to another tuning, it will still be bent. Permabends are stored in Reaper presets and project files like everything else.

A permabend pedal also affects tuning taps. If you hold the permabender down while tapping, the direction of the tap will be reversed. This lets you tap up and down with only one tap zone.

EDOtap: allows you to explore an EDO directly. Instead of tapping from one ratio to the next, you tap from one EDO-step to the next. The graph view is the most helpful. Set the EDO slider and click on the EDOtap button in the upper left. Tap away and watch the left edge of the screen. Tapping up too high and bumping up against the next higher key will cause the key to jump down to just above the next lower key. Tapping down too far will likewise jump up. In EDOtap, you can always tap the center note, even if "allow center note taps?" is set to no in the prefs/misc screen.

If the EDO number is less than the number of keys (like 7-EDO in a 12-note octave), two adjacent keys can share the same pitch, and a key tapped up too high will jump down to match the next lower key.

You can't permabend while EDOtap is on. You can't tap a note to silent when EDOtap is on, but you can tap it silent before turning EDOtap on, and it will remain silent.

You can't cycle when EDOtap is on. You can't cycle to 12-ET from EDOtap, and you can't turn on EDOtap from 12-ET. You can however switch to 12-ET while EDOtap is on. EDOtap can be on in one custom tuning and off in another.

EDOtap changes the format of the scale sliders. They show the number of EDO-steps the key spans, plus one; or 0 if the key is silent. If you go to another EDO, EDOtap is automatically turned off, as the EDOtap slider settings are meaningless in a different EDO. However, if you change the # of keys on the keyboard screen, EDOtap remains on.

When you leave EDOtap, either by clicking the EDOtap button or by moving the EDO slider, alt-tuner sets each scale slider to whichever ratio is nearest to the current tuning. Any previous permabends are taken into account.

Holding down a permabend pedal reverses the direction of EDOtaps, very handy for one tap zone and large EDOs.

More about EDOs: 0-EDO means that there is no EDO and the tempering sliders can freely take on any value. 1-EDO is a special setting that resets the temper sliders. If there is a linkage, the sliders will be set to that linkage, otherwise they will be set to JI. If the stretch slider is not 1200¢, the sliders will be stretched. If you want to slightly tweak an EDO, perhaps flattening the white slider slightly, you must go directly to 0-EDO without passing through another EDO. Do this by double-clicking the EDO slider, not dragging it.

When you go to an EDO with the EDO slider, you create the nearest EDO-mapping, with the tempering sliders set to the EDO-step that most closely approximates the untempered rung. Individual ratios are not always approximated as well. For example, 20-EDO has EDO-steps of 60¢. Tw5 is 720¢ = 18¢ sharp, and Ty3 is 360¢ = 26¢ flat. But Tg7 = 1080¢ which is 72¢ sharp of g7 = 9/5 = 996¢. Why isn't Tg7 tuned to the nearest 20-EDO-step, which would be 1020¢? Because that would mistune one of the rungs that make it up. Tg7 is tuned to the sum of the tempered rungs, each of which are set to the nearest EDO-step. You could set the yellow slider to 420¢, making Tg7 = 1020¢, but that would make other ratios "miss" their best fit. For example, y3 = 386¢, but Ty3 would be 420¢, not 360¢.

If the octave is stretched, moving the EDO slider sets the tempering sliders to the EDO-step closest to the stretched, but otherwise untempered rung. If the stretch slider is off to the right at 1300¢, the tempering sliders will also tend to be off to the right. However, double-clicking a tempering slider makes it jump to the EDO-step nearest to the center.

Alt-tuner distinguishes between 12-ET and 12-EDO. You can only get to 12-ET by cycling or switching. You get 12-EDO by moving the EDO slider. The only thing you can do to 12-ET is adjust the cents offset. 12-ET is useful mostly as a quick way to compare alternative tunings to the standard tuning. 12-EDO can be stretched, tempered, linked, offset, etc. 12-EDO is much more useful. For example: set the EDO slider to 12 and set the tempering strength slider to 0%, and modulate to the chord root on every chord change. Now no matter how far out your chords are, the root is always in tune with 12-ET!

EDOs and linkages: If a comma disappears in an EDO's nearest EDO-mapping, that EDO is said to temper out that comma. You can see which EDOs temper out which commas by watching the lattice as you drag the EDO slider. Just look for certain notes to resonate. For example, if the white 4th is selected, and its key is held down, the green 4th will be circled in any EDO that tempers out the green comma, and the blue 4th will be circled when the red comma is tempered out. Make sure you don't have any linkages active when you do this, or you may not get the nearest EDO-mapping. It's important to hold down the white 4th's key (and no other keys) because the green 4th and the blue 4th may be resonating with other notes. For example, 11-EDO doesn't temper out either comma. But because the blue 4th resonates with the yellow 3rd and the green 4th resonates with the blue 3rd, both 4ths will light up. However, they won't be circled. Move the EDO slider to 12-EDO, which does temper out both commas. Now both 4ths will not only resonate but also be circled.

In alt-tuner, the EDO slider always takes precedence over the linkage, breaking it if need be. If a given EDO's nearest EDO-mapping breaks a linkage, alt-tuner will try to find a more distant one that doesn't. You can see this in action by going to the first linkage and unOKing every comma. Next set the EDO slider to 22-EDO. The tempering sliders will jump to the nearest 22-EDO-mapping, which doesn't temper out g1. Now OK the g1 comma, and the yellow slider will jump from 382¢ to the more distant 436¢. Un-OK the comma and the yellow slider will jump back. There are usually other valid EDO-mappings as well, which can be found by adjusting the sliders. In this example, moving the white slider to the left produces Tw5 = 654¢ and Ty3 = 218¢, making an even more distant 22-EDO-mapping which tempers out g1.

If the clear slider is linked directly to another rung, an EDO is created by the linkage itself. This EDO is displayed below the linked rank. A linkage that creates an EDO is easily broken by the EDO slider. For example, the white comma LLw-2 creates 12-EDO and is broken by every EDO that isn't a multiple of 12, like 24-EDO, 36-EDO, etc. When this happens, you'll see a bluish message like "warning: LLw-2 = 240¢" on the linkages screen.

Tempering sliders: If a tempering slider is at its default value, the interval is completely untempered, exactly just. Thus 702.0¢ really equals 701.955¢. And 701.9¢ also equals 701.955¢. However, 702.1¢ is exactly 702.1¢ and not 702.055¢. The interval is only tempered if the slider value is more than minTemper cents from just. MinTemper is usually set equal to the step size of the tempering sliders, so that the two nearest slider settings will produce just intervals. See the chapter on customizing alt-tuner.

You can type in any number you want in the little box to the right of a slider. Thus 3/2 can be tempered to, say, 600¢, even though you can't drag the slider all the way down to 600¢. Also, you can type in, say, 696.578¢ (the quarter-comma meantone 5th), even though the slider can only be dragged to 696.5¢ or 696.6¢.

Preset scales: The scale sliders don't say exactly which ratio to use for a scale degree. They merely point to a spot in your current tapnotes table, which is derived from the rows screen. A slider5 value of 2 means "use the 2nd narrowest ratio for the major 2nd". In other words, the pattern of gray circles in the prefs/tapnotes screen IS the preset.

If you record a song and later change the lattice rows, your song's tuning will change accordingly. This is usually desirable if you substitute one nearby ratio for another, like p4 for y4. But removing a ratio or adding a new one may inadvertently retune that ratio's note. For example, go to the rows screen and shorten the bbg row by setting its "from" to -1. This will remove the purple 5th from the tapnotes screen, which will change your selected fifth from w5 to r5. You can experiment freely with retuning a recorded song by changing the ratios (or tempering, linking, etc.). The change will be temporary unless you save the alt-tuner preset or the Reaper project.

If you change the # of keys in the keyboard screen, or the keyspans in the rungs screen, or if you add or remove rows in the rows screen, or if you change the keyspans in the tapnotes screen, your preset scales will generally be changed, and will have to be re-entered by going to the prefs/misc screen and using "save current scale to preset" repeatedly.

Interval display: alt-tuner displays the size of the interval between the first two keys played after a lull in playing. In other words, to see the display, you have to stop playing entirely, then play only two notes. This is to prevent flashing numbers from distracting you as you play.

If any of the tempering sliders is not at the default setting, the interval display will include the difference the temperament makes in cents. The color and degree will have a "T" for tempered. The only exception is when the tempering strength slider is at 0%. Also, all the cents displays will include tenths of a cent. Note that in meantone, even though TLw3 = Ty3, alt-tuner will still distinguish between them in the display. The interval from wF to wA will be displayed as TLw3 = 81/64 - 21.5¢.

When in an EDO, alt-tuner also displays the interval as the number of EDO "semitones". Playing a 5th in 22-EDO displays the octave fraction "13\22", which equals 13/22 of an octave = 13/22 · 1200¢ = 709¢. The backslash distinguishes the octave fraction "13\22" from the frequency ratio "13/22", which uses a forward slash.

If you're not in an EDO, and you play a ratio that cannot be played from the center note, like a deep yellow ratio, alt-tuner will look for a more accurate and/or simpler alternative ratio. For example, in C, play yB and bgG♭ to get bgg6 = 112/75 = 694¢. Because there is no bgg row in the lattice, alt-tuner will also display Tw5 = 3/2 - 8¢.

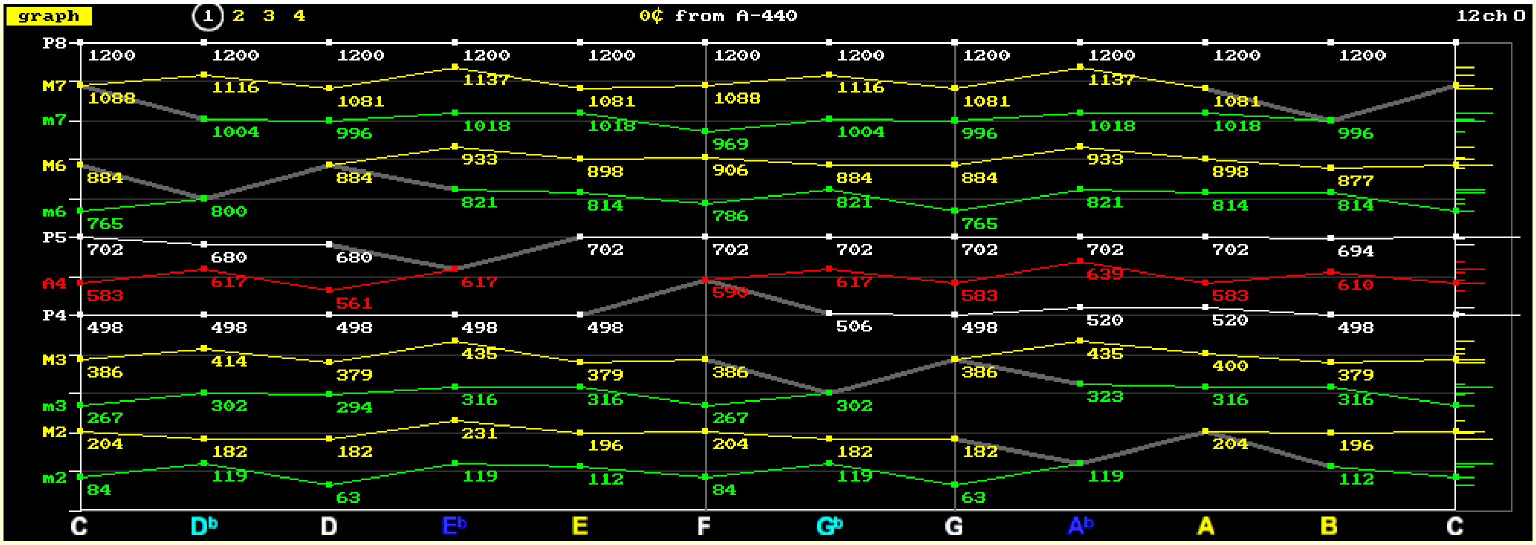
If you want to be in an EDO and see the alternative ratio, just set the EDO slider, and then double-click it to send it back to zero with the tempering sliders unmoved. You will now be non-EDO in name only, and in effect in an EDO.

To find the ratio between two tapnotes, play and hold a note, then tap that note, then play and hold the same note an octave higher. For example, in C, play yD, tap D to white, play a high wD, and the interval display will read "P8 +22¢ = 1222¢ = g8 = 81/40". Double the ratio's bottom number to convert it to g1 = 81/80.

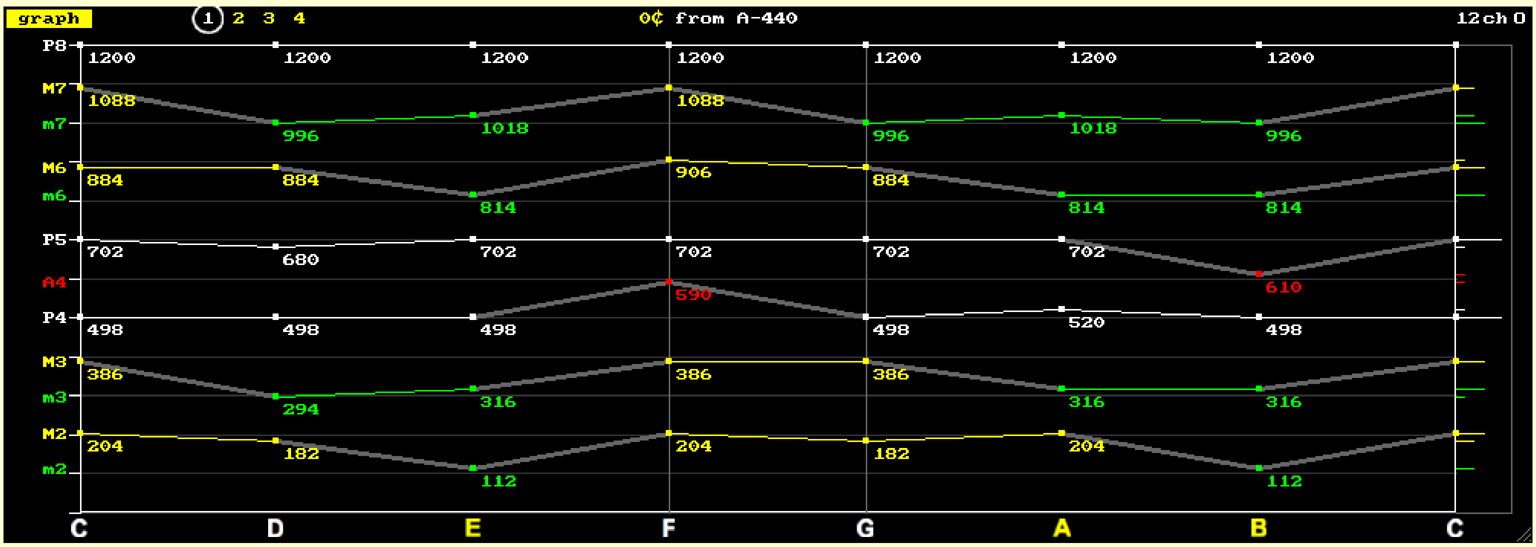
Silent keys: Silent keys are used for two reasons. One, in case a small number of tapnotes or a large number of keys cause ratio-less keys. Two, so that the graph and table views can be used to analyze, say, 7-note scales when the # of keys is 12.

For example, to create a diatonic scale, either go to prefs/misc, allow silent taps, and tap 5 notes silent, or else go to the rows screen and delete all but the first row (which has length 7).

Silent taps allow you to use the graph and table views to analyze scales with less than 12 notes, like diatonic scales. Once you've set up such a scale, you can turn the silent tap option back off, and then click the notes to see different versions of your scale. In the graph and table views, tapping or clicking a note silent makes its whole column disappear, as well as certain notes in other columns. In the graph, the other missing notes create breaks in the colored lines. These breaks will be spanned wherever possible with gray lines linking similar degrees. For example, the two lowest lines are for the major 2nd and minor 2nd. Since they're both 2nds, these two lines will be "stitched together" into one. Here's what the graph looks like for the default scale with B♭ tapped silent:



This graph will instantly reflect any changes to the tempering or EDO sliders. If you tap all the black keys silent, you get a graph of the 5-limit just major scale:



Modulation: There are four ways to modulate in alt-tuner:

> use a modulating foot pedal or physical keyswitch to modulate by a certain interval
> right-click on a note on the screen to modulate to that note
> press a key in a modulating tapzone to modulate to that note
> press an auto-modulate modulator and play a note during the auto-mod window

The first method says what interval to modulate by, the others say what note to modulate to. The second method could be thought of as specifying an interval in the sense that repeatedly clicking on a certain spot on the screen will repeatedly modulate by a certain interval.

The last method is called automatic modulation, as opposed to the other three which are called manual modulation.

Adaptive tuning: Playing certain chord progressions, called comma pumps, creates tuning problems. There will be either mistuned intervals (wolves like y5 = 40/27) or commatic pitch shifts or tonic drift. Tempering solves the problem somewhat by making some or all of the intervals slightly mistuned, making tempered chords. Adaptive tuning lets you play comma pumps without wolf intervals or tonic drift or tempered chords. This is done by keeping whatever chord you're playing in the center of the lattice, to avoid the mistuned intervals at the edges. See chapter 4.2 for more.

A rung has a default cents value which is usually at the black center mark on its slider. It has a tempered value at its slider's fader knob. It has an adaptive value, somewhere in between the two, depending on the tempering strength slider. You hear the adaptive value, but you modulate by the tempered value.

For example, the default value for the white slider is 702¢. Suppose the white slider is set to 710¢ and the tempering strength slider is set to 25%. Playing a fifth makes an interval of 704¢ (25% of the way between 702¢ and 710¢). Modulating by a fifth changes the cents offset by 10¢ (the difference between 710¢ and the 700¢ 12-ET fifth).

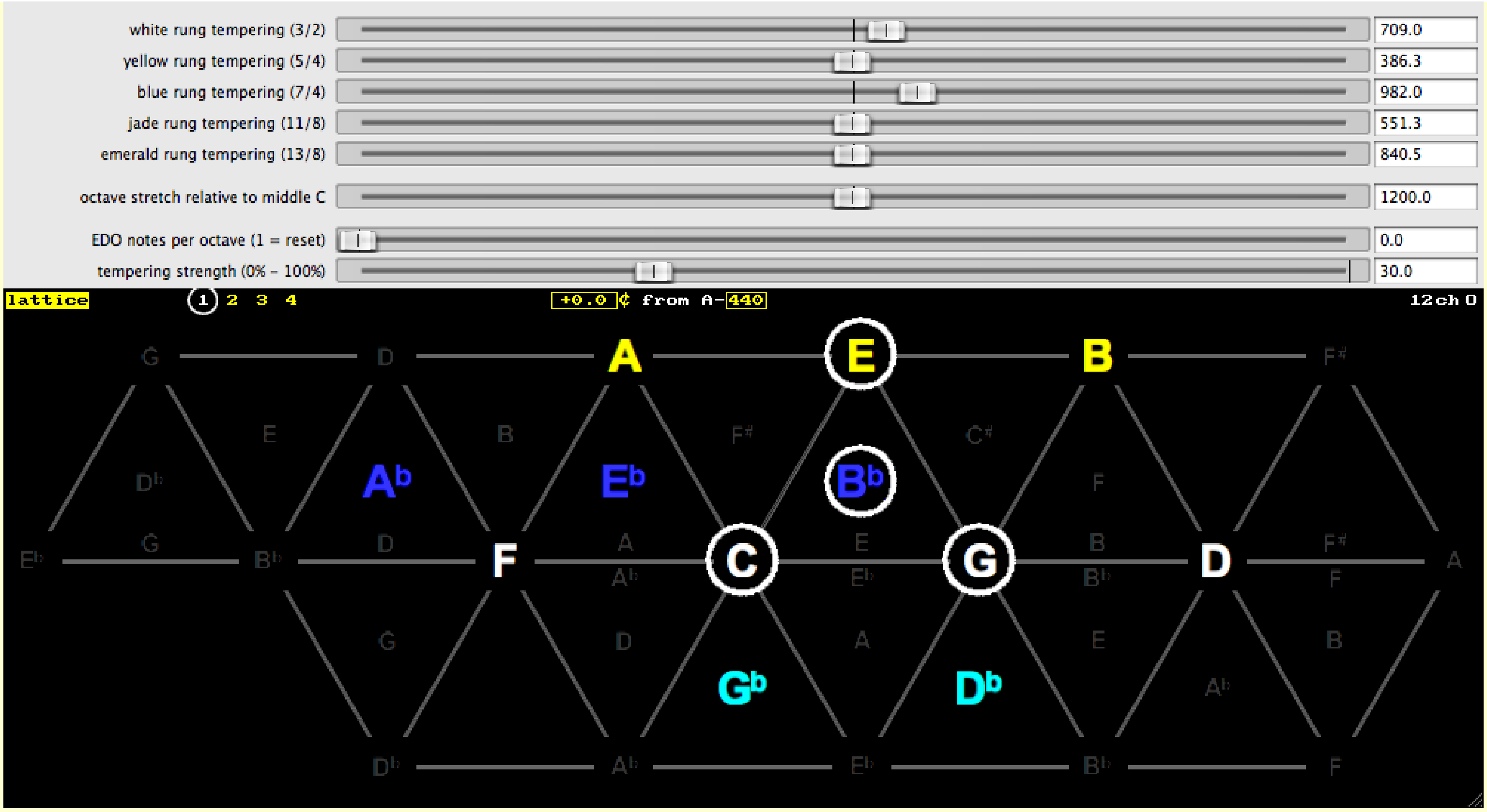
The default value can be changed by redefining the rung (chapter 6.7), and the center mark can be changed by customizing the slider (chapter 6.12).

Auto-modulate: The downside to adaptive tuning is that you have to "steer" alt-tuner through the comma pump so that the lattice is always centered on the current chord. You have to use various modulating pedals on every chord change. Auto-modulate does the steering for you. You'll still be pedaling on every chord change, but only with one pedal. You still have to tell alt-tuner which comma you're pumping. If your piece uses several different comma pumps, set up several custom tunings, each linked by a different comma, and switch from one to the other as you play. Set "preserve the tonic when switching?" to "yes".

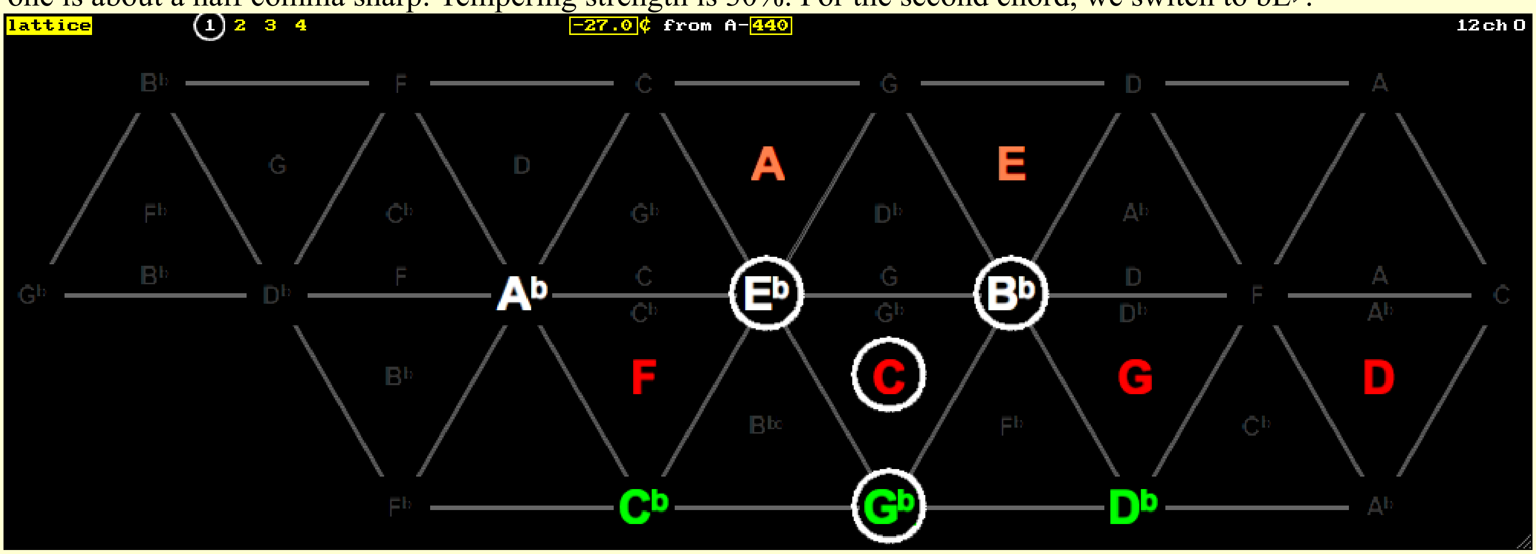
There are two ways to trigger auto-modulate. On the CCs screen, set the function for a pedal to auto-mod. Pressing this pedal triggers auto-mod; releasing it leaves auto-mod. There is also an option in prefs/misc to trigger auto-mod when the sustain pedal is released, which piano players commonly do on chord changes. Pressing the sustain pedal again leaves auto-mod.

When you trigger auto-modulate, alt-tuner modulates to the lowest note played during the auto-modulate window of time. This note will usually be the root of the chord you are playing. If not, it will usually be close enough to the root to keep the chord away from the edges of the lattice, where the wolf intervals lurk. If your retroactive retuning window is 1/10 of a second, the auto-mod window opens 1/10 second before the triggering. In other words, alt-tuner looks back into the recent past for a bass note. Only notes still sounding are considered; you can't auto-mod to a note that has already been released. The auto-mod window closes 1 second later, or when auto-mod is left, whichever comes first. Alt-tuner first modulates to the lowest note in the retuning window, then modulates to any lower note that is subsequently played while the window is open. To see this in action, zip your finger down the keyboard while in auto-mod.

Shift/unshift and auto-shift: The above methods work well for 5-limit comma pumps like Iy – yVIg – y=wIIg – Vy (with Tg1 = w1), but won't work for 7-limit comma pumps like Ih7 – bIIIs6 – b=wVIIh7 – IVh7 (with Tr1 = w1). When you modulate to the b3 for the second chord, there is no g3 or r6 for that chord, and you're forced to use a b3 and a y6. The tonic is a common tone to the first and second chords and is supposed to stay constant, but instead disconcertingly goes flat by about 50¢. The solution is to use shifting rather than modulating. Starting in C, and playing Ch7 – bE♭s6 – b=wB♭h7 – Fh7:



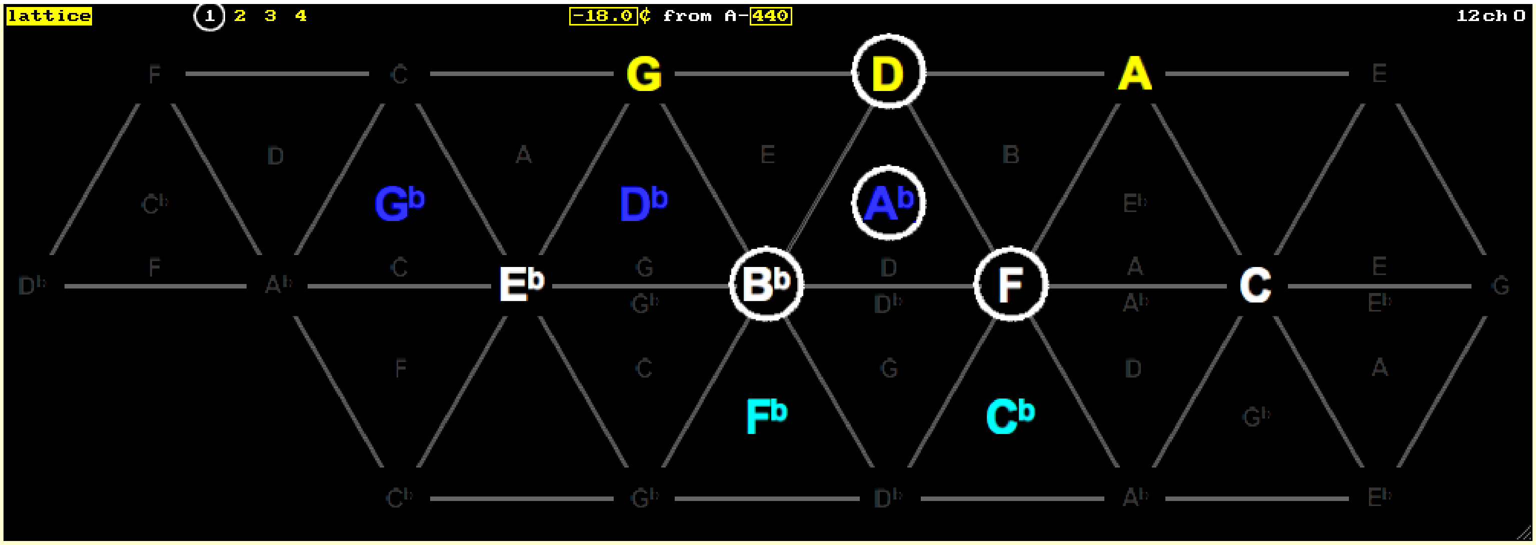
The white and blue sliders are linked by the red comma. The white one is about a quarter comma sharp and the blue one is about a half comma sharp. Tempering strength is 30%. For the second chord, we switch to bE♭:



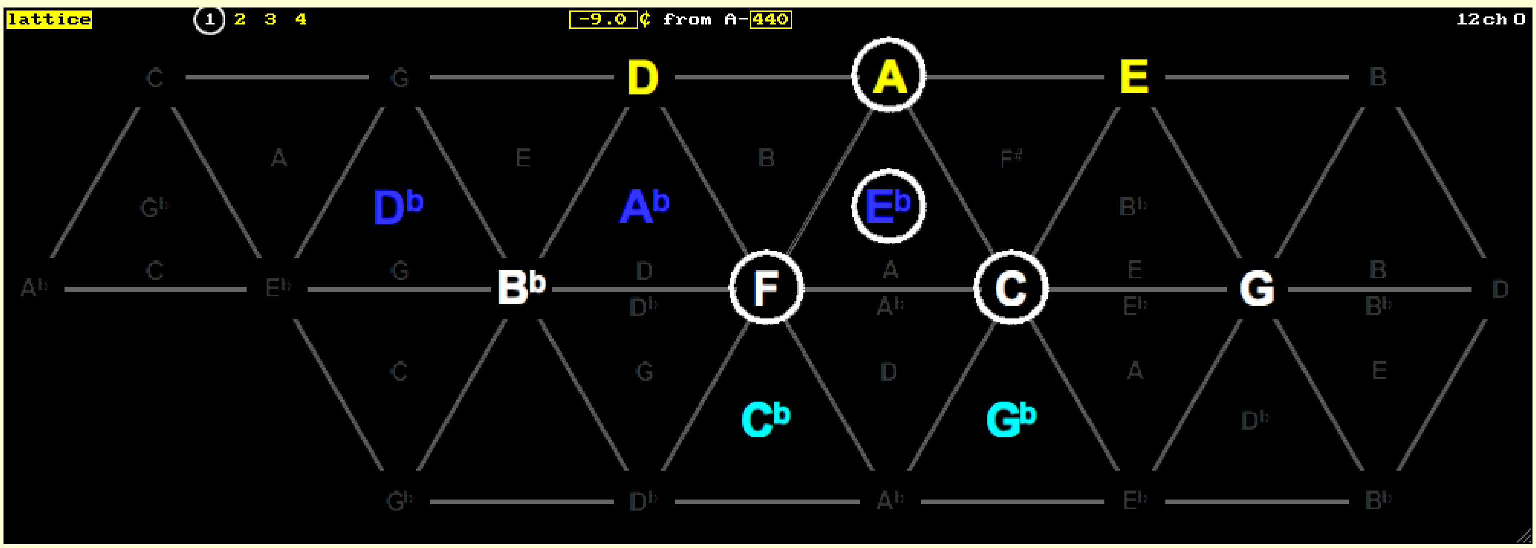
All the notes have shifted down and to the right by an interval of r6, which is the inverse of the modulating interval of b3. There is no ry5 in the lattice, so alt-tuner substitutes the nearest ratio, which is g6. Instead of a reddish B in the upper right, there is a green C♭ on the bottom row. The cents offset is a quarter comma sharper than if we had modulated without tempering. Because the tempering strength is 30%, the C note in the second chord is only 70% of a quarter of a red comma sharper than the C in the first chord. That comes to about 5¢, nearly imperceptible.

After shifting, the gray lines of the lattice have become noticeably brighter. If we were on the graph or table screens, the outer box would become brighter. On the tapnotes screen, the column numbers become black on a white background. On the keyboard screen, the key numbers and the outer box become brighter.

Next we shift again to B♭. Every other time you shift, you actually unshift back to the original set of ratios:



The lattice lines become dim again. This B♭ is about halfway between b7 and w7, and is written b=w7. The cents offset reflects this. To get to F, modulate without shifting:



The cents offset is gradually returning to zero. To get back to C, again modulate without shifting. The cents offset will be zero, and the lattice will look exactly like the first picture.

The complete process is: shift-mod to E♭, unshift-mod to B♭, mod to F, and mod to C. Instead of shift, unshift, mod, mod, it could be shift, mod, unshift, mod. This would result in the B♭7 chord being an r,g7 chord instead of a y,b7 chord. Different combinations of mods and shift-mods will produce different chords.

Notice that when we set up this example, nothing tied us to the key of C. Once you create your linkage and set the tempering sliders and the tempering strength slider, you can play this chord progression in any key. Notice also that if the tempering strength slider were set to 100%, there would be no point to shifting to E♭. Instead you could just play an E♭ chord without any modulating. Shifting is only useful for adaptive tunings.

The prefs/CCs screen lets you set the CC usage to either a shifter or an auto-shifter. A shifter pedal functions like the shift or control key on the computer keyboard. It does nothing by itself, but it modifies the action of other pedals.

There are three ways to shift. The first way is to hold down a shifter pedal while modulating via a modzone tap, a modulating pedal, an auto-mod pedal, or a sustain pedal release if that option is set in prefs/misc. The second way is to press an auto-shift pedal, which is in effect a simultaneous auto-mod pedal press and a shift pedal press. The third way is with mouse clicks: The alt, shift and control keys are called modifier keys. When you hold one of these keys down while clicking, say shift-clicking or alt-clicking, you get a modified click. We have seen how clicking on a lattice note will tap it and either right-clicking or modified clicking will modulate to it. To shift-modulate to a note, use either a modified right-click, like shift-right-click, or a double-modified left click, like alt-shift-click or control-shift-click, or press both mouse buttons simultaneously.

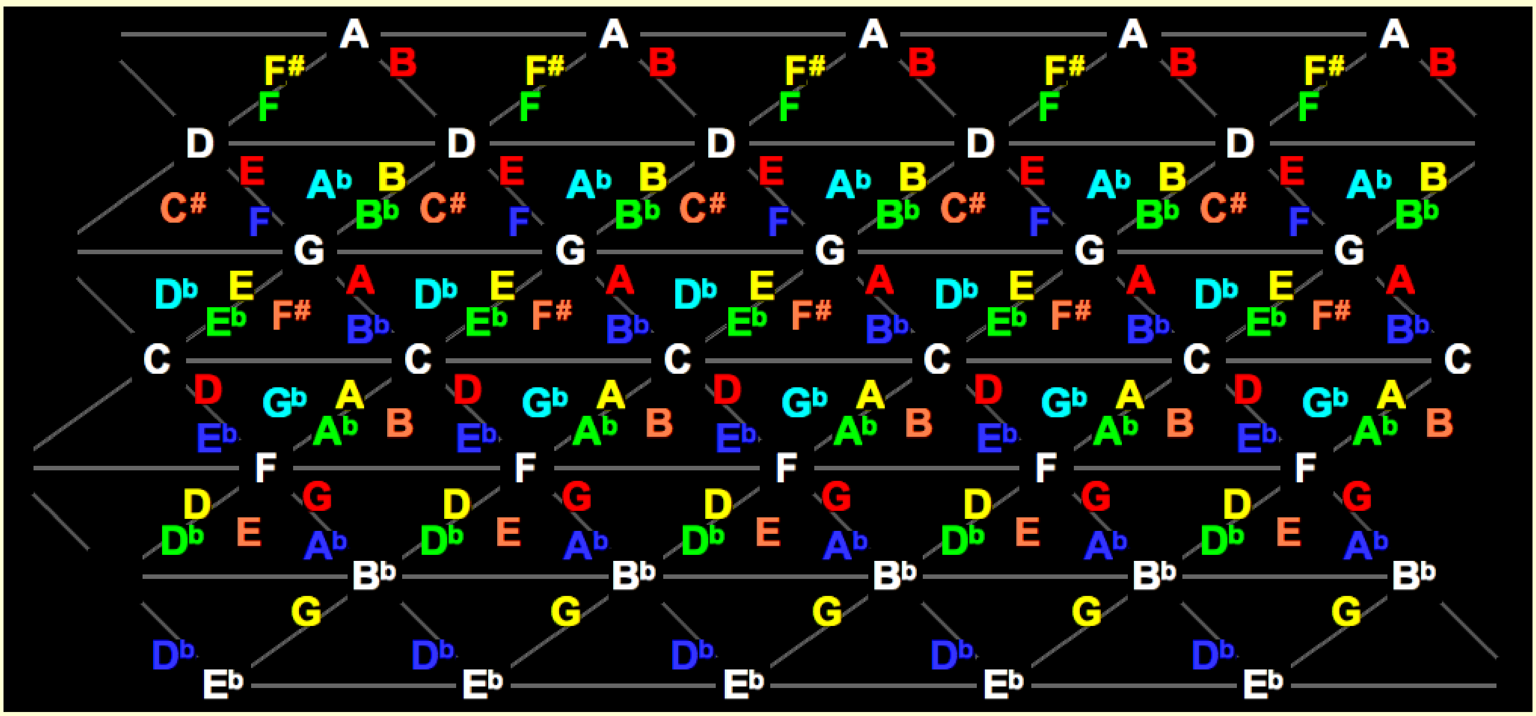
Of the three possibilities, auto-mod, shift/unshift, and auto-shift, which one(s) should you use? Here are three setups:

Auto-mod only: if shifting is not needed, but auto-modulation without using the sustain pedal is needed.
Shift only: if shifting is needed, and auto-modulation either isn't needed or is done via sustain pedal release.
Auto-mod and auto-shift: if shifting is needed, and auto-modulation without using the sustain pedal is also needed.

In addition, all three types of pedal may be used in certain advanced situations.

<u>Octave lattices</u>: If on the prefs/layout screen you set rung #1 to be visible, your lattice becomes a visible-octave lattice, or octave lattice for short. (If rung #1 is not 2/1, just substitute "period" for "octave" in this section.) When you update your CC #s, alt-keyswitcher reports the highest and lowest playable (not a physical keyswitch and not in a tapzone) keys to alt-tuner. These are used for octave lattices to decide how many keys of each pitch class to display. These keys are not reported if note filtering is on. To use keyboard A's range with keyboard B's CC #s, activate A's alt-keyswitcher, update alt-tuner to load the range, activate B's alt-keyswitcher, set filtering on, and re-update alt-tuner to load all the other info. You can set filtering back off after updating, if desired.

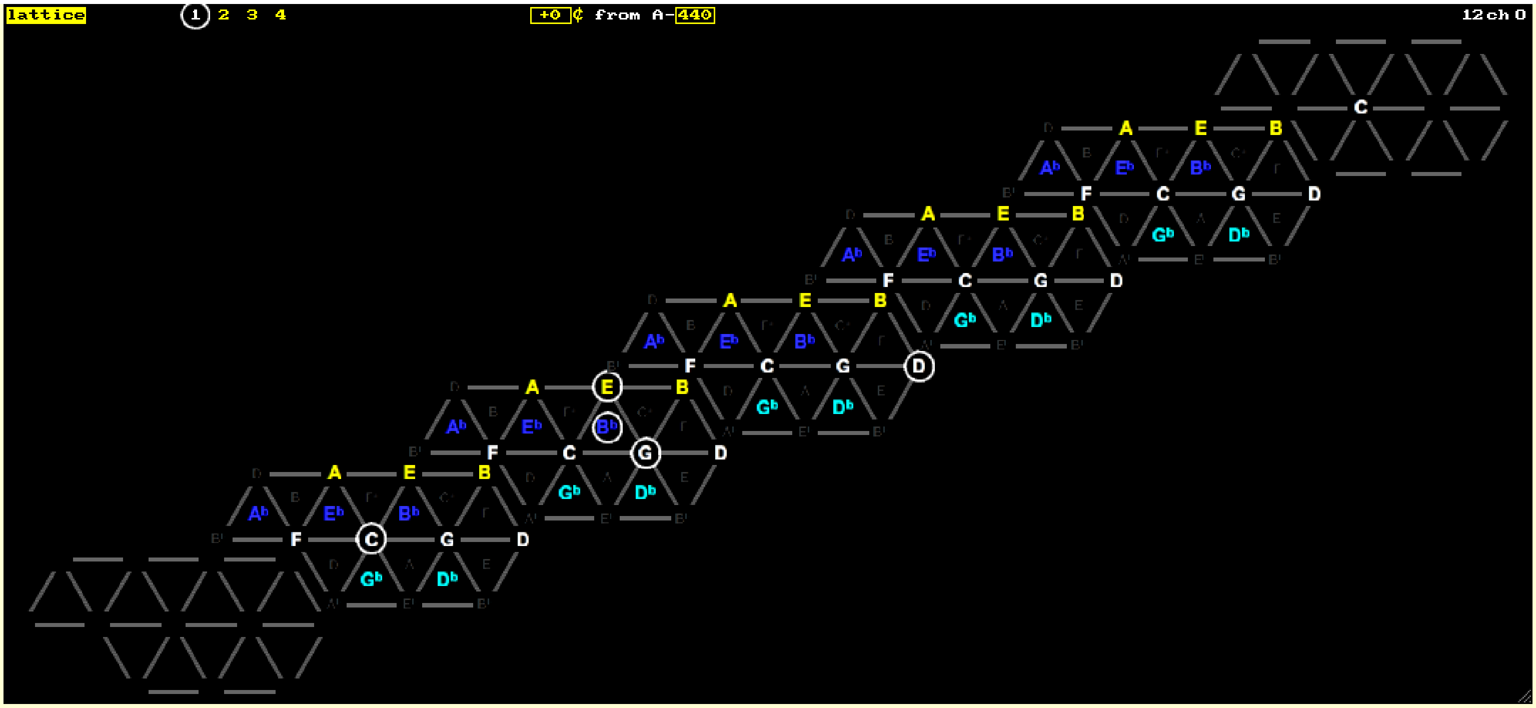
With octave lattices, you can make the horizontal distance between notes exactly equal to the melodic distance. On the layout screen, set each rung's "horiz" to be proportional to the rung's cents on the rungs screen. For example, set rung #1's horiz/vert to 240 & 0, rung #2 to 140 & 100, rung #3 to 77 & 67, and rung #4 to 194 & 27. Set "1st line" to rung #1 and "2nd line" to rung #2, and set the "octave reduce?" option to "no". For a five-octave keyboard that runs from C2 to C7, you would get this lattice:



When octave reduction is off, two white rungs add up to 9/4 = a ninth, but when on, they add up to 9/8 = a second.

Another approach is to use octave layers. Set "octave reduce?" to yes. Set the rungs' horiz/vert as usual. Set "1st line" to rung #2 and "2nd line" to rung #3. Set rung #1 very large, large enough to place each octave of notes in its own cluster. Set the "octave layers" option on, so that each cluster has its own network of lines. Each octave is now a layer of the lattice. The "layers" option doesn't have any effect when "1st line" is rung #1.

You may want to increase the size of the notes on the prefs/misc screen. You may also want to remove unneeded ratios, to make the lattice more readable. Below is a layer lattice for the same five-octave keyboard. The ratios in the lower left of the lattice have been removed, so that the layers can fit together snugly. The lower notes are to the lower left. Five octaves requires seven layers to accommodate other tonics. For example, if you modulate to D, the lowest layer would contain C and C#, and the highest layer would be empty. A C9 chord is being played. The voicing can be read directly from the lattice, it's C2 – E3 – G3 – B♭3 – D4.



<u>Sharing alt-tuner presets with others</u>: All your alt-tuner presets (Reaper presets, not scale presets) are stored in an .ini file, see the last section of the customization chapter. However, because there is no easy way to merge one .ini file with another, sharing this file isn't very useful. To share a preset with others, use a project file. The project need only have one track with no midi or audio in it, with one instance of alt-tuner in the FX chain, set to the relevant preset. You can type the preset's name in the very top of alt-tuner's window, in the comment area just above the presets menu. To share multiple presets in a project file, use multiple instances. The recipient can load the project and then save the alt-tuner settings as their own preset, thus adding it to their existing presets. See also the next two sections:

<u>Versions</u>: The file names for alt-tuner and alt-keyswitcher don't have version numbers. That's because Reaper can't tell that "alt-tuner 1.0" and "alt-tuner 1.1" are related. So it can't transfer presets from one to another, or substitute one for another in your Reaper projects, unless the files have the same name. The version number is shown on the top of alt-tuner's screen, just below the preset list and just above the sliders. In addition, the release date is shown in the screen that appears when you hit the "Edit..." button.

Every version of both alt-tuner and alt-keyswitcher can load all presets created by earlier versions, and can even load those made by future versions, up to a point. Past this point, you'll get an error message, "can't load this preset or project, it's from a newer version of alt-tuner" (or alt-keyswitcher, as the case may be). Click on the message to make it go away. If your alt-keyswitcher is a different version than your alt-tuner, when you try to update the CC #s, you'll get an error message, "can't update the midi CC #s, wrong version of alt-keyswitcher".

<u>Presets from customized versions</u>: You can change alt-tuner's maxNum limits to increase its power, see the customization chapter. For example, suppose you want to make an alt-tuner preset that tunes a 128-note array keyboard to the first 128 notes of the harmonic series. You'll need to increase the maximum number of rungs from 25 to 31, because 127 is the 31st prime, and your preset will reflect this. If someone with a standard version of alt-tuner loads your preset, they can only load settings for the first 25 rungs. Any settings for the other 6 rungs will be ignored. So if you want to share such a preset with a friend, make sure they know what maxNum limits are required, perhaps by naming it "harmonic series scale (31 rungs)".

Conversely, if you load one of your friend's presets into your enhanced alt-tuner, only your first 25 rungs will be modified. The higher rungs will remain unchanged. Likewise if your version of alt-tuner has more than 8 scale presets, loading your friend's alt-tuner preset will not change your extra scale presets. However, this doesn't hold for the other maxNum limits. If you increase the number of tunings from 30 to 40, when you load a preset from a standard version of alt-tuner, tunings #31-40 get initialized to 12-ET. Likewise any extra rows or ratios or switch modes are initialized.

<u>Other files</u>: These files come with alt-tuner and alt-keyswitcher:

<u>Aftertouch Converter</u> converts your keyboard's aftertouch messages to pitch bends and/or mod wheel moves, for greater expressiveness. Aftertouch Converter should go <u>before</u> alt-tuner in the effects chain. The settings are:

channel aftertouches become upward pitch bends
channel aftertouches become downward pitch bends
channel aftertouches become mod wheels
channel aftertouches become mod wheels and upward pitch bends
channel aftertouches become mod wheels and downward pitch bends
filter out all channel aftertouches

Mod wheel output affects all notes played, but pitchbend only affects some. As with the pitchbend wheel, the last note played is bent. In octave-equivalent mode, that note's octave mates in the same pitch class are also bent. Notes can be bent up or down by the full synth bend range, usually 200¢. Small bends can be filtered out, to prevent accidental mistuning. The filtering threshold is based on a percentage of the synth bend range.

<u>Pedalboard fixer</u> corrects a problem with the FBV pedalboard. See "Hardware & Software Issues".

The <u>AltTunerGfx</u> folder contains alt-tuner's graphics files. See "Customizing alt-tuner" for more details.

The <u>solo play</u> and <u>solo play with ReaSynth</u> Reaper projects are designed for a quick start.

The <u>pitch bend sweep.mid</u> file is for synth testing. See "How Retunable Is Your Synth?" in chapter 6.11.

These files are available for all at www.TallKite.com:

<u>Alt-tester</u> is for testing your hardware and software for tune-ability, see "Hardware & Software Issues".

<u>Rechanneler</u> is a utility that reroutes midi from one channel to another, or blocks certain channels. You can even combine two channels into one by sending them to the same channel.

<u>Midi_template</u> is for creating your own Jesusonic effects, see below:

<u>Jesusonic</u>: Reaper has a built-in programming language, Jesusonic, that lets you create your own audio and midi effects. You can run Jesusonic in other DAWs with the free VST ReaJS, which is part of ReaPlugs. ReaPlugs is Windows only. The midi template at www.TallKite.com makes it fairly easy to create midi-only effects for use with alt-tuner. Here is the template:

midi_template.txt

```
desc: describe your midi effect here
/********* Jesusonic midi template by Kite Giedraitis / TallKite Software
MIDI messages: 1 status byte, 1 or 2 data bytes (usually). 1 byte = 8 bits.
Base 16 (hexadecimal): 0 1 2 3 4 5 6 7 8 9 A B C D E F, two hex digits per byte.
Status bytes define the message type, always starts with 8, 9, A, B, C, D, E or F.
16 midi channels, 0-15, all messages (except "F" ones) only affect 1 channel.
A midi cable carries 1 port of 16 channels, a usb cable can carry multiple ports.
Data bytes are only 7 bits long, so data values range 0-127. Examples:
    Mod wheel: 0 = off, 127 = fully on. Keys: 128 total, 60 = middle-C.
8x kk vv = note-off for channel x for note k with velocity v (rare)
9x kk vv = note-on for channel x for note k with velocity v (v=0 for note-offs)
Ax kk vv = polyphonic aftertouch for channel x for note k (rare)
Bx nn vv = controller change for controller #n, changes to value v
    #64 = sustain pedal, #1 = mod wheel, #7 = volume, #10 = pan, etc.
    #120-127 are reserved for all sound off, local control off, etc.
Cx nn = program change = changes the sound for channel x to patch #n
Dx vv = channel aftertouch for channel x of value v (somewhat rare)
Ex vvvv = pitch bend for channel x of value vvvv (0-16383, 8192 = center)
F0 nn nn nn nn ... F7 = sysex message (talks to the sound module)
F1, F2, etc. = start, stop, play, timing clock, etc. No data bytes.
*********************************************************************/

// MIDI-only effect
in_pin:none
out_pin:none

slider1: 0 <0, 16, 1> midi in channel (0 = all channels)
// add more sliders here if needed

@init
samplecount = 0;
NO = 9;           // note-on message
NF = 8;           // note-off message
CC = 11;          // controller change message
PB = 14;          // pitch bend message

@slider
// perhaps do something here to respond to slider movement

@block
while (midirecv (blockOffset, status, databytes)) (
   midiPass = 1;                                    // flag to pass midi through
   msgTime = samplecount + blockOffset;             // time elapsed since the effect started
   msgNum = (status & 240) / 16;                    // message # portion of the status byte
   channelNum = status & 15;                        // channels 1-16 are really 0-15
   note = CCnum = LSB = databytes & 127;            // LSB = least significant byte
   velocity = CCvalue = MSB = (databytes / 256) | 0; // MSB = most significant byte
   isCCmsg = (msgNum == CC);                        // control change message #
   isNoteOnMsg = (msgNum == NO) && (velocity > 0);  // note on message #
   isNoteOffMsg = (msgNum == NO) && (velocity == 0); // zero velocity means note-off
   isNoteOffMsg |= (msgNum == NF);                  // note off message #
   (channelNum == slider1 - 1) || (slider1 == 0) ? ( // midi on "our" channel?

      // do something here with midiPass, blockOffset, msgNum, channelNum or databytes
      1; // placeholder command to prevent an error, delete it after adding other lines here

   );
   midiPass ? midisend (blockOffset | 0, (16 * msgNum + channelNum) | 0, databytes | 0);
);
samplecount += samplesblock;             // keep track of time
```

By changing only a few lines, you can use this template to do almost anything you want to your midi. For example, suppose you're using two keyboards, and want to use the sustain pedals for both as alt-tuner controls. They both send CC #64, so alt-tuner can't distinguish between them, but you want to use them for different purposes. Suppose the first keyboard is on channel 1 and the second one is on channel 2. You can write a short effect that converts sustain messages on channel 2 to something else, perhaps CC #63. All you need to do is replace these lines:

```
// do something here with midiPass, blockOffset, msgNum, channelNum or databytes
1; // placeholder command to prevent an error, delete it after adding other lines here
```

with this line:

```
isCCmsg && CCnum == 64 ? databytes -= 1;
```

Put this effect before alt-keyswitcher and alt-tuner in the effects chain and set the input channel slider to 2. You could make this effect more flexible by adding two sliders to this effect, just before the `@init` section, like so:

```
slider2: 64 <0, 119, 1> input CC #
slider3: 63 <0, 119, 1> output CC #
```

Two more sliders will appear, ranging from 0 to 119 in steps of 1. They will start out set to 64 and 63. The "do something here" line becomes:

```
isCCmsg && CCnum == slider2 ? databytes += slider3 - slider2;
```

In Jesusonic, "a += b" means "a = a + b". You can transform any midi message into any other midi message. See the other effects in your JS/MIDI and JS/IX folders to get an idea of the possibilities. Here's some more:

Reverse the polarity on your sustain pedal:

```
isCCmsg && CCnum == 64 ?            // 64 = sustain pedal message
   databytes = 64 + 256 * (127 - CCvalue);  // change 0 to 127 and 127 to 0
```

Turn your mod wheel into a pitch bend wheel:

```
isCCmsg && CCnum == 1 ? (           // mod wheel message?
   msgNum = PB;                     // pitch bend message
   databytes = 256 * CCvalue;       // CC value becomes MSB of pitch bend value
);
```

Turn your pitch bend wheel into a mod wheel:

```
msgNum == PB ? (                    // pitch bend wheel message?
   msgNum = CC;                     // CC message
   databytes = 256 + MSB;           // CC #1 = mod wheel message
);
```

The next example is a little more involved. It makes any foot pedal work as a bass drum pedal by converting the pedal's CC message to a midi note corresponding to the bass drum sound in your drum VSTi or synth patch. First set up some sliders just before the `@init` section:

```
slider2: 64 <0, 119, 1> input CC #
slider3: 36 <21, 108, 1> output note
slider4: 96 <0, 127, 1> threshold
```

The input CC defaults to the sustain pedal. The output note defaults to note #36, which is C2, which is usually the bass drum. If the input pedal is a simple on/off footswitch, the note's velocity will always be the maximum, 127. But if it's an expression pedal, the note's velocity will be set according to the speed that the pedal was pressed. The threshold determines how far down you have to press the pedal to trigger the note. Here's the "do something here" part:

```
isCCmsg && CCnum == slider2 ? (     // CC message for "our" pedal?
   CCvalue >= slider4               // is the pedal pressed down far enough?
   && oldCCvalue < slider4 ? (      // and it wasn't just before?
      msgNum = NO;                  // note-on message
      v = CCvalue - oldCCvalue;     // calc velocity from the speed of pedal press
      databytes = 256 * slider3 + v;
   ) : CCvalue < slider4            // or else, is the pedal released?
   && oldCCvalue >= slider4 ? (     // and it wasn't just before?
      msgNum = NO;                  // note-on message
```

```
        databytes = 256 * slider3;    // a note-on with velocity zero is a note-off
      ) : midiPass = 0;               // or else, suppress all other pedal messages
      oldCCvalue = CCvalue;           // use the old CC value to calculate velocity
   );
```

More possibilities:

Make a single pedal send out two or more CC messages
Make a key on your keyboard send a CC message
Make a key on your keyboard send one CC message when played and another when released
Midi compressor: make loud midi notes quieter
Midi gate: block quiet midi notes
Route certain midi messages to certain channels
Delay certain midi messages by a certain amount of samples

Jesusonic tips:

1) Each statement should end with a semicolon. The only exception to this is within if-then-else statements.

2) Parentheses must always come in open/close pairs. Lines are usually indented to help keep track of this.

3) Use "=" when assigning a value to a variable and use "==" when checking a variable in an if-statement.

4) Midi channels, which appear on your controls as 1-16, are actually numbered 0-15.

Jesusonic reference:	http://www.cockos.com/reaper/sdk/js/js.php
more Jesusonic info:	http://forum.cockos.com/showthread.php?t=15833
	http://forum.cockos.com/showthread.php?t=6027
Reaper forums:	http://forum.cockos.com/forumdisplay.php?f=20
A good midi guide:	http://www.gweep.ne t/~prefect/eng/reference/protocol/midispec.html

Chapter 6.9 – Advanced Examples

<u>Quick & dirty</u>: Use these methods if you want to input a tuning from a .tun file or a .scl file.

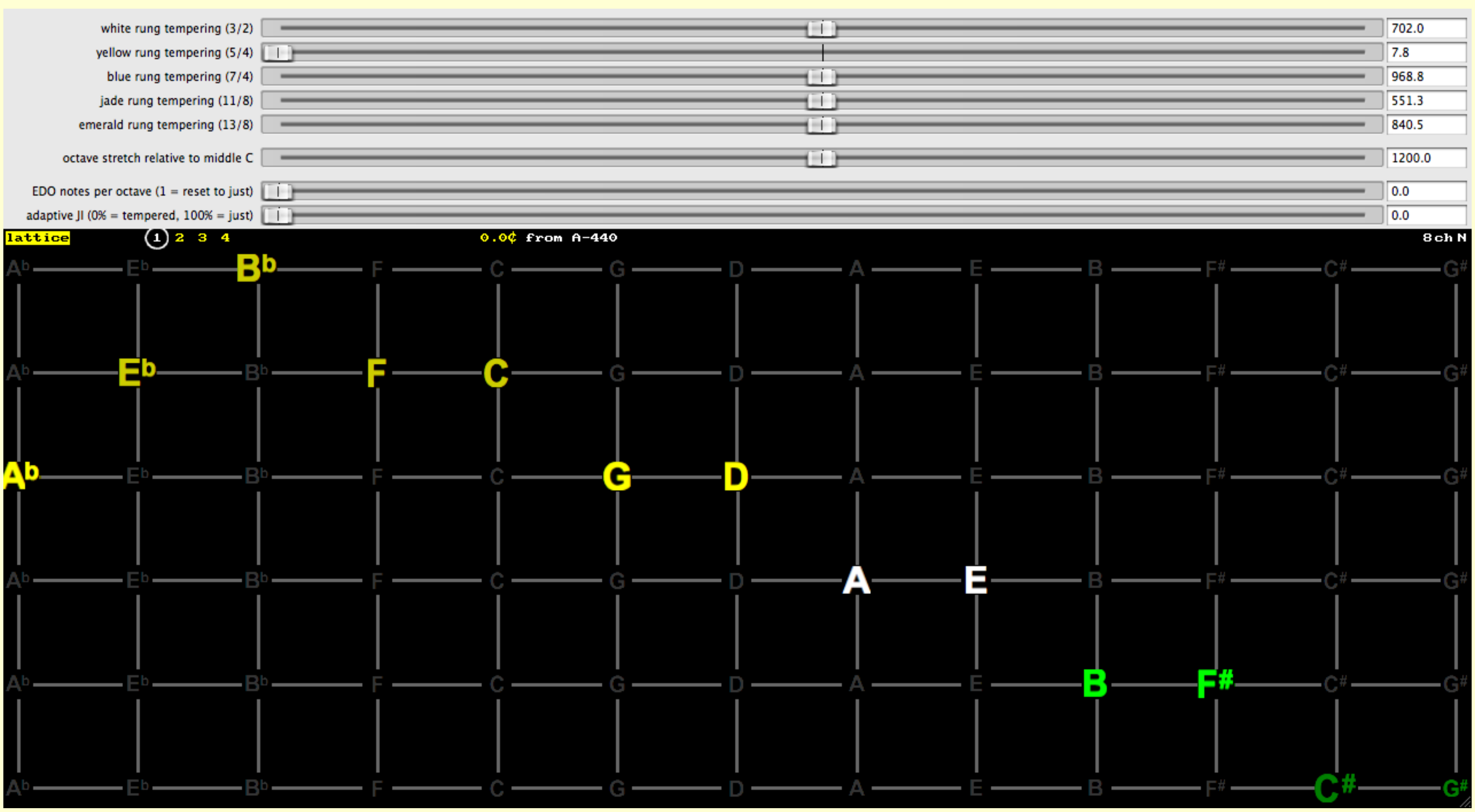
Method #1: Set the EDO slider to 12 and use ratiobend or keybend to tune each note individually.

Method #2: Set the EDO slider to 240 by typing in the number box on the right, turn EDOtap on, and tap each note up/down 5 cents at a time. Or, set the EDO slider to 1200 for 1¢ increments. In prefs/CCs, either set the low tapzone to tap down, or set up a pemabend pedal and use it to reverse the direction of the taps.

Method #3: Use higher rungs for other ratios. To quickly hear a 19/16 minor third = 298¢, set the 6th rung's ratio to 19/12, tap to the emerald 6th, and play a 4th and a minor 6th. Use 19/12, not 19/16, because it has the same keyspan as 13/8 and won't change the preset scales.

<u>Well-temperament</u> refers to historical tunings which tame meantone's wolf fifth. Start in the key of D. Go to the rungs page and set the 3rd rung's ratio to 74/73 as an approximation of the white (pythagorean) comma. Make sure its keyspan is 0 and its degree is 1. Go to the rows page and delete all but the white, yellow and green rows. Add a deep yellow, a triple yellow, and a deep green row. Set the row starts to: w = -6, y = -5, g = -7, yy = -4, y^3 = -3 and gg = -8. Set the ends to w = 6, y = 7, g = 5, yy = 8, y^3 = 9 and gg = 4. Go to the layout screen and set the yellow rung's horizontal to 0. Uncheck the "3rd line" option. Alt-tuner will assign appropriate colors to the new rows automatically. Verify this with the rows slider, and adjust the colors if desired. Go to the misc screen and OK "allow center note taps". Set the size of the colored notes to 20 and the size of the gray notes to 12. Allow gray to turn colored. Save all this as an alt-tuner preset, to use as a starting point for any well-tempered tuning.

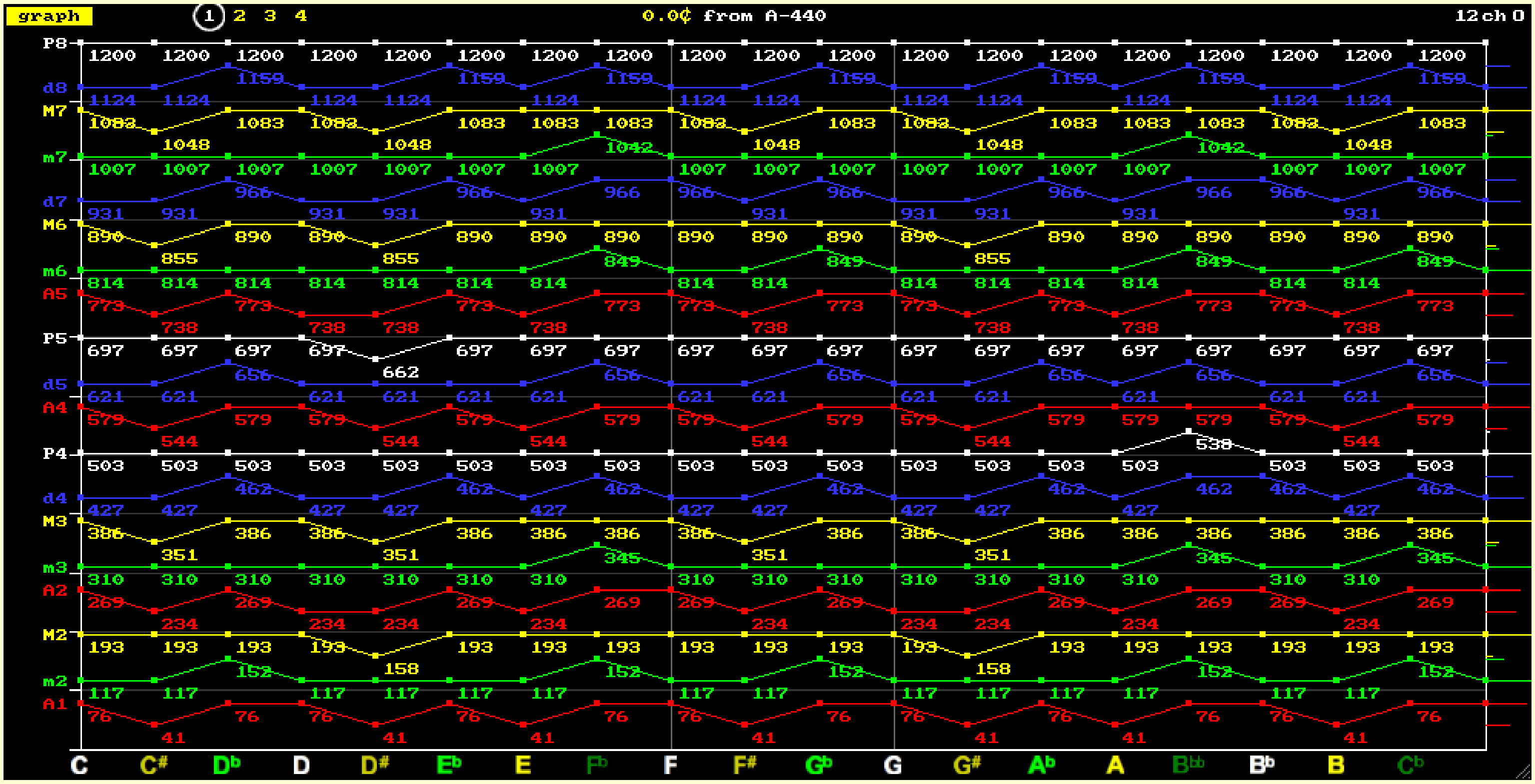
In Werckmeister II the fifths C – G, D – A, E – B, F♯ – C♯, and B♭–F are flattened by 1/3 comma, and the fifths A♭–E♭ and E♭– B♭ are sharpened by 1/3 comma. The other fifths are pure. Set the yellow tempering slider to 7.8¢ = 1/3 of a white comma by typing in the box. Go to the lattice screen and tap the notes to get this tuning:



The flat fifths slope down on the lattice, and the sharp ones slope up. Just fifths run straight across. The G♯ is 3 yellow rungs lower than the equivalent A♭ because it takes 3 yellow rungs to equal a white comma.

Alternatively, set the white rung to 700¢ and the yellow to 1/12 of a comma = 1.955¢, to bring G♯ up to the same row as A♭. A just fifth will slope up one yellow rung, a sharpened fifth slopes up by 4 rungs, and a flattened fifth slopes down by 3 rungs. Your lattice will need 9 rows, but it'll accommodate 1/4-comma and 1/6-comma temperaments too.

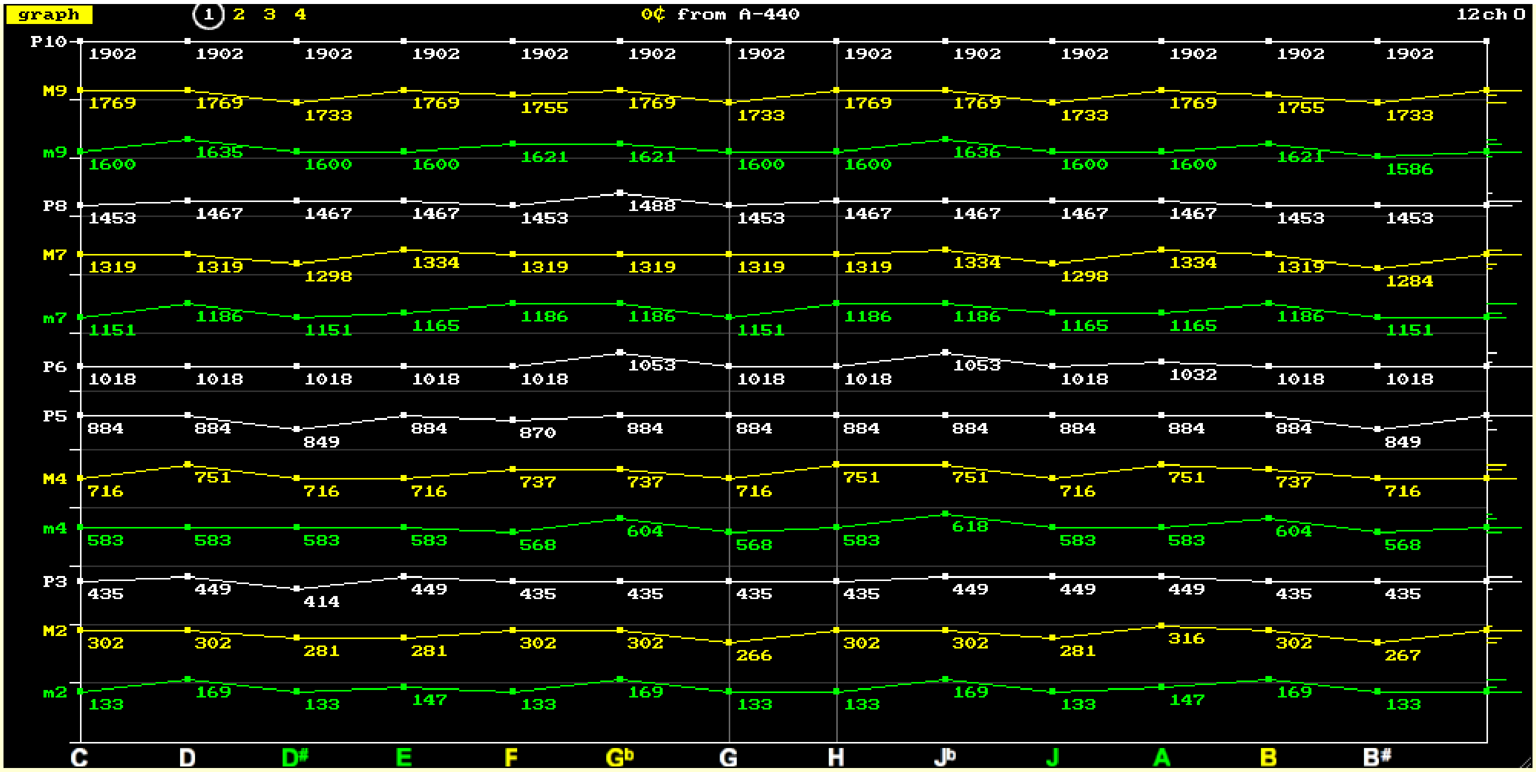
19 keys per octave: A historical tuning that extends meantone using split black keys. Go to the rows screen and delete all but the first 3 rows. Set the green row's "from" to -3. Add a deep yellow row that runs from -1 to 3 and a deep green row from -3 to 1. The new rows' colors will default appropriately. In the linkage screen, OK the 1st comma, 81/80, for meantone. On the keyboard screen, set "# of keys" to 19. Check the tapnotes screen. Note that C♯ and D♭ are different notes. If you're in C, you may see unexpected things like w4 = F♭. Do not set the keyspan here, instead go to the keyboard screen. Drag the white F key's slider to bring it in line with the w4 ratio. Fix other problems similarly. You can click on the colored letters below the keyboard to tap. In this particular linkage, tapping has no effect on the sound. You can verify that in the tapnotes screen by looking at the sizes and steps. The graph has 19 lines per octave:



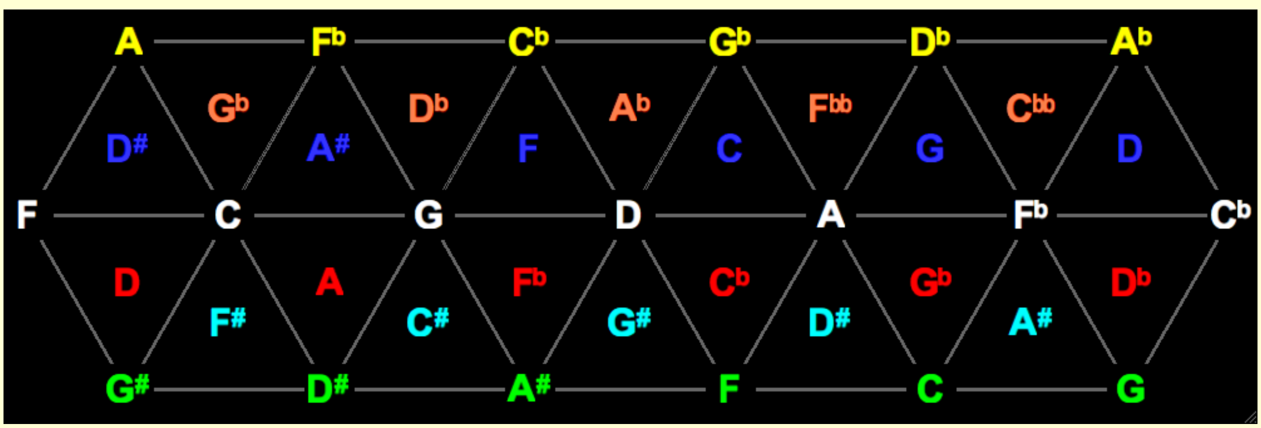
88cET is a non-octave equal tuning made up of an endless series of 88¢ steps. To set it up, first find the 2 nearest EDOs to 88cET. Go to the table view, drag the EDO slider and watch the bottom row for something near 88¢. You will find that both 13-EDO or 14-EDO have "semitones" of approximately 88¢. Start with 14-EDO and stretch the octave until the table shows 88.0¢ for the semitone and 880.0¢ for the ten-semitone interval. Or, start with 13-EDO and compress the octave until you see 88.0¢ and 880.0¢. A stretched 14-EDO octave (88 · 14 = 1232¢) is closer to 2/1 than a compressed 13-EDO octave (88 · 13 = 1144¢), so you might prefer 14-EDO.

Bohlen-Pierce: Another non-octave tuning, it repeats at every wide fifth aka tritave = 3/1 = 1902¢ instead. The tritave is usually considered to be 13 "semitones" wide, and there are usually 9 note names, so a tritave could be considered a "decave". Twos are not used in any ratios. Quick & dirty method for equal tempered Bohlen-Pierce: set the EDO slider to 12 and set the stretch slider to 12/13 of a tritave by typing in 1755.7¢. Every 13 keys will span a tritave, but the note display will be inaccurate. Longer method: go to the keyboard screen and set "# of keys" to 13 and "# of names" to 9. Click the last white key once to use "J" instead of "I". Adjust the note's positions as desired. Go to the rungs screen and set rung #1 to 3/1 and set rung #2 to 2/1. The stretch slider should now be 1902¢ for 3/1. The white tempering slider should be 1200¢ for 2/1, the yellow slider should be 884¢ for 5/3, the blue slider should be 1467¢ for 7/3, etc.

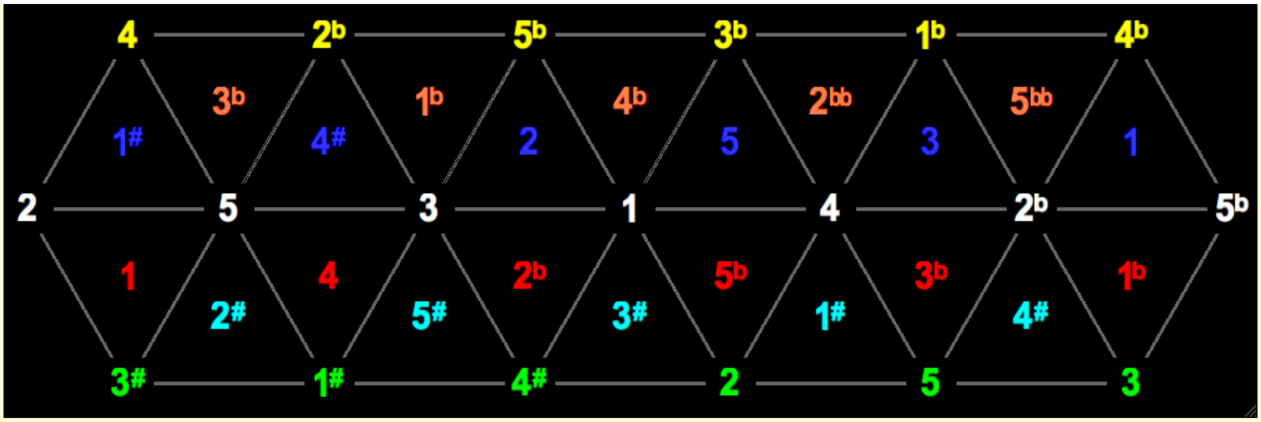
For equal tempered Bohlen-Pierce, set the EDO slider to 13 and you're done. For 7-limit JI B-P, we have to reconstruct the lattice to fill in the 13 keys. For one particular lattice, set "# of rungs" to 3. Set rung #1 to 3/1, rung #2 to 5/1, and rung #3 to 7/1. Go to the rows page and delete all but the first three rows. The 1st row should run 0, -3, 3. The 2nd row should run 1, -2, 3. The 3rd should be -1, -2, 2. Here's the graph for 7-limit JI B-P; see the end of chapter 6.7 for an explanation of the line qualities:



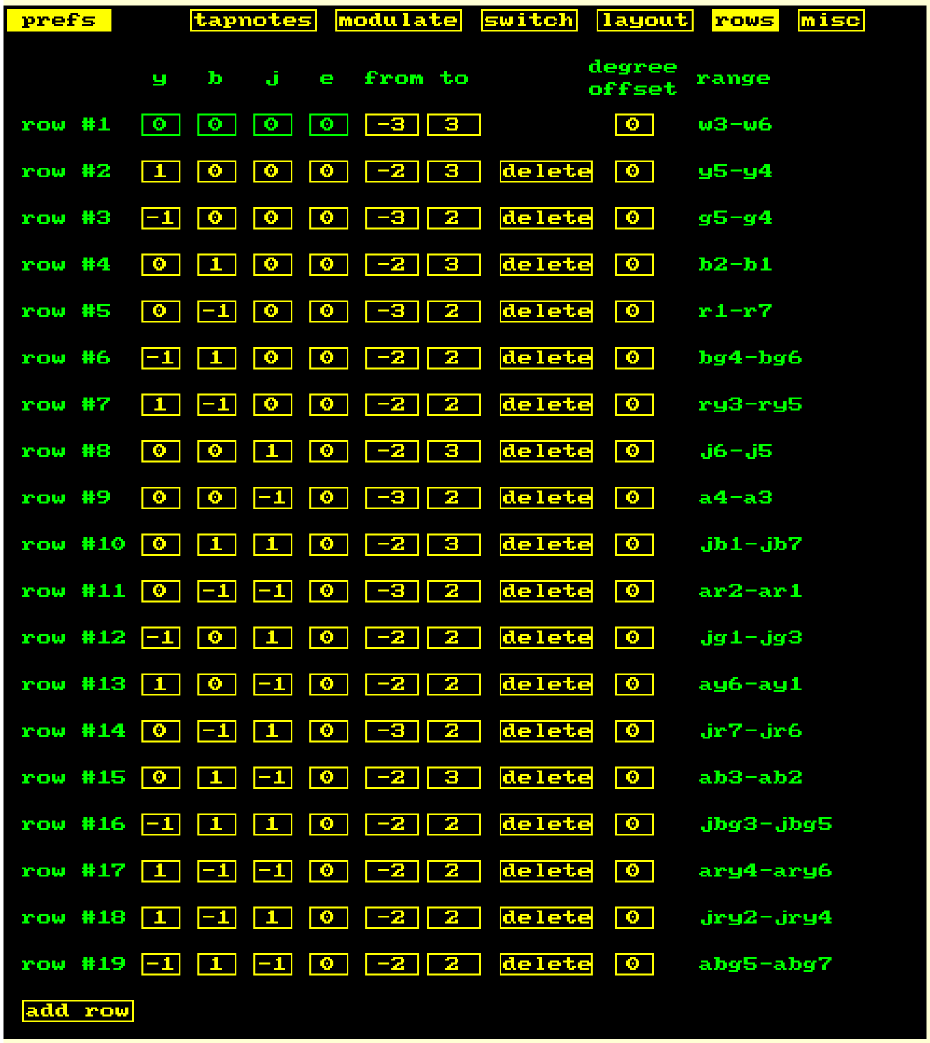
Pentatonic display: not a tuning, just an alternate naming framework. On the keyboard screen, set the # of names to 5. Click on the white keys to set up a scale like "A – C – D – E – G", or "1 – 2 – 3 – 4 – 5". You may want to name your keyboard's black keys and have the white ones unnamed. In other words, have black be natural and white accidental. To do this, mentally reverse the key colors in the keyboard diagram. To name the black keys "H – I – J – K – L", set up your diagram as "* H * I * * J * K * L", and set middle-C to the first key. In the layout screen, set the purple (bbg) row's vertical offset to -265, and in the rows screen, set the purple row's degree offset to +1. The flats are now on the right side of the lattice, because A to F♭ is a fifth, as is A♯ to F.



For the "bingo-card lattice", I like setting 2 to a minor third, 3 to a perfect fourth and 5 to a minor seventh:

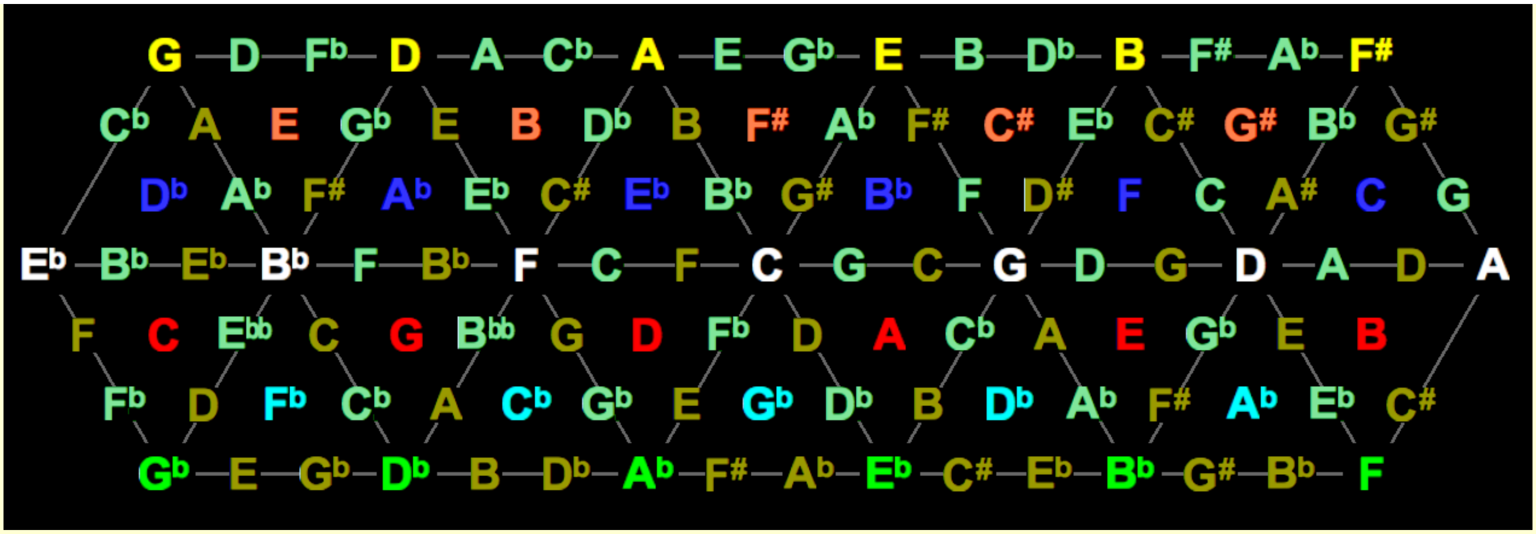


Full 11-limit: Go to the layout screen and set the jade rung's horizontal to 167 and vertical to 58. Set the blue rung's vertical to 58. Go to the rows screen and delete the bluish-blue and emerald rows. Add these 11 rows:



	y	b	j	e	from	to		degree offset	range
row #1	0	0	0	0	-3	3		0	w3-w6
row #2	1	0	0	0	-2	3	delete	0	y5-y4
row #3	-1	0	0	0	-3	2	delete	0	g5-g4
row #4	0	1	0	0	-2	3	delete	0	b2-b1
row #5	0	-1	0	0	-3	2	delete	0	r1-r7
row #6	-1	1	0	0	-2	2	delete	0	bg4-bg6
row #7	1	-1	0	0	-2	2	delete	0	ry3-ry5
row #8	0	0	1	0	-2	3	delete	0	j6-j5
row #9	0	0	-1	0	-3	2	delete	0	a4-a3
row #10	0	1	1	0	-2	3	delete	0	jb1-jb7
row #11	0	-1	-1	0	-3	2	delete	0	ar2-ar1
row #12	-1	0	1	0	-2	2	delete	0	jg1-jg3
row #13	1	0	-1	0	-2	2	delete	0	ay6-ay1
row #14	0	-1	1	0	-3	2	delete	0	jr7-jr6
row #15	0	1	-1	0	-2	3	delete	0	ab3-ab2
row #16	-1	1	1	0	-2	2	delete	0	jbg3-jbg5
row #17	1	-1	-1	0	-2	2	delete	0	ary4-ary6
row #18	1	-1	1	0	-2	2	delete	0	jry2-jry4
row #19	-1	1	-1	0	-2	2	delete	0	abg5-abg7

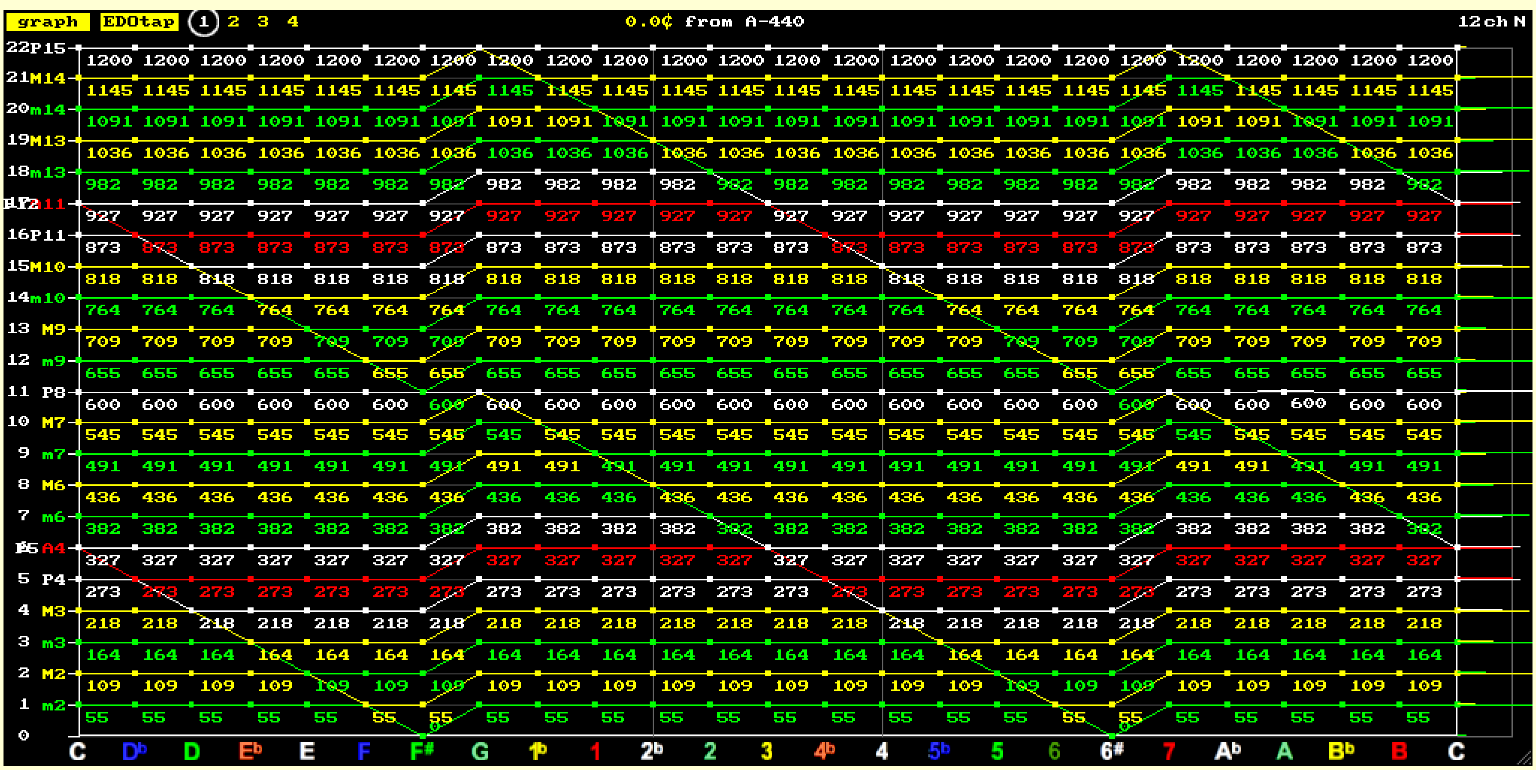
Go to the misc screen and allow center taps. For a pretty picture, set "size of gray letters" to 12, same as colored letters, and set the EDO slider to a low number like 5, so that all the letters turn colored:



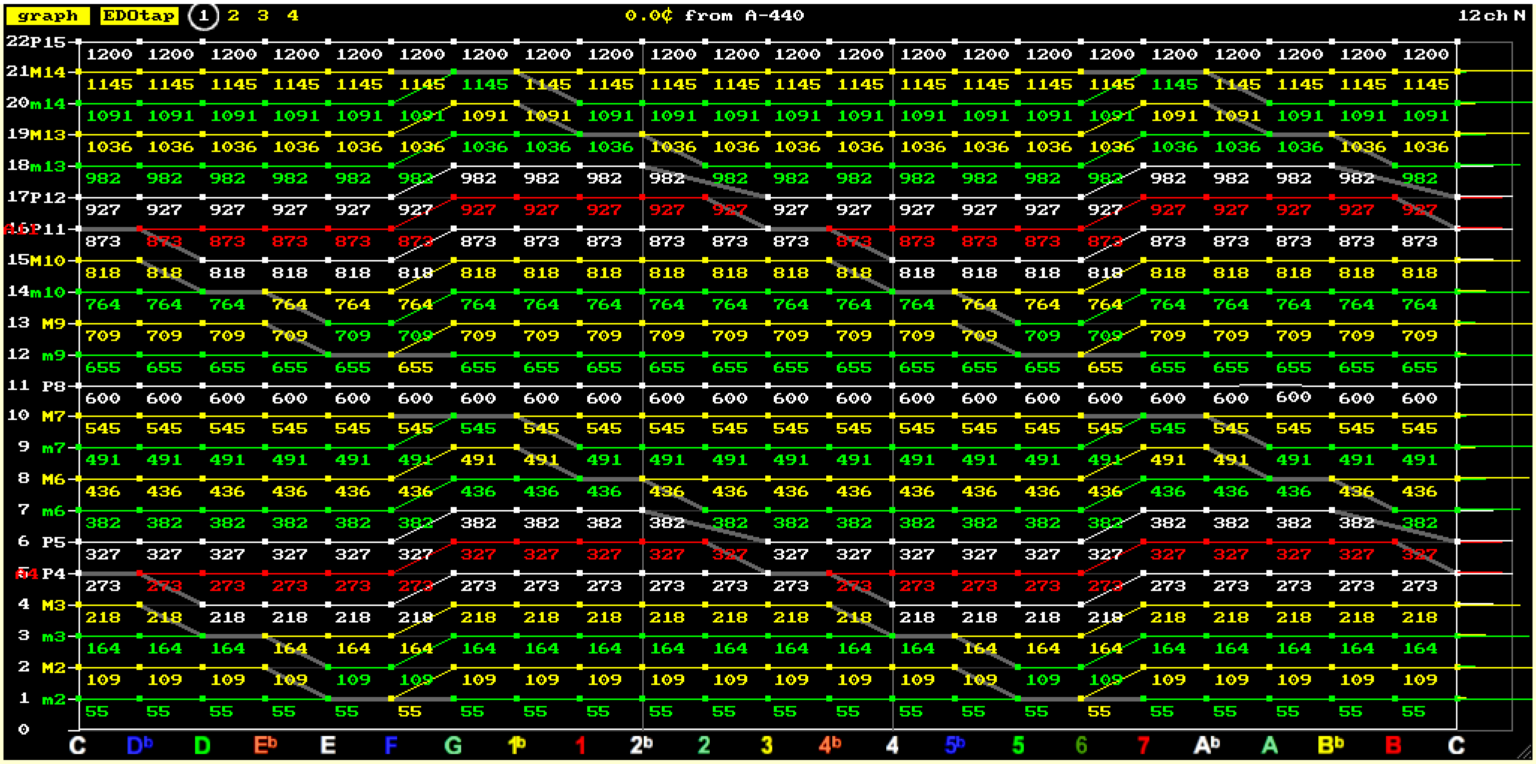
Freely adjustable Blackwood: A 10-tone tuning consisting of two 5-EDO scales, one on the black keys and one on the white keys, with B and C tuned the same, and E and F also tuned the same.

Go to a blank linkages row and enter 5 in the first column. When you OK the comma, you should get sw2 = 90¢. The white slider should be 720¢. Go to the rows screen and set the reddish row's "to" to 2. Go to the lattice and modulate to D. Tap all the white keys to the white row and all the black keys to the reddish row. Go to the graph view and watch as you move either the yellow or the blue slider. The scale will have alternating large and small steps, and will reproduce certain scales in 10-EDO, 15-EDO, 20-EDO, etc. You can even use midi-learn to control the yellow slider with a pedal, to vary the scale as you play. Use a footpedal, not a footswitch.

22-EDO over 24 keys: This tuning has 2 keys per double octave that are either silent or duplicates of neighboring keys, so that it can fit onto a standard keyboard. On the prefs/keyboard screen, set "# of keys" to 24. You may want to set "# of names" to 14, set H-N to 1-7, and adjust F and 6. On the prefs/misc screen, perhaps set "allow silent taps" on. Set the EDO slider to 22. Go to the graph screen, perhaps tap 2 keys silent, then turn EDOtap on, and tap the keys to fill in the gaps. In this picture, the F♯s are duplicates of the Fs:



In this picture, the F♯s are silent:



<u>More than 88 keys</u>: The previous example of 22-EDO over 24 keys gives you less than 4 octaves total range on a standard 88-key keyboard. You can get a bigger range by using two 88-key keyboards at once, one for the lower notes and one for the higher notes.

<u>One-track method</u>: Set the 1st keyboard's output to channel 1 and the 2nd one to channel 2. Open the solo play project file and open the effects chain. Add another instance of alt-keyswitcher and of alt-tuner. Right-click on each effect and choose "Rename FX instance". Name them like so:

1st alt-keyswitcher
2nd alt-keyswitcher
1st alt-tuner
2nd alt-tuner

Set the 1st alt-keyswitcher's midi in channel to 1 with the slider. Set the 2nd one to 2. Go to the 1st alt-tuner's prefs/misc screen. Set the midi input channel to 1, the "first output channel" to 9 and the # of channels out to 8. Set the 2nd alt-tuner to input channel 2, first output channel 1 and # of channels 8. (The 1st alt-tuner's output channels are 9-16 to avoid mixing its output with the 2nd alt-tuner's input.) Set up your synth to handle 16 channels of midi.

Set the 1st alt-tuner's calibration frequency to 110 = -2400¢. Set the 2nd one to 1244.5 = +1800¢. Set the max # of decimal places to 0 and use speed-scrolling to save time! Set each alt-tuner to 24 keys per octave and 22-EDO. The highest C on the 1st keyboard and the lowest C on the 2nd keyboard will both be retuned to middle C, so there will be a 4-key overlap between the two keyboards.

For 31-EDO, 41-EDO, etc., adjust the # of keys and the calibration frequencies accordingly. To handle keyswitches and pedals, set the 2nd alt-keyswitcher and alt-tuner both to a different register block.

The maximum polyphony <u>per keyboard</u> is only 8 pitch classes. You can alternatively use two tracks, one for each keyboard. This makes it more cumbersome to switch between the alt-tuner instances, but it allows more polyphony.

<u>Two-track method</u>: Set the 1st keyboard's output to channel 1 and the 2nd one to channel 2 as before. Open the solo play project file. Select the track and duplicate it with the track menu. Set the input for the 1st track to all midi inputs, channel 1. Set the input for the 2nd track to all midi inputs, channel 2. Set the # of keys, calibration frequencies and register blocks as before.

Softsynth users: set the # of midi output channels to 16 on both alt-tuners. If your synth is sysex-retunable or multi-timbral or multi-midi-channel, add your softsynth to the end of each track's effects chain and you're done. Otherwise set up two multi-track groups like the one group in the "solo play with ReaSynth" project. Your polyphony will be 16 pitch classes per keyboard.

Hardware synth users: set the output mode of both alt-tuners to 10 channels. Set the 1st one to channels 1-10 and set the 2nd one to 7-16. Add the Rechanneler effect to the 1st track at the end of the effects chain. Reroute channels 7-10 to 13-16. Your polyphony will be 6 pitch classes per keyboard, with an extra 4 channels (13-16) to handle the overflow. So you can play up to 10 pitch classes per keyboard as long as you're playing 6 or less pitch classes on the other keyboard at the time. You can alternatively have 2 overflow channels for a polyphony of 7 to 9 pitch classes, 6 overflow channels for a polyphony of 5 to 11, 8 for 4 to 12, etc.

Extended pythagorean: Go to prefs/rows and delete all but the white row and set its "from/to" to -20 and 20. Go to prefs/layout and set the white rung's horizontal to 50. Go to prefs/misc and allow center note taps.

Alternative method that sounds exactly the same but uses a more compact lattice: On prefs/rows, delete all but the white, yellow and green rows, add a deep yellow and a deep green row, set all "from"s to -4 and all "to"s to 4. On prefs/rungs, set the yellow rung's degree to a 4th. On prefs/misc, allow center note taps. On prefs/linkages, set up Ly1 as a linkage (8 Tw5 + 1 Ty3), and double-click the white slider's fader to make it untempered. Optional: on prefs/keyboard, set the # of keys to 41. Optional, to make a note's name reflect its position in the chain of fifths: go to prefs/keyboard and set the # of names to 1, and create these accidental png image files:

sharp: "+7", sharp2: "+2", sharp3: "+9", continue on with "+4", "+11", "+6", "+1", "+8", "+3", "+10", "+5", "+12", etc.

flat: "-7", flat2: "-2", flat3: "-9" etc.

Set "1 sharp = [] keys" to 1. The lattice chain of fifths should run A-2, A-1, A, A+1, A+2, etc.

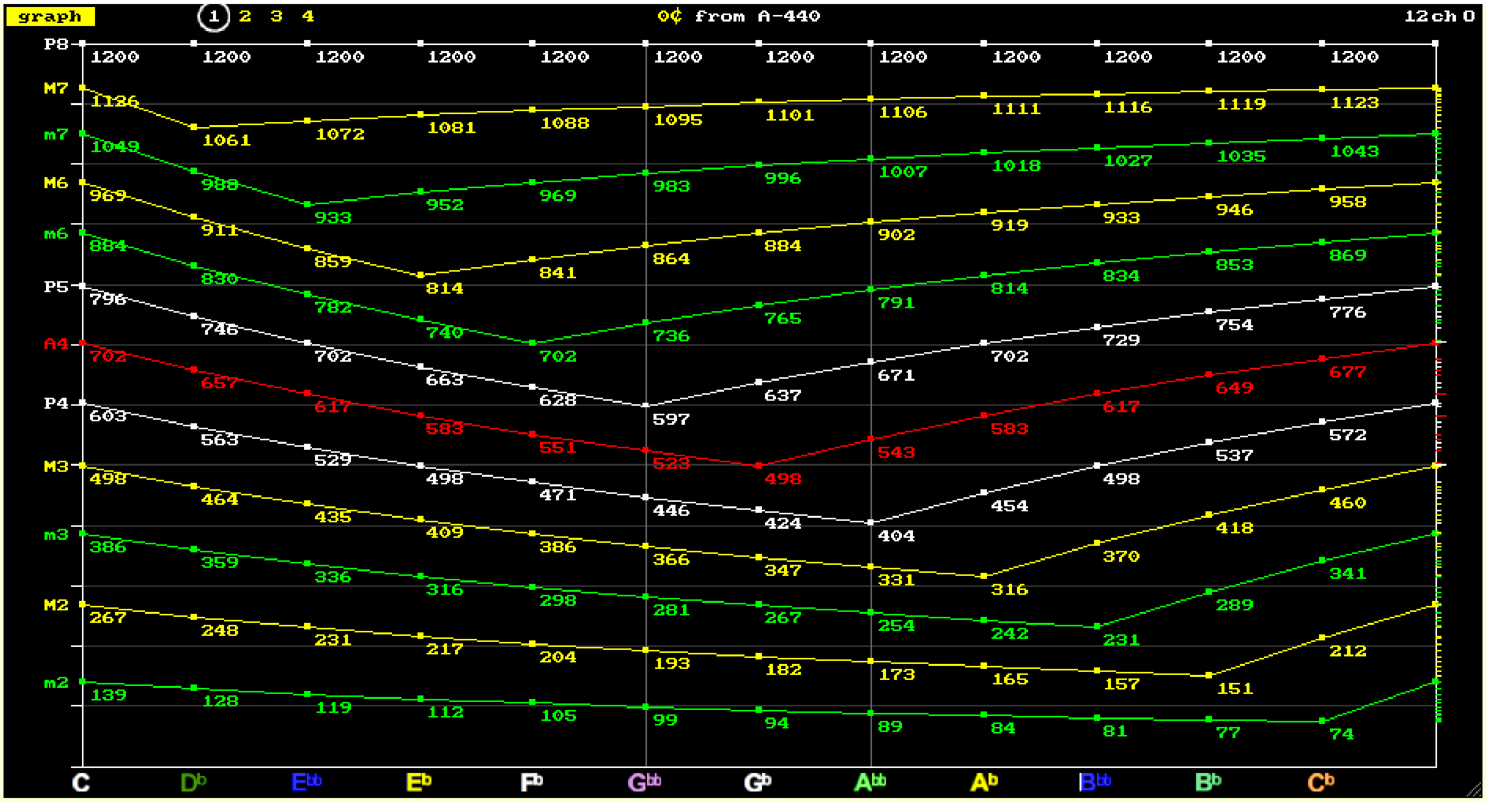
Harmonic-series tuning: Suppose you want a 12 note scale running from harmonic 12 to 24. First go to the rungs screen and add 3 more rungs, which will conveniently default to exactly what we need: 17/16, 19/16 and 23/16.

Next go to the rows screen and add these new primes to the lattice. First let's make some room: delete the green, red, bluish, reddish and purple (bbg) rows. Also shorten the rows by setting all starts to 0, except the white start, which should be -1. Set the white, yellow and blue ends to 1 and all other ends to 0. Add three new rows. The first one will be set to the last row you deleted, because alt-tuner tries to undo your last delete. Set it to all zeros, including the start and end, and put a 1 in the t column. Set the next row to all zeros, with a 1 in the f column. The 3rd row has all zeros and 1 in the i column.

Now go to the layout screen. Set the rung slider to the 17/16 rung. Set the rung's vertical slider to 100. Likewise, set the 19/16 rung's vertical to 170. Set the 23/16 rung's horizontal to -100 and vertical to 100. Using the row slider to access the t, f and i rows, pick some pretty colors for these rows.

Next go to the tapnotes screen. You should see 12 notes, with no maj 2nd and two min 7ths. Starting with 7/6, adjust the keyspans of most of the ratios downward one semitone to spread them out evenly, one per column. 7/4 should be 9 semitones.

Now go to the lattice screen. Most notes are shown with more flats than usual, because their position on the keyboard has shifted to the left. The graph screen has a slanting "creased" look unique to harmonic series scales. The histogram lines on the right are very short and numerous, reflecting the diversity of interval sizes.



Keyboard splits with alt-tuner: Starting with the solo play project, add two new tracks, "low zone" and "high zone". In the "midi input" track's I/O box, add two sends to the two new tracks. Add Reaper's included midi_note_filter effect to both new tracks. In the low zone track, set the note filter to the lower half of the keyboard, and in the high track, the upper half. Softsynth users: if your synth is sysex-retunable or multi-timbral or multi-midi-channel, add your softsynth to the end of each track's effects chain and you're done. Otherwise set up two multi-track groups like the one group in the "solo play with ReaSynth" project. Hardware synth users: set alt-tuner's output mode to 8 or less channels. Add the Rechanneler effect to the "high zone" track at the end of the effects chain. Set the channel shifter slider so that the two tracks output to different channels. For example if the first alt-tuner has 6 channels of output, set the second track's channel shifter to 6, so that channels 1-6 go to channels 7-12. Send the output of the "low zone" and "high zone" tracks to your synth, and set channels 1-6 on your synth to produce one sound and set channels 7-12 to produce another.

You can set up keyboard layers the same way, just don't use note filtering.

Tuning zones use different tunings in different keyboard ranges. Open the solo play project file, and set alt-tuner's output mode to 8 or less channels. Add Reaper's included midi_note_filter effect just before alt-keyswitcher. Duplicate the track (select it, then menu/track/duplicate tracks). In the original track, set the note filter to the lower half of the keyboard, and in the new track, the upper half. In the new track, set alt-tuner's first midi output channel to one more than the number of channels output. Each instance of alt-tuner now controls only half the keyboard. You can tuning-tap each instance of alt-tuner independently by setting the first instance of alt-keyswitcher to have a low tapzone and the second one to have a high tapzone. If you are using a second keyboard for tuning taps, it will be split just like the main keyboard.

You can have more than two zones, but hardware synth users will only have 16 channels total to be shared by all the zones. (Unless you have two identical-sounding synths, that is.) Actually you may be able to squeeze an extra channel or two in. Say you have three zones, using five channels each. However, you occasionally need a sixth channel in each zone, but never in more than one zone at a time. You can put Rechanneler on all three tracks and set the output of the first track's 1-6 to 1-5 and 16. Set the second track's 1-6 to 6-10 and 16, and the third to 11-16. Channel 16 will handle the overflow from all 3 zones.

If you omit note filtering, you'll get "tuning layers", with each key producing two independently tunable sounds! This could be useful for subtle detuning effects like the slight beating of gamelan notes. If you don't mind staying close to 12-ET, an easier way to achieve this effect with hardware synths is to leave local control on.

<u>48 keys permabendable</u>: This method lets you retune each individual key in each octave independently via ratiobend or EDOtap for up to 48 keys. Suppose we want a 12-note scale that fills four 2/1 octaves. In the keyboard screen, set "# of keys" to 48. Set "# of steps" to 28, so each octave has 7 notes. Alt-tuner will distribute these 28 notes evenly across the 48 keys. The bottom octave runs A through G as usual. But the F key will be in the wrong place, so use its slider to adjust it. The higher octaves will run H thru N, O thru U, etc. You should also adjust F's counterparts M, T and 0, so that the black keys have the usual 2-and-3 pattern. You may want to name the first seven keys 11-17, the next seven 21-27, then 31-37 and 41-47. Alternatively, you can create 28 custom note symbols that run "A1", B1"... "G1", "A2", "B2"... "G4".

In the rungs screen, set rung #1 to four octaves, which is 16/1 = 4800¢. Set the rung #2 keyspan to 1 and the rung #2 degree to 2. In the rows screen, delete all the rows but the first (white) one. Set that row's "from" to 0, and its "to" to 47. In the layout screen, set the white rung's horizontal to 50. The lattice screen should show a long line of notes. Set the white slider to 100¢ by typing in the little box on the right. You should now be in 12-ET, with a wildly inaccurate ratio display. Check the keyboard for dead keys and lengthen the white row if needed. You will have insane accidentals like $\sharp^{12}$. You can go to the tapnotes screen and adjust each ratio's degree downwards to get the familiar A B♭ B C D♭ D etc. The next octave should run H I♭ I J K♭ K etc. Save all this as an alt-tuner preset.

All 48 keys can now be independently retuned by permabending. If you don't want to permabend, you can go to 960-EDO and EDOtap up and down in 5¢ increments, or even 4800-EDO to EDOtap 1¢ at a time.

Your tuning will be repeated every 48 keys, so some permabends or EDOtaps will tune two keys at once. To make your whole keyboard permabendable, use two tuning zones.

For other types of tunings, divide the period by the # of keys per period to get the semitone size. For example if your period is the tritave = 3/1 = 1902¢, and you want 13 keys per tritave, the semitone size is 1902¢ / 13 = 146.3¢. If you want a 39-key tuning that fills 3 tritaves, set # of keys to 39, and set the stretch slider to a triple tritave = 5706¢. An easy way to do this exactly is to set rung #1 to 3/1 cubed = 27/1. For a 48-key tuning, you must set the stretch slider to 48/13 of a tritave = 7023¢. The "ratio" for this interval is the 13th root of three to the 48th power. You can't enter this into the rungs screen, so it's easiest to leave rung #1 alone and just set the stretch slider to 7023¢. Set rung #2's keyspan and degree as above, and set the rows as above, so that the white row's end is one less than the # of keys. Set the layout as above, and set the white slider to the semitone size, 146.3¢. Verify that a 13-key interval is 1902¢, and save this as an alt-tuner preset. If you want to EDOtap, set the EDO slider to the stretch slider's cents divided by the increment size. For example for 39 keys and a 5¢ increment, use 5706 / 5 = 1141-EDO. For 48 keys and a 2¢ increment, use 7023 / 2 = 3512-EDO.

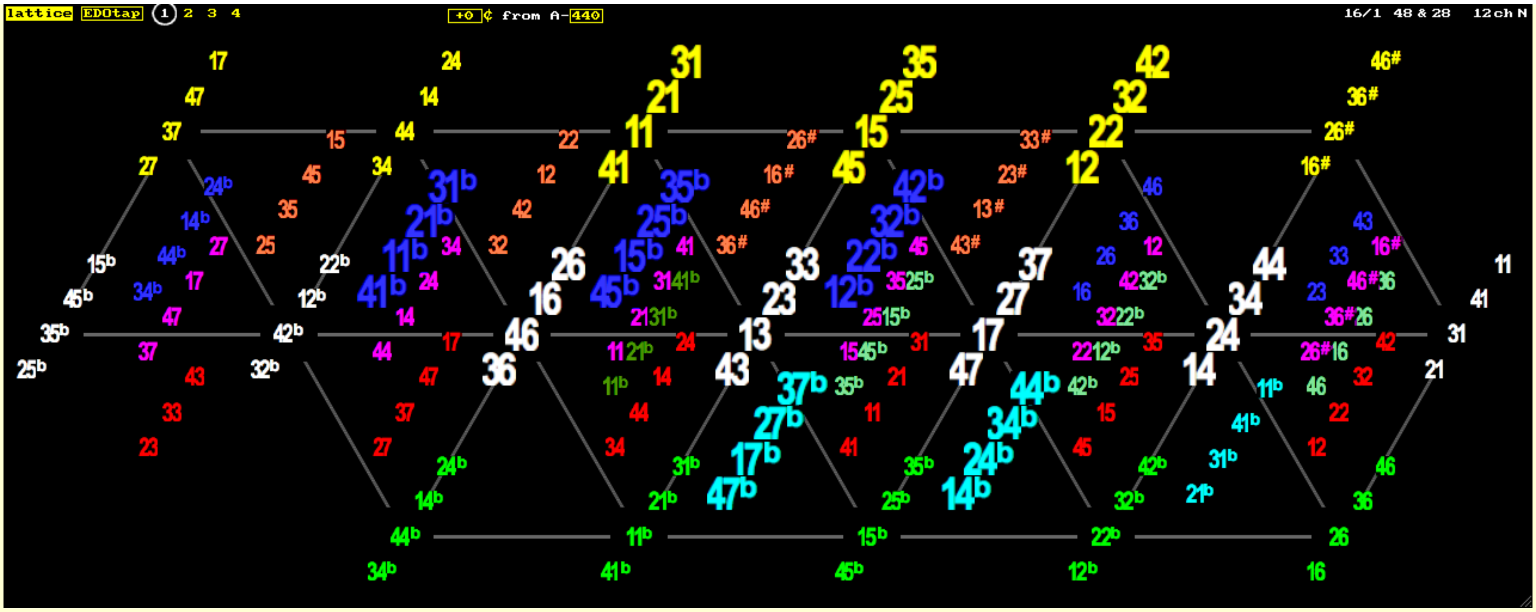
<u>48 keys fully retunable</u>: The last tuning is untappable because there's only one ratio per key. To make a tappable 48-key 4-octave tuning, we need to construct the appropriate lattice "inside" the larger 4-octave interval.

Set the midi output mode to non-octave. In the rungs screen, set rung #1 to 16/1. Notice that alt-tuner won't reduce 5/1 to 5/4 like it used to. So you must set rung #2 to 3/2, #3 to 5/4, #4 to 7/4, etc. Then set the number of rungs to 7 and set rung #7 to 2/1.

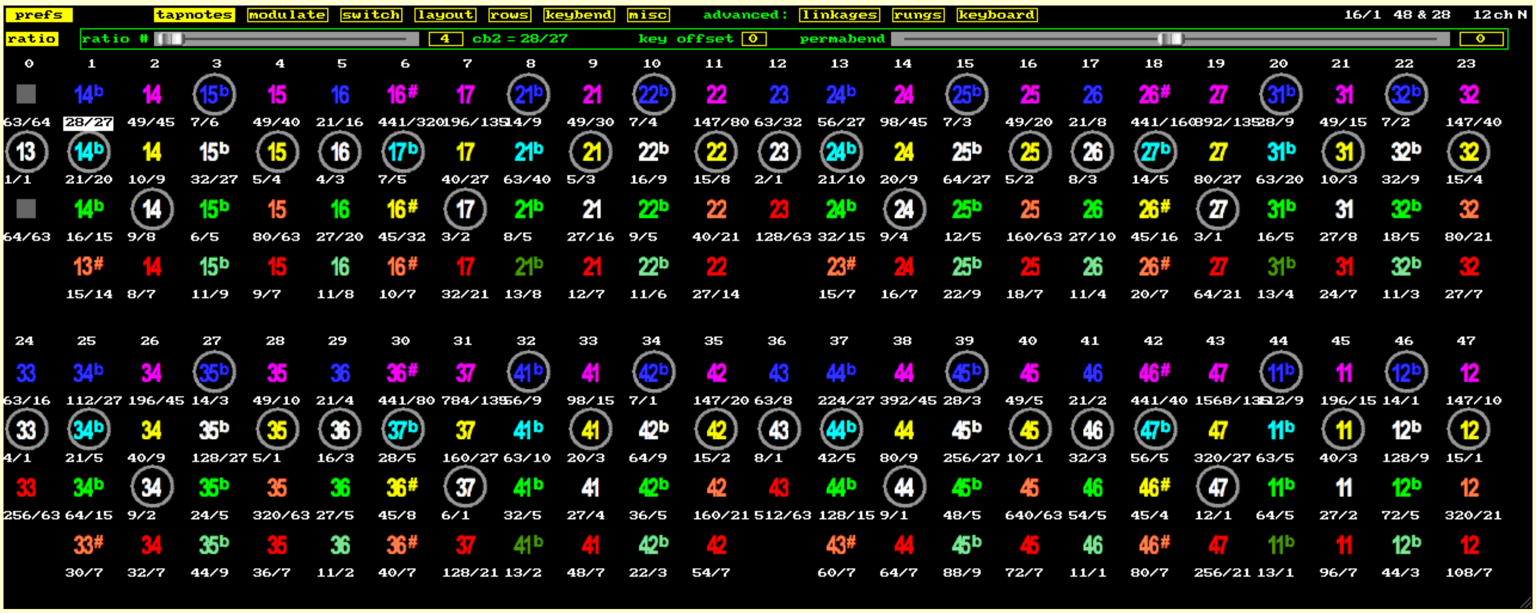
Now go to the keyboard screen and set "# of keys" to 48 and set "# of steps" to 28. See the first paragraph of the previous example for naming advice.

Next add rows to fill the 4-octave range. For each existing row, there will be 3 more rows with rung #7 set to -1, 1 or 2. The "from" and "to" must be adjusted by this number as well. For example, if the white row runs from -3 to 3, the white row with rung #7 set to 1 will run from -2 to 4.

Now go to the layout screen and set rung #7's horizontal to 20 and the vertical to 30. For the 3 new purple rows, set the vertical to 55. The lattice will be crowded but intelligible. Here it is with middle C = 13 as the center note:



Go to the tapnotes screen and choose whatever ratios you want. Each choice will only affect one of the four octaves.



If you want to use tuning taps to select ratios, your tapzone must be 48 keys wide, so you'll probably want to use a second keyboard. The tuning will be repeated every 48 keys, so some taps will tune two keys at once. To make your whole keyboard fully retunable, use two tuning zones.

Non-microtonal uses of alt-tuner: In octave and non-octave modes, alt-tuner redirects notes to multiple channels, even when set to 12-EDO or 12-ET. This can have useful musical effects. For example, by altering the pan settings in each instance of your softsynth (or channel of your hardsynth), you can make your notes jump around in the stereo image pseudo-randomly.

Another example: On the prefs/misc screen, set the mode to octave and the # of channels to 12. Watch the midi monitor as you play one note repeatedly. Only one channel should be active. Now either enter non-octave mode or set the # of channels to less than 12. The same note will jump from channel to channel. Does it sound different this way? It may, depending on your softsynth, especially with sounds with a long decay. You can even do this with percussion sounds to create more realistic snare rolls.

Udos: See the chapter in Part V about udos.

Combining radically different tunings in one song: Suppose you want part of your song to use the 22-EDO over 24 keys tuning, and part to use the Bohlen-Pierce tuning. You can't switch between them, because the # of keys is not switchable. One solution is to use two instances of alt-tuner, right next to each other in the effects chain, with one set to

22-EDO and one set to Bohlen-Pierce. Choose one of your pedals to be a "meta-switcher". Use a footswitch, not a footpedal. In alt-keyswitcher, set its usage to blank. In both alt-tuners, in prefs/misc, update the midi CC #s. Then use Reaper's midi-learn feature to connect the meta-switcher to the bypass parameter of both instances of alt-tuner. Click the "Param" box, near the presets menu. Then select "FX parameter list", "Learn", and "Bypass". Select toggle mode. Move the meta-switcher. Its name should appear in the "Command" window. If not, go to Reaper preferences/audio /midi devices, and set the pedal's input to "Enabled+Control". Do this for both alt-tuner instances.

Manually bypass one alt-tuner by clicking the checkbox on the upper right and un-bypass the other one. Pressing the meta-switcher pedal should now toggle the bypass on both alt-tuners, allowing only one to run at a time.

To switch among three or more tunings, set up three meta-switchers, each one toggling the bypass of one of the instances. Then press two pedals simultaneously to switch between instances.

To do this with automation, set up the bypass envelopes for all instances so that only one at a time is enabled.

Here's another method using a single alt-tuner instance and using Reaper's actions menu to change presets while playing (requires Reaper, won't work with reaJS):

Download the SWS extension for Reaper from http://www.standingwaterstudios.com/index.php. Then restart Reaper and click on Actions/Show Action list in the main menu.

In the "Filter" box, type "preset". You'll see various actions relating to effects presets. For some reason you get different results if the "Selection" box on the upper right says "Main" vs. "S&M extension".

Click on an action, perhaps "Trigger next preset of selected FX for selected tracks", to highlight it. In the lower left is a "Shortcuts for selected action" box. Click the "Add" button. You can assign a QWERTY keyboard shortcut if you want. Or you can move a pedal or knob and assign it to the action. Then choose absolute or relative. For an on/off switch, relative seems to work better.

Do the same with "Trigger previous preset of selected FX for selected tracks", using a different keystroke or knob/pedal.

Next select the track(s) that contains alt-tuner. Open the FX rack for that track and click on alt-tuner to select it. Now use the keyboard or the pedals to move through the presets. When you reach the beginning or the end of the preset list, it wraps around. If the pedal is also assigned a use in alt-keyswitcher, the action use will override it, as long as the track is selected. if the track isn't selected, the alt-keyswitcher use will happen.

One way to automate this is to use marker actions. Go back to the actions list and right-click on the actions. Select "Show action IDs". Now look off to the right for the ID of your action. My example has ID #54185 (next preset) and #54186 (previous). Next insert a marker where you want the preset to change and name it !54185 or !54186. Hit the play button, and when you get to that part of the song, the preset will change.

Beware, the IDs might change in future Reaper versions, so keep an eye on that. Also be sure that the right tracks and effects are selected when you render. Also be sure the right effect preset is selected when you start. There may be a better way to do this with ReaScript.

Chapter 6.10 – Hardware & Software Issues

Alt-tuner can retune all audio hardware and software in real time, except certain things that are by design untunable:

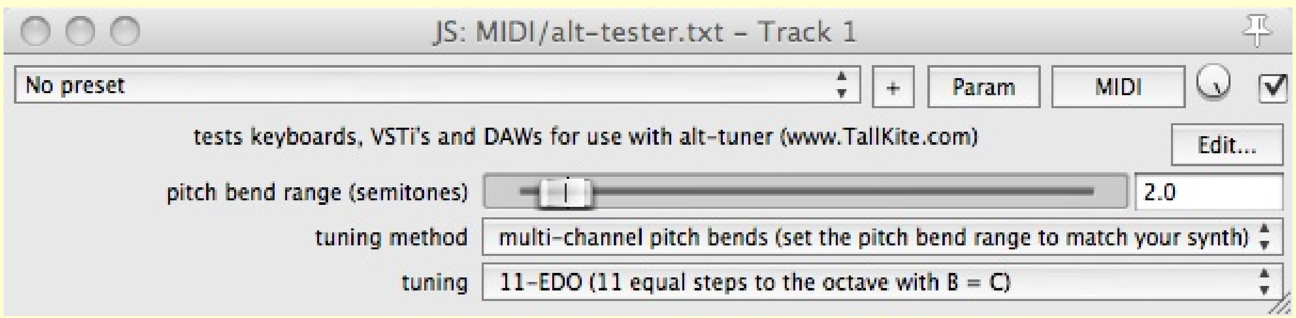
certain DAWs that don't support midi well (like GarageBand or Audacity)
certain softsynths that are not DAW plug-ins (like those in some music notation programs or media players)
certain synths that ignore pitch bend messages (like Unmet Ozcan's GenesisCM or the Technics PX-201)
certain keyboards that are not fully multi-timbral (like the Nord Stage)
certain keyboards that don't allow you to turn off local control (like the Yamaha PSR-282)

See chapter 6.1 for basic setup info. See "Basic MIDI Guide" below for a tutorial. See also "Synth is out of tune" in the troubleshooting section of chapter 6.11.

To configure alt-tuner properly for various synths, it's important that you fully understand alt-tuner's midi channel options and midi output modes. These are explained at the end of chapter 6.4, in the prefs/misc section.

Alt-tuner does not output scala files, TUN files, etc., because these file formats don't allow retuning while playing. They require your hands to move from the keyboard to the computer to load the files. Hopefully, any software that currently supports these file formats will add support for retuning either via midi sysex #82 or via Kontakt-style virtual keyswitches.

Alt-tester: If you want to make sure that alt-tuner will work with your midi set-up before you buy it, and the alt-tuner forum doesn't list what you have, you can test your hardware, your softsynth, or your DAW's ReWire connection with the free download alt-tester, available at www.TallKite.com. If your set-up works with alt-tester, it'll almost certainly work with alt-tuner. If it doesn't work with alt-tester, let me know, I may be able to fix the problem.



Alt-tester is not full-fledged tuning software like alt-tuner, it's only for testing purposes.

Unlike alt-tuner, alt-tester doesn't support creating your own scales. You are limited to 2 preset scales.
Unlike alt-tuner, alt-tester doesn't support using the sustain pedal, the pitch bend wheel or the mod wheel.
Unlike alt-tuner, alt-tester doesn't support holding down more than 2 keys at once in mono mode.

Installation for Reaper users: first download and install Reaper. Windows users, select ReWire under "Additional functionality". Run Reaper, and with all effects windows closed, in the menu choose "Options/Show REAPER resource path in explorer/ finder". Go to the Effects/MIDI subfolder and move alt-tester to this folder. To uninstall alt-tester, just delete it.

Installation for ReaPlugs/ReaJS users (Windows only): If you're a Windows user and you don't have Reaper, you must use ReaJS, which is part of ReaPlugs. ReaPlugs is free, get it here: http://www.reaper.fm/reaplugs/. Use the latest version, ReaPlugs 2.36, the graphics are broken in earlier versions. To install ReaPlugs, just double-click the exe file.

(If you've already installed ReaPlugs, and you're not sure which version you have: There is no version number on the reaJS effect screen, so check the "date modified" of the reaJS.dll file. Version 2.36 is January 2016. Or read reaplugs_readme.txt. See next paragraph for the location of reaJS.dll)

Next, find your VST folder, which contains all your VSTs, as well as the ReaPlugs folder (which contains reaJS.dll). Assuming it's C:\Program Files\VstPlugins\, move alt-tester to C:\Program Files\VstPlugins\ReaPlugs\JS\Effects\MIDI\. Alt-tester should show up in the "Load" menu. If not, you may need to remove and reload ReaJS, or possibly even quit and relaunch your DAW.

Reaper users: Open the "tester.RPP" Reaper project. It's all set up with ReaSynth, Reaper's included VSTi. Verify that

ReaSynth is being retuned by playing a few notes. Verify that moving your pitch bend wheel bends the notes. You should hear an 11-EDO scale in which all the semitones from C up to B are slightly wider, and B and C are the same note. If you choose 10-EDO, the semitones will be even wider, and not only B and C but also F and F♯ will be the same note. You can switch EDOs as you play. If there's no sound, check the audio device settings in Reaper's preferences.

You have the choice of seven different tuning methods:

multi-channel pitch bends (assumes the pitch bend range = 2 semitones = 200¢)
multi-channel pitch bends (same but a "safe" mode requiring 12 instances)
single-channel pitch bends (for monophonic synths)
sysex82 (for Xen-Arts synths among others)
sysex88 (for Roland synths among others – retuning is limited to -64¢ to +63¢)
keyswitch (for Kontakt – requires the microtonal keyswitch script)
multi-channel RPN coarse/fine channel tuning (let me know if anything works with this!)

The first two are the most likely to work with your synth. The "safe" method is for sounds with a lengthy release time. It keeps the notes from "getting their tails bent" by notes that follow. Use the third method for monophonic synths. Most synths don't respond to the sysex methods, but if yours does, it will greatly reduce your set-up time. The sysex88 method limits the retuning range. The keyswitch method is for Kontakt only. It requires the Kontakt instrument receiving the midi to contain the microtonal keyswitch script, available at www.TallKite.com. This script uses virtual keyswitches on midi notes 123-127 (D#9 to G9) to retune the instrument. The RPN method uses midi registered parameter numbers and is a possible alternative to the pitch bending methods. This method is not part of alt-tuner because I don't know of any softsynths or hardsynths that respond to RPNs but not pitch bends. If yours does, let me know, and the RPN tuning method can be added to alt- tuner.

To use any of the three pitch bend methods, first use the pitch bend wheel to determine what your synth's bend range is. For example, usually G can be bent all the way up to A and all the way down to F, and the bend range is 2 semitones or 200¢. Alt-tester's pitch bend range ***MUST*** match your synth's pitch bend range. If they don't match, change either alt-tester or your synth so that they do. If your synth has a range of only half a semitone, you can type "0.5" in the little box next to the pitch bend range slider. If your synth's range is two octaves, type "24" in the box.

If your synth doesn't respond to your pitch bend wheel, either the pitch bend range of your synth is set to 0, or it completely ignores all pitch bend messages. If you can't find a way to set the pitch bend range, your synth is unbendable, and untunable with pitch bends. If it doesn't respond to sysex messages or RPNs either, your synth is simply not retunable.

If you use the sustain pedal, the mod wheel, or the pitch bend wheel, it will only affect some of the notes. Alt-tester doesn't support these or any other controls, however, alt-tuner does.

To test softsynths (VSTi's, AUi's, etc.): Try the easy way first: Solo the 1st track and put your synth in the effects chain right after alt-tester. If your softsynth is mono, or if it's polyphonic but you've set it to mono, set alt-tester to the third tuning method and you're done. Otherwise, set alt-tester to the first method. If your synth is multi-timbral, set it up with multiple channels all set to the same sound, and you're set. Otherwise, test it by holding down one key and playing short random notes over it. Listen for pitch shifts in the drone note. This will be easier to hear with a sustaining sound. If there are no shifts, your softsynth is multi-midi-channel, and you're good to go. PianoTeq falls into this category. Otherwise, try the two sysex methods. If either one retunes your softsynth, it's sysex-retunable, and you're A-OK. Otherwise, remove it from the effects chain and un-solo the track. You'll have to retune it the hard way by setting up multiple instances of your synth on the other tracks.

Choose one of the multi-channel methods. As long as your synth responds to pitch bends, one of the first two methods should work. Use at least 6 channels. That's usually enough to cover normal playing. Replace the first 6 instances of ReaSynth with your softsynth. "Safe" mode requires you to use 12 channels and replace all 12 instances.

To test a hardware synth: Solo the 1st track and in its I/O box, set the midi hardware output to your hardware. On your keyboard, turn local control off. For mono patches, set alt-tester to the third method. Otherwise, try the two sysex methods first. If either one works, you're done. Otherwise, put the synth in multi-timbral mode. Set at least the first 6 channels of your tone generator to be all the same instrument. For "safe" mode, set the first 12 channels. Your synth must be sufficiently multi-timbral to receive all these channels, see the "specs" section of your synth's manual.

To test ReaJS in your DAW: Put ReaJS in your effects chain. Load alt-tester into ReaJS. Send the midi to either your keyboard or to your softsynth.

To test your DAW's ReWire connection: Windows users, first download loopMIDI, a free virtual midi cable (VMC) from here: http://www.tobias-erichsen.de/software/loopmidi.html. After installation, run loopMIDI and create 6 ports. OS X users: first run Audio Midi Setup and make sure the built-in VMC called IAC has at least 6 ports.

Rewire Reaper (the slave) into your DAW (the host). In your DAW, create two tracks, "raw midi" and "tuned midi". Set the "raw midi" track's input to your keyboard or controller. Set the output to a VMC midi bus, aka port. Open the appropriate "alt-tester ReWire wrapper.RPP" in Reaper. Set the input of the 1st track to that same VMC bus. Set the midi hardware output to another VMC bus. In your DAW, set the input of the "tuned midi" track to this 2nd bus. Send the midi from this track to either your keyboard or to your softsynth. You may need to use multiple instances of your softsynth on multiple tracks. See "tester.RPP" for an example of how to do this. To tune more instruments, add more "raw midi" and "tuned midi" tracks to your DAW, and use additional VMC busses to connect them with the 2nd and 3rd tracks in the wrapper.

Basic MIDI guide: Midi is not music but instructions on how to create music. Midi is to audio like sheet music is to a CD or an mp3 file.

There are two kinds of retuning software: those that retune audio, and those that retune midi. Audio retuners like Autotune, Melodyne, Waves Tune, etc. usually require monophonic audio like vocal or sax, as opposed to polyphonic tracks like piano or guitar. Midi retuners like alt-tuner do fine with polyphonic material.

To use alt-tuner, you need a keyboard with a midi out jack or a usb jack. There are two kinds of midi keyboards: controllers, which can't make sound without being attached to a computer, and synths, which can. Nowadays synths are called hardware synths, or hardsynths, to distinguish them from softsynths. Hardware synths like Yamahas, Korgs, Rolands, etc. contain 2 or 3 basic components: a controller, perhaps a sequencer, and a tone generator. The controller consists of things you play music with: the keyboard, the pitch bend and mod wheels, and various pedals. Playing music generates midi messages. The sequencer records this stream of midi messages for later playback. The tone generator takes the midi stream and converts it into actual audio. Most midi messages have to do with the controller part. They report what the musician plays. But sysex (short for "system exclusive") messages control the tone generator's settings. This kind of midi message is hardware-specific. Each brand and model of keyboard has its own sysex messages. A few hardware synths can be retuned by sysex messages. However, most hardware synths can't. Alt-tuner retunes these keyboards by intercepting the midi stream and adding midi pitch bends to it. These pitch bends are the same message that moving the pitch bend wheel generates. Alt-tuner actually "fools" the synth into thinking that someone is moving the pitch bend wheel every time a note is played. To intercept the midi, you must disconnect the controller from the tone generator by turning local control off on your synth. Now your synth will send midi to the computer and the computer will send it back retuned. Unfortunately, midi pitch bends don't bend individual notes, they bend all the notes at once. The solution is to use midi channels. Almost every midi message is on one of 16 channels. The tone generator in a hardware synth is actually many (usually 16) separate tone generators, each one capable of having a different instrument sound, and each one capable of having its notes pitch-bent independently. When you set your synth to multi-timbral mode, each one will "listen" to a different channel. This mode was originally designed to allow one synth to sound like up to 16 different instruments at once. Channel 1 would be set to piano, channel 2 to flute, etc. Instead, we're going to set channels 1-12 of your synth to be all the same instrument, one for each of the 12 notes in the octave. Check that alt-tuner is in the default 12-channel octave mode, so that it can then retune each pitch class independently via pitch bends. Some hardsynths have less than 16 separate tone generators, see the timbrality discussion in the hardware section below. Often you don't need all 12 channels and can get by with only 6 or 8.

Software synths are actually software tone generators. Most softsynths are plug-ins, which are apps that run inside a DAW (digital audio workstation = music program) like Reaper. Plug-ins come in several formats: VST, AU, DX and RTAS. Reaper works with all of these formats except RTAS, which is ProTools-only. AU plug-ins only run on a mac. DX plug-ins only run on a PC. VST softsynths are called VSTi plugins or VSTi's (i is for instrument). Softsynths can accept midi from a hardware synth, or a midi controller (a hardware synth minus the tone generator), or a sequencer. Local control is obviously not an issue. There are lots of free VSTi softsynths to be found on the internet, see the "softsynth" section below.

Many notation programs like Finale or Sibelius have built-in softsynths, as do music players like Quicktime. Standalone softsynths like these may possibly work with alt-tuner if they can receive midi from Reaper, including pitchbends, and if they are multi-timbral.

Some DAWs like Logic have softsynths bundled with them. They can only run inside that DAW. You can tune another DAW's softsynths with alt-tuner by using either ReWire or a wrapper. ReWire is free and is already built into your DAW. Run both DAWs simultaneously and ReWire will pass both midi and audio back and forth between them. For midi, ReWire uses ports, aka busses, each of which contains 16 midi channels.

A wrapper is a special kind of plug-in that can itself host other plug-ins and convert them from one format to another. For example, ProTools can't host VST plug-ins, only RTAS plug-ins, so there are VST-to-RTAS wrappers available. Alt-tuner is a Jesusonic plug-in. Jesusonic plug-ins could originally only run inside Reaper. But ReaJS is a Jesusonic-to-VST wrapper that allows alt-tuner to run inside most DAWs. ReaJS is not available for OS X.

The following information is highly subject to change.
See the forum at TallKite.com for the latest information.

Softsynths: To get started quickly with free synths, I suggest the Pianoteq demo version, or Helm, or for PC users the Xen-Arts synths. For acoustic sounds, Sampletank 3 Free is multi-timbral, fairly lightweight (about 500 MB) and runs on both PCs and macs. Many more softsynths can be found at www.kvraudio.com.

There are five kinds of softsynths: sysex-retunable, keyswitch-retunable, "multi-midi-channel", multi-timbral, and the most common kind, "multi-instance". A very few softsynths accept sysex retuning messages, and you only need one instance (copy) running for each sound. Make sure that alt-tuner's output mode is the proper sysex mode. Some DAWs block sysex messages, see below. Keyswitch retuning is a new method of retuning that works with Kontakt. Only 1 instance is required, and identical sounds are not needed. For all other synths, set the output mode to octave or non-octave or mono. Some synths are what might be called "multi-midi-channel", which means pitch bends will retune each channel independently. (Not to be confused with"multi-channel", which means multiple channels of audio output, as in Quadrophonic or SurroundSound.) As with sysex-retunable softsynths, you only need one instance. These three types are the most convenient to work with. Multi-timbral synths can produce more than one sound. You only need one instance, but you need to load multiple identical sounds into that one instance. All four of these types might be called "single-instance" synths. All other synths are what might be called "multi-instance" synths, the least convenient type to work with. You must load multiple instances of the softsynth, each one on a separate track, one for each channel of midi output by alt-tuner. See the included Reaper file "solo play with ReaSynth" for an example of this, using Reaper's built-in softsynth ReaSynth. Once you set up your multiple tracks, select the tracks by clicking the first one and shift-clicking the last one, and save them as a Reaper track template. Now you can quickly use your softsynth in any Reaper project with the "Insert track from template" command.

softsynth type	example synth	# of alt-tuner midi channels out	# of synth instances / DAW tracks
sysex-retunable	Xen-Arts synths	1	1
keyswitch-retunable	Kontakt	1	1
multi-midi-channel	Pianoteq, Helm	many	1
multi-timbral	Kontakt, SampleTank	many	1 track with 1 instance containing many identical instruments
multi-instance	ReaSynth	many	many tracks, each with 1 instance

See chapter 6.1 for how to determine your softsynth type. Of course, if you're recording a monophonic track, you only need one instance, no matter what kind of softsynth you have.

There are two other kinds of synths. Some are "file-retunable" because they can load scala files or .tun files. Alt-tuner doesn't output these files. A file-retunable synth can always be retuned by one of the other methods. A very few synths are "untunable" because they don't respond to pitch bend messages at all. If you absolutely must retune one of these, as a last resort, you can set up 12 instances with alt-tuner as if it were multi-instance, using octave mode with 12 channels. Then put a pitch-shifting effect after each instance, set appropriately. You will be retuning the audio, not the midi, which will be much more CPU-intensive, and may have audible tuning artifacts. If the synth has a calibration or detuning feature, use that instead of a pitch-shifter to retune the audio more directly and reliably. You may not need all 12 instances. For example if your piece is diatonic you'll only need 7. You won't be able to modify the tuning on the fly of course, because alt-tuner isn't actually affecting the tuning, just routing the midi to different channels.

Unless a softsynth is sysex-retunable or keyswitch-retunable, you must set the pitch bend range to match alt-keyswitcher, usually 2 semitones.

Using multiple softsynths can overload your computer. To ease CPU usage, turn off the reverb on each individual VSTi, and run them all through one reverb effect. Reducing the number of voices in the VSTi is also an option. You can also set the output to fewer channels in the prefs/misc screen. Another possibility is using Reaper's included ReaMote utility to run softsynths on multiple networked computers.

When using just intonation, to appreciate its beauty you may want to turn off all chorusing.

Windows users: If you experience latency, try downloading the ASIO4ALL driver from www.asio4all.com

By the way, when demoing softsynths without a controller, it's worth noting that Reaper's Virtual MIDI Keyboard is velocity sensitive in the sense that clicking on the back of the key (higher on the screen) produces a softer note. This also sets the volume of subsequent notes played by typing on the computer keyboard.

When using multi-timbral or multi-instance synths, the multiple instruments should all have identical settings – the same volume, pan, EQ, ADSR envelope, etc. Especially important is the pitch bend range. If you overlook any of these settings, the sound of the instruments will vary in a random way. Once you have set up your instruments, it's a good idea to double-check the settings by ear. Alt-tuner provides an easy way to do this. If you're in octave mode, set the number of midi channels to less than 12 (or more generally, to less than the number of keys on the keyboard screen). In non-octave mode, the number of channels doesn't matter. Now play one note repeatedly. Alt-tuner will send this note to each channel in turn, and each instrument will respond in turn. You can verify this by watching the midi channel monitor on the prefs/misc screen. Most DAWs let you use the computer keyboard like a virtual MIDI keyboard. This "musical typing" creates notes of uniform volume, good for our purposes here. As you play, listen carefully to the tone of that note. If you hear any variation, check your settings.

ReaSynth: Retunable. Requires multiple instances. Extremely basic, free with Reaper.

Ivor, Xenharmonic FMTS and XenFont (Xen-Arts): Retunable. Free. PC only. Must use sysex82 mode. Set it to global MTS microtuning, 12TET. Only need to use one instance. Don't need to set the pitch bend range. XenFont comes with 128 general MIDI sounds and 75 presets for those sounds. It also lets you load any SF2 format soundfont sound into it.

Helm (Matt Tytel): Multi-midi-channel, requires only one instance. PC and Mac. Free and open source!

Kontakt (Native Instruments): Retunable and multi-timbral. PC and Mac. Not hard to set up.

The easiest way to retune Kontakt is with alt-tuner's virtual keyswitch midi mode. Only one instance of each instrument is required. The virtual keyswitch midi protocol was developed in 2013 by Robert Walker, Ozan Yarman and myself, as well as others. Robert has written a KSP script, a version of which is included with alt-tuner. (Authors of instruments for Kontakt can add this script to their instruments. The license permits it to be used in this way, in any project, free or commercial.) The full version of Kontakt is required, Kontakt Player will not work. First install the keyswitchRetuner.nkp file into Kontakt by copying it to the appropriate folder:

> OSX: users/<your username>/Documents/Native Instruments/Kontakt 5/presets/scripts
> Windows 7: C:\Users\<your username>\My Documents\Native Instruments\Kontakt 5\Presets\Scripts
> Windows XP: C:\Documents And Settings\<your username>\My Documents\Native Instruments\Kontakt 5\Presets\Scripts

Then run Kontakt and install the script in each instrument you want to retune:

1. Click the wrench icon in the upper left
2. Click on the "Script Editor" button over on the right
3. Look through the 5 tabs that appear and find an empty one to click on
4. Click on the Preset button, over on the left, select the user submenu and select keyswitchRetuner
5. Save the instrument as "Clavinet keyswitched" or somesuch

To uninstall the script, in step 4, select Preset, factory, and -Empty-. You often have to bypass Kontakt's Options script to get keyswitchRetuner to work, so if there are no empty slots, just use the Options tab in step 3. You may need to bypass other scripts as well. The "release samples" script retunes the release samples separately. Use this script if there are problems with the other script. These scripts are simple versions that assume the use of alt-tuner or alt-tester. Robert Walker has written more elaborate and powerful Kontakt scripts, available here:

> http://robertinventor.com/ftswiki/Velocity_keyswitches_retuning

Kontakt can also be retuned with standard pitch bends. For octave, non-octave and mono modes, put one instance of Kontakt immediately after alt-tuner. Load an instrument. Click "Options" and under "PB Range", check that down is set to -2 and up to 2. Shift-drag the knobs for extra accuracy. Under "Tuning", check that "Equal Tempered" is selected. Save the instrument as "Clavinet tunable" or somesuch. Then load many instances of the instrument into the rack. Each new instance will automatically receive on a different channel. Save this multi for future use as "Clavinet multi" or somesuch.

You can also set the pitch bend range this way:

1. Click the wrench icon in the upper left
2. Unclick the Group, Mapping, Wave and Script buttons as needed
3. Click the Mod button on the left to open it up
4. Look for the pitch bend modulator and move its slider to 2
5. If there is no pitchbend modulator, click the "add modulator..." button
6. You may need to bypass some or all of the scripts as well

Kontakt has a built-in script called "Dynamic Pure Tuning" that claims to do adaptive just intonation. However, when presented with such simple chord progressions as I-IV-V, I-ii, ii-V or i-V, it stops working until you reload it. The order of the chords doesn't matter, so I-IV-V includes I-V-IV. The tonic of these chord progressions is arbitrary, so ii-V includes i-IV, I-ii includes both V-vi and i-VII, and I-IV-V includes ♭VII-IV-I.

Pianoteq (Modartt): Retunable. PC and Mac. Multi-midi-channel; only one instance is needed. Usable demo version. Click options/midi and confirm that pitch bend range is -200 to +200. Confirm that "temperament" is set to "equal" (stretched octaves) or "flat" (unstretched). For unstretched octaves, confirm that "octave stretching" is set to 1. ("Equal" is always stretched even if this slider is at 1.) Click on "Diapason 440 hz" and then "detune notes" to further control stretch.

Sampletank 2 (IK Multimedia): Retunable and multi-timbral. PC and Mac. Free, dozens of decent sounds, acoustic and electronic. Good "starter" kit. The free version comes with 471 MB of samples, the commercial versions with either 2 GB or 6 GB of samples. The free version is upgradable. Not very easy to install. Here's how:

Free version 2.5.5: You'll need to register on their website to download it. When you click on the download button, you'll see a dialog box with a mac download and a PC download. Download the appropriate version, and also click on the red "download sounds" text in this dialog box to get the 471 MB zip file. Also download the newest authorization manager from the site. Also go to your user area, go to "my products", find ST2, click on the red "authorizations" text, and get the serial number. The sounds could be downloaded here as well.

The manual is included in the download. For OS X, look in /Library/Documentation/IK Multimedia. For Windows, look in C:\Program Files\IK Multimedia.

Install ST2 and install the sounds. Note the folder that you installed the sounds in. Then run the ST2 plugin, click on prefs, click on browse, go to the instruments folder where you installed the sounds, and click OK. The instruments should appear in the ST2 window in the center part. Even though ST2 is free, you still have to authorize it, otherwise it expires after 10 days. The website allows you 5 authorizations. Run the authorization manager, click registered user, and enter your user name and password. On the next screen enter the serial number.

To load an instrument, first click on one of the 16 parts on the left to select it, then double-click an instrument. The red numbers (polyphony, pan, vol, etc.) can be changed by clicking on one and dragging it up and down. You may want to reduce the polyphony. Click the"synth" button and make sure the "bender" knob is set to 2. This sets the synth bend range to 200¢. Load multiple copies of the same instrument and save it as a combi or a Reaper preset.

Superwave: Retunable. Free, windows only. Multi-instance. Doesn't respond to pitch bends correctly unless they're on midi channel 1. So you must make each track send go to channel 1. See the alt-tuner forum for details.

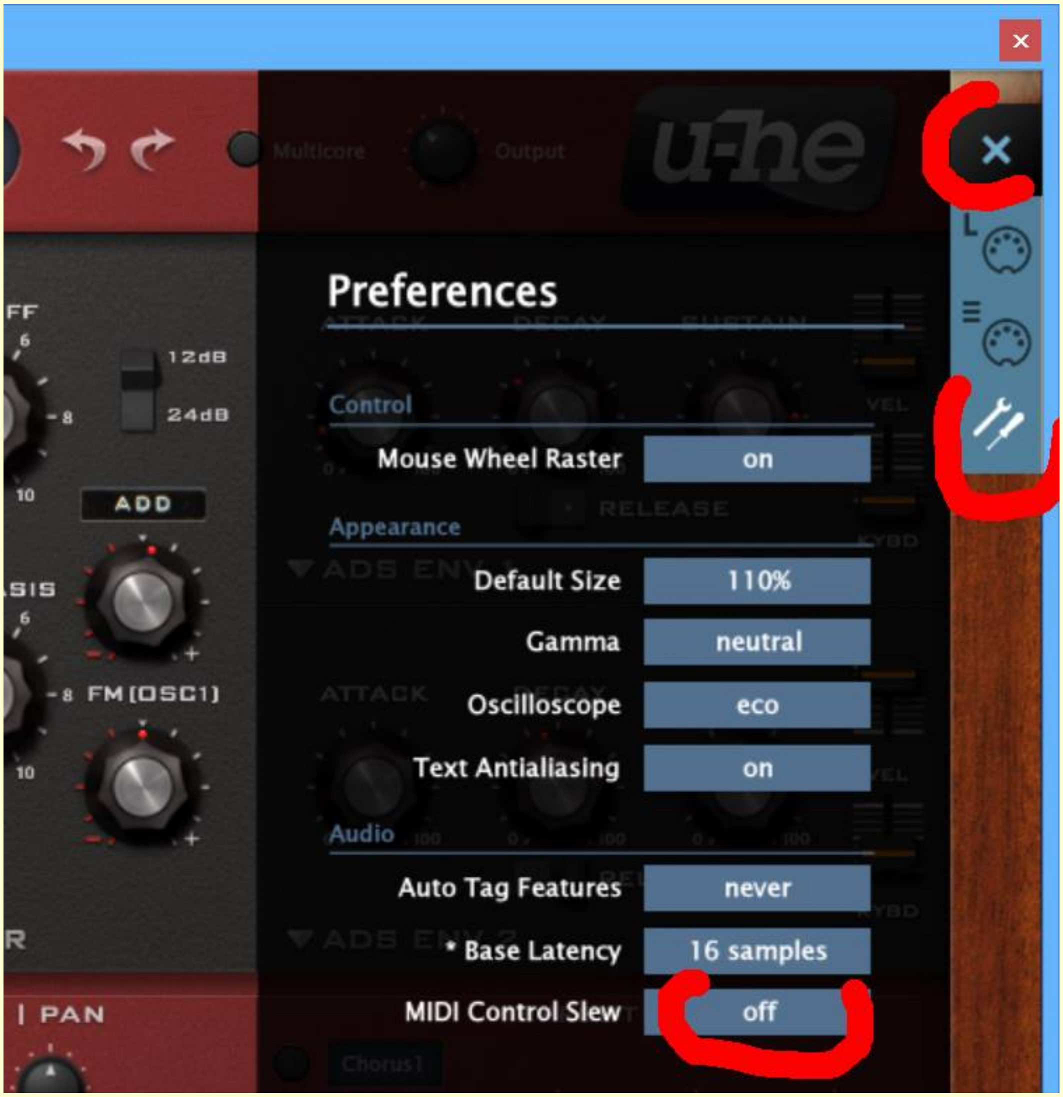
Curve (CableGuys): The mac version is fine, but the PC version unfortunately has a sluggish response to pitch bends. It's fast enough to keep up with someone physically moving the wheel, but a little too slow for alt-tuner. You can hear audible "scoops" at the start of certain notes. Alt-tester's safe mode or alt-tuner's 12-channel mode will solve the problem, but only if the tuning doesn't change over the course of the song.

Genesis CM (Unmet Ozcan): unfortunately completely untunable because it ignores pitch bend messages. See above for a workaround, 4th paragraph of the "softsynths" section.

Finale: Mostly untested. Finale Notepad 2009 seems to be multi-timbral. Set it to receive channels 1-16 from a virtual midi cable, and set Reaper's MIDI hardware output to match. Doesn't make sound while you play, you have to replay the score to hear it. Doesn't seem to receive pitch bends.

Quicktime (Apple): Not a great-sounding softsynth, but it's free and included on all macs. Quicktime version 7.6 is multi-timbral, but can't receive midi directly from Reaper. It can however replay a midi file exported from Reaper.

Diva (U-he): To avoid a sluggish response to pitch bends, click the gear in the upper right corner of the wooden frame. Then click on the wrench-screwdriver icon. Set "MIDI Control Slew" to off.



DAWs: To use alt-tuner in another DAW, either use ReaJS, the free Jesusonic-to-VST wrapper, or connect your DAW to Reaper via either ReWire or a **virtual midi cable (VMC)**. ReaJS is part of ReaPlugs, which can be downloaded here: www.reaper.fm/reaplugs. It's easiest to use ReaJS, but ReaJS is Windows only, and requires a DAW with VST support, so sometimes ReWire or a VMC must be used. Also ReaJS doesn't currently transmit sysex messages. Cantabile Lite is a free standalone app (Windows only) that can host ReaJS and thus alt-tuner. Cantabile Lite is handy if you want to use alt-tuner without a DAW. It's also useful for certain DAWs that don't support multi-channel midi. www.cantabilesoftware.com/download. An alternative host is VSThost: www.hermannseib.com/english/vsthost.htm

Using a VMC is easier than using ReWire, but ReWire may be needed if alt-tuner is using automation envelopes.

VMC Instructions: Mac users can use IAC (Inter-Application Communication), which is included in OS X 10.3 and above. Users below 10.3 can possibly use MIDI Patchbay. First run Audio Midi Setup and enable IAC. Set up many pairs of ports, otherwise known as busses, and call them "IAC bus 1", "IAC bus 2", etc. Each pair of ports allows one instrument at a time to be retuned. 20 ports would allow you to have 10 softsynths in your DAW being simultaneously retuned by 10 instances of alt-tuner in Reaper. The maximum number of IAC ports seems to be 64.

PC users can use loopMIDI (free, 32- and 64-bit compatible, supports XP/Vista/Win7/Win8). Get it here: www.tobias-erichsen.de/software/loopmidi.html. After installation, run loopMIDI and create as many pairs of ports as you think you'll need. Then right-click the traybar icon and turn on auto-start. Other VMCs besides loopMIDI:

LoopBe1 (1 port only, Windows 2000/XP/Vista/7, $14, free for non-commercial personal use)
LoopBe30 (30 ports, only 9 ports in windows 2000, $20, demo version runs for 1 hour)
Maple Midi (doesn't support sysex)
MIDI yoke (old, last update was in 2007, only runs on XP)

In Reaper, go to Options/Preferences/Audio/MIDI Devices, and enable the odd-numbered ports for input and the even-numbered ports for output. This will prevent the accidental creation of midi feedback loops. You can now select these ports in the MIDI input and hardware output menus. In your other DAW, do the opposite: enable the even-numbered ports for input and the odd-numbered ports for output. If you plan on using Reaper as a ReWire host and your other DAW as a slave, you may want to reverse these instructions: in Reaper, enable even ports for input and odd for output, and in your other DAW, enable odd for input and even for output. That way the first VMC port used is always port #1.

General ReWire instructions: Windows users: when installing Reaper, confirm that ReWire is selected under "Additional functionality". When using ReWire, one of the DAWs will be a host and the other will be a slave. It's possible to have more than one slave, but there can only be one host. Many DAWs can't be ReWire slaves and must be hosts. Reaper can be either. Start the host DAW first, and start the slave DAW next. The two DAWs will automatically be connected, or "rewired", when the slave DAW is started. The slave is said to be rewired "into" the host. Windows users should launch their VMC first, then start the two DAWs. The host DAW sets the tempo. Both DAWs are linked, so hitting play or stop on one will affect the other one too. When you quit, quit the slave DAW first.

There are many ways to set up ReWire, depending on how you want to divide the workload (recording, editing, creating audio from midi, mixing, and mastering) between the two DAWs. Let's look at two possible configurations. Suppose Reaper is your primary DAW, the one you are most familiar with, and the one in which most of the work will be done. Suppose you're only using the other DAW for its built-in instruments. Reaper will be the host and you'll send alt-tuner's tuned midi to the slave and receive audio from the slave's instruments. But suppose the other DAW is your primary one, and the only reason you're using Reaper is for alt-tuner. Reaper will be the slave, and you'll send the untuned midi to Reaper and receive the tuned midi from Reaper's alt-tuner instance(s).

Some DAWs can't be a slave. A host can't send audio to a slave. If Reaper is your primary DAW, you may need to use a VAC, a virtual audio cable, like ReaRoute or Soundflower.

If Reaper is the secondary DAW, set up a midi-only rewire. In Reaper, for each midi track, set the midi input and the midi hardware output to either a VMC bus or a ReWire bus. Do the same for the midi track(s) in your DAW.

Not all DAWs support ReWire. If all else fails, you can transfer midi via file transfer. Export the midi from your DAW and import the midi into Reaper (or else record it there in the first place). With alt-tuner in the track's FX chain, create a tuned midi file by right-clicking the midi item and selecting "apply track FX to item as new take (MIDI output)". Export the midi with file/"export project midi"/"entire project, selected items only", and import the midi file into your DAW.

Reaper as a ReWire host: This is a little tricky, using Reaper as a slave is easier. Most DAWs implement ReWire by having ReWire connections show up directly in track input and output choices. Reaper as host works differently. It uses a single rewire effect on a track dedicated to handling the data flow between the two DAWs. You must load this effect before launching the other DAW. The effect takes as input all the midi that Reaper is sending to the other DAW, and it outputs all the audio and midi the other DAW is sending to Reaper. Each midi stream uses either its own channel or its own 16-channel midi bus. There are 16 midi busses per Reaper track. Each audio stream is usually stereo, so the rewire track is usually multi-channel, as in quadrophonic or higher.

Instead of simply choosing a rewire connection from the track's output menu, you must set up a send from the track to the rewire track, and then set up a send in the rewire effect to a track on the other DAW. I find this cumbersome, and for midi I prefer to simply use a VMC, which will appear as a midi input and as a midi hardware output.

To get audio from the slave, you must set up the rewire effect to send the data to a certain audio channel (or channels) of the rewire track, then set up a send for that audio channel(s) from the rewire track to the target track. Click on the "Audio From ReWire" tab. On the left are 64 audio channels coming from the other DAW. Right-click a row to set a channel's destination. The pop-up menu lists all the channels in the rewire track.

You can process the other DAW's sounds with Reaper's effects, for example by putting ReaEQ on a track that receives ReWire audio. If you want to send audio from Reaper to the slave DAW, you may be able to use ReaRoute, or a virtual audio cable like Soundflower. Muting the rewire track or disabling the rewire effect will break the connection between the two DAWs. You can have two different ReWire tracks containing two different ReWire effects, sending to two different DAWs. It's possible to load other effects before or after the rewire effect. Effects after don't seem to affect the sound. It's also possible to rewire Reaper to itself. I have no idea why anyone would want to do this! See also Reaper/Preferences/Plug-ins/ReWire.

Reaper as a ReWire slave: This is very straightforward. To receive midi from the host, just set the track's input to a bus of your VMC. To send midi to the host, in the track's I/O box, set "MIDI Hardware Output" to a VMC bus, sending to the original channels. You can use the ReWire midi bus instead of a VMC bus, but there's only one ReWire bus, not enough for multiple instances of alt-tuner running in multi-channel mode. To send audio to the host, in the track's I/O box, set "Audio Hardware Output" to a ReWire output or pair of outputs. Audio cannot be received from the host, except perhaps with ReaRoute, or a virtual audio cable like Soundflower.

Useful links from the Reaper forums:

Reaper compatibility: http://forum.cockos.com/forumdisplay.php?f=25
Reaper compatibility with VSTi's: http://forum.cockos.com/showthread.php?t=800
Reaper compatibility with OS X: http://forum.cockos.com/showthread.php?t=35196

Other DAWs:

Ableton Live (Ableton): see below.

Audacity: has very limited midi capabilities and can't be used with alt-tuner, or any other midi-based software.

Cubase (Steinberg): untested.

Digital Performer (MOTU): untested, OS X only.

FL Studio aka Fruity Loops (Image Line): see below.

Garage Band (Apple): see below.

Logic Pro (Apple): see below.

MuLab (MuTools): see below.

Pro Tools (Avid aka Digidesign): see below.

Reason (Propellerhead Software): untested, reportedly works with alt-tester. Reason doesn't support VST's, so ReaJS is not an option. Use Reaper and ReWire, or use ReaJS with a VST host like Cantabile Lite and a VMC like LoopMIDI.

Sonar (Cakewalk): untested, PC only.

Studio One (Presonus): untested. Doesn't allow recording or playback of sysexes. Nor multi-channel midi files.

Standalone mode using Cantabile Lite or VSThost (Windows only) Cantabile Lite and VSThost are free programs that can host ReaJS, and hence alt-tuner. Latency may be an issue.

Abelton Live is retunable, but requires additional software to host alt-tuner, unless using mono mode or keyswitch mode. Unfortunately, AL merges all midi channels in track sends, effect outputs and max4Live outputs into channel 1, and it filters out all sysex messages. Therefore all multi-channel modes and sysex modes simply *will not work* in AL. In these modes, alt-tuner must be run outside of AL, either inside Reaper, or inside another DAW via ReaJS. Reaper or this DAW can be rewired to AL, or perhaps just linked with loopMIDI. Windows users may want to use Cantabile Lite, a free app that's simpler and easier to learn than a full-on DAW. However, automation envelopes are not possible in Cantabile, and latency can be an issue.

To use alt-tuner in mono or keyswitch mode: To use ReaJS (or any third-party midi effect, for that matter) in AL, you must put it on its own midi track, set up a 2nd midi track, set the 2nd track to receive from the 1st track, and change "Post FX" to the name of the effect. To use both alt-tuner <u>and</u> alt-keyswitcher, you must set up 3 tracks.

AL can be a ReWire host or slave. When AL is a host, it can't receive midi, so use a VMC. Set the AL midi track's "MIDI To" to your VMC's bus #1, channel 1. AL's instruments are multi-instance, requiring multiple tracks. To use one, set up multiple AL midi tracks with the instrument, with the "MIDI From" set to your VMC's bus #2. Each one should receive from a different channel. You can right-click a track and use the "Duplicate" option to save time.

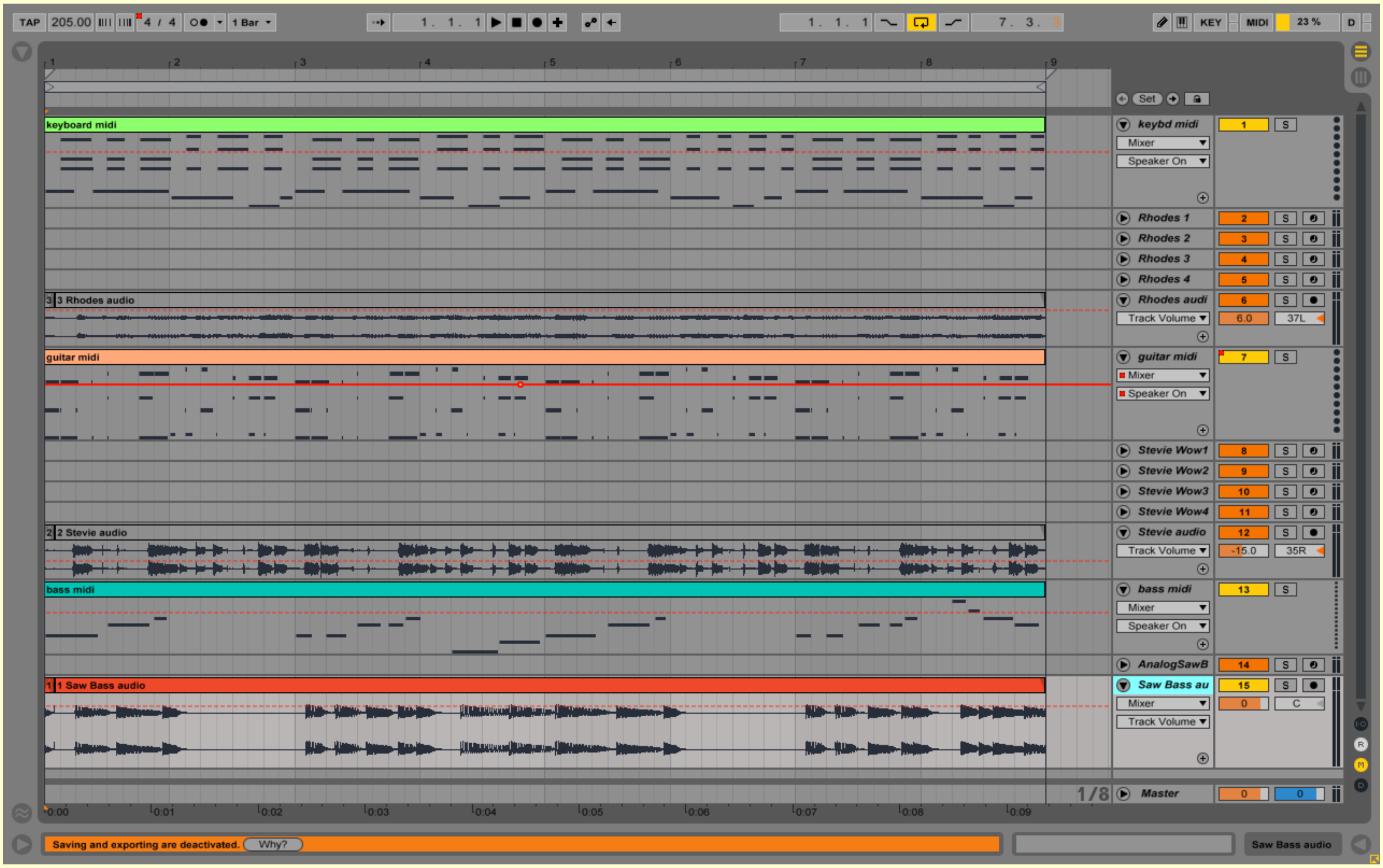
To use primarily Reaper, run AL as a slave. Set up multiple AL midi tracks with the AL instrument, with the "MIDI From" set to your VMC's bus #2. Use "ReWire with Reaper as host.RPP".

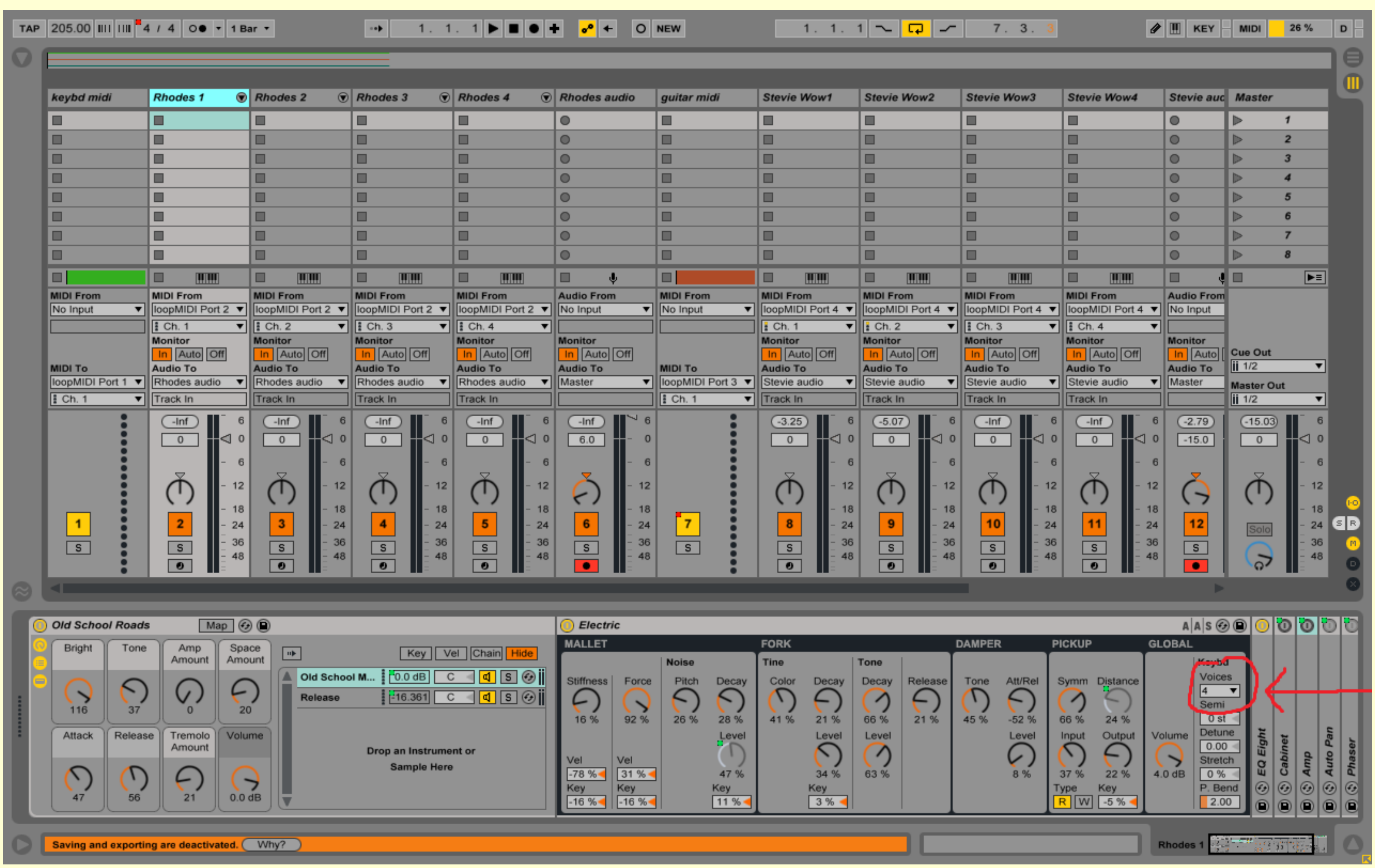
The next few pictures show a Live set with three midi clips. The first two require 4 voices and the third is a mono bass track. In this example, the number of voices in each softsynth is set to 4 or 1. In non-octave mode, each softsynth would only need 1 voice. In octave mode, you need enough voices to handle all the notes of one pitch class. For example, if a midi part contains a low C, a middle C and a high C, all sounding simultaneously, you'll need 3 voices.

AL is the ReWire host and Reaper is the slave. AL is sending midi out the VMC busses 1, 3 and 5, and receiving on the VMC's busses 2, 4 and 6. Reaper should be running either the alt-tuner wrapper or the alt-tester wrapper.

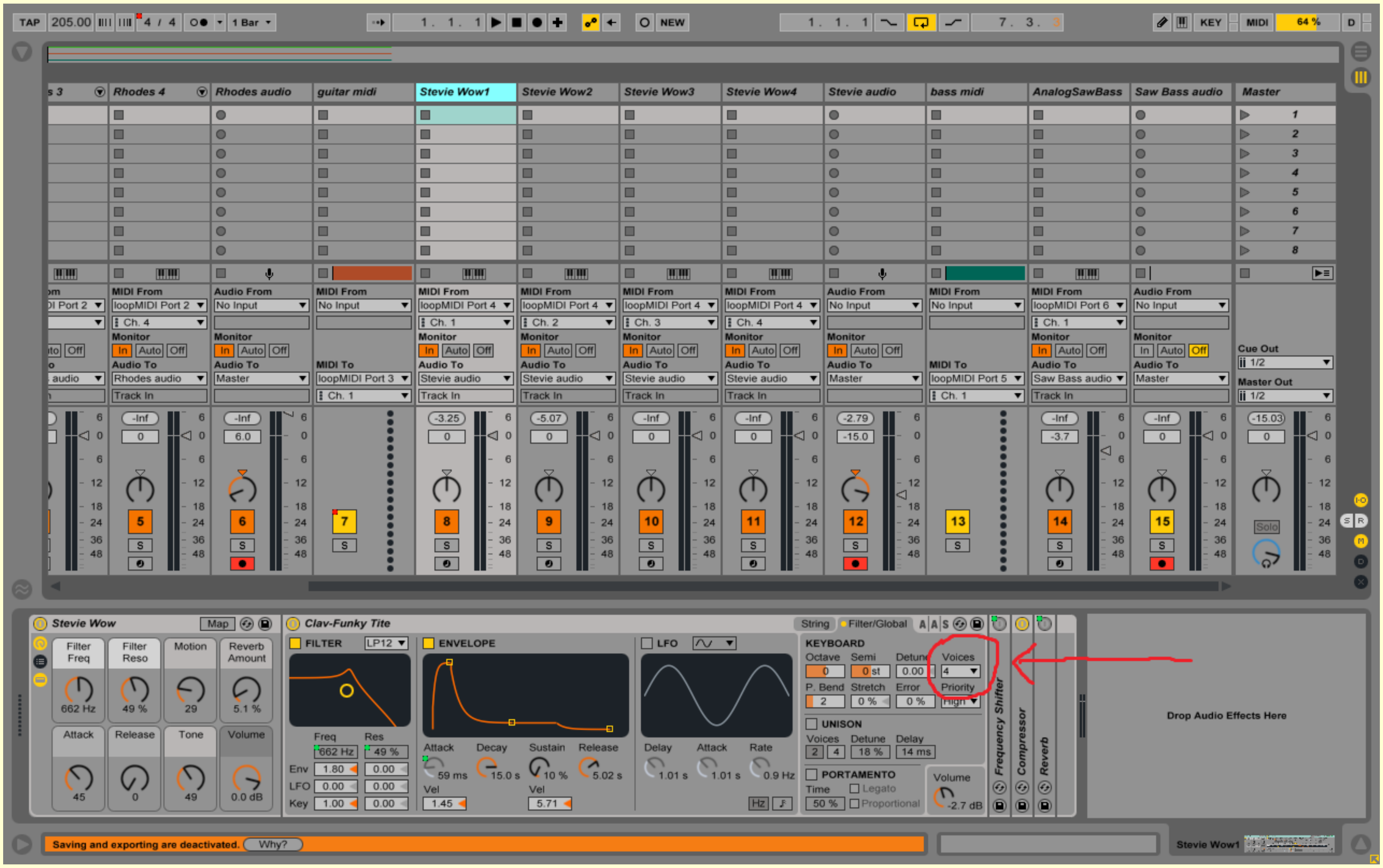
This example uses AL's instruments, which are multi-instance. The number of voices for each instrument has been reduced to 4 for the keyboard and guitar and to mono for the bass. To use other synths instead of AL's instruments, follow this example exactly. <u>All</u> softsynths, even sysex-retunable or multi-timbral or multi-midi-channel synths, require multiple instances of the synth (unless it's a mono track, of course). The only exception is Kontakt, which will work with keyswitch mode. To retune a hardsynth from AL, it's easiest to send the retuned midi from Reaper/Cantabile/VSThost/whatever directly to your keyboard. In any sysex mode, you <u>must</u> send the midi directly to the keyboard, to avoid AL blocking the sysexes.

Ableton Live screenshots:





Ableton Live screenshots:



FL Studio aka Fruity Loops (Windows and OS X, only the Windows version has been tested) is retunable via ReaJS or ReWire. FLS filters out all sysex messages, so alt-tuner can't use sysex modes. On my system, Reaper can't use FLS as a VST plug-in because pitch bend messages seem to get filtered out.

Good news: all instruments, whether FLS instruments or VSTi's, become multi-midi-channel in FLS, so only one instance is needed! They all automatically have a pitch bend range of 12 semitones. Instructions:

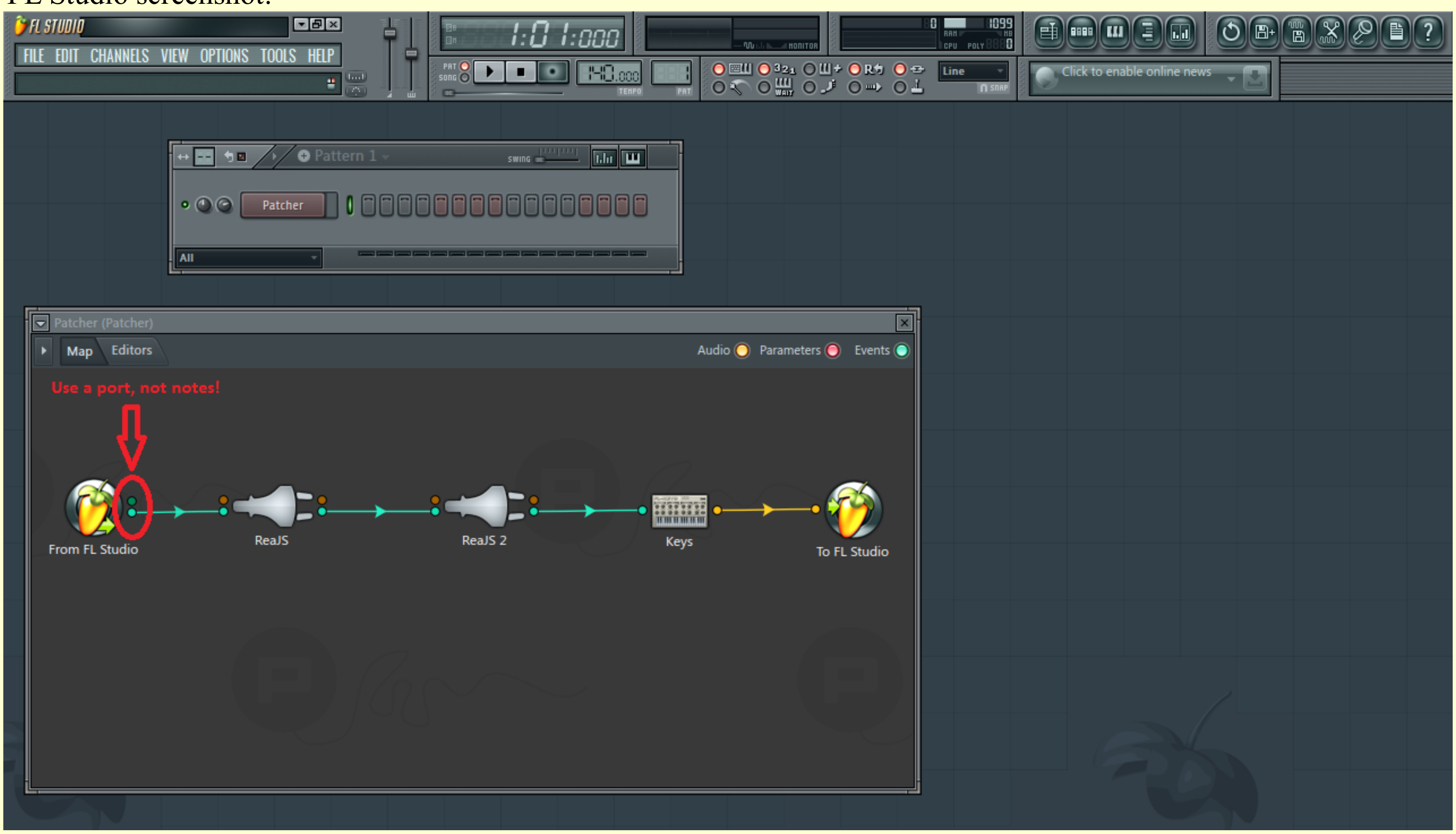
> In options/midi settings, enable all midi devices and set their ports to 0, 1, etc.
> Insert Patcher in a channel of a pattern.
> In Patcher, right-click on "From FL Studio", disable "Notes" and enable the appropriate port(s), e.g. port 0.
> Insert ReaJS two times, and load alt-keyswitcher in the first ReaJS and alt-tuner in the second one.
> Set the pitch bend range to 12 semitones in alt-keyswitcher and update the CC #s in alt-tuner.
> OR, if just testing, insert ReaJS once, load alt-tester, and set the pitch bend range to 12 semitones.
> Set both ports on all ReaJS's to 0, using the second button on the upper left, then using "Settings".
> Use the "Presets" button next to the "Settings" button to load and save alt-tuner and alt-keyswitcher presets.
> Insert an instrument. Don't insert a VSTi as an effect, insert it as a generator via "More Generators..."
> Connect "From FL Studio" to ReaJS to ReaJS to the instrument to "To FL Studio" the obvious way.

You don't need to set the midi input port on a VSTi. The "Send pitch bend range" option seems to function even when it's unchecked.

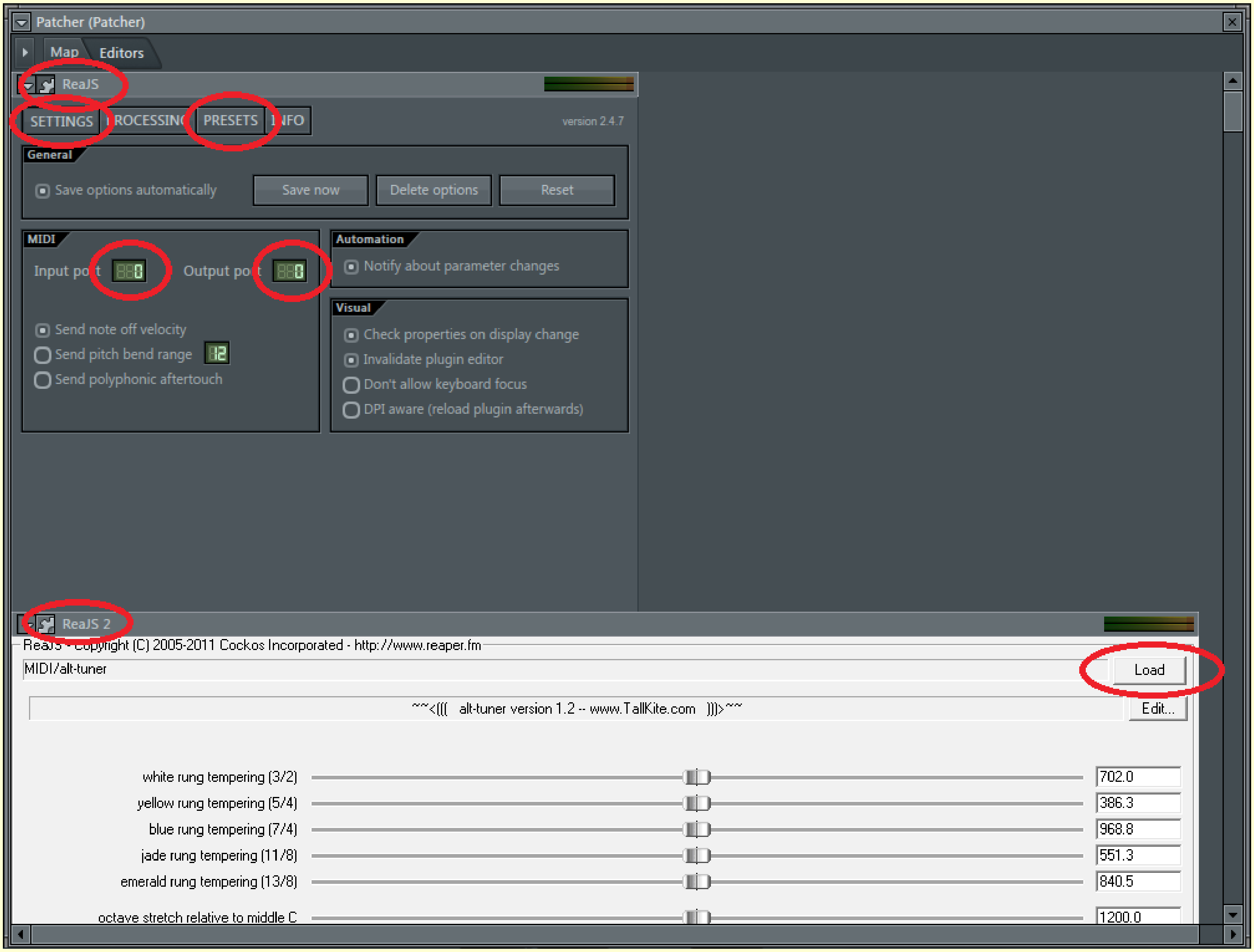
Here's a demo video about using Patcher from another FLS user: http://screencast.com/t/jxDI2gV7hee.

To record the mid output from alt-tuner, use a VMC to send the midi out of FLS and then back in again.

FL Studio screenshot:



FL Studio screenshot:



GarageBand (OS X only) is retunable, but not always in real time. It can't be a ReWire slave.

First the good news: GB is easy to use, comes free with your mac, has decent built-in sounds ("instruments") and lots of useful midi loops for your rhythm tracks.

Now the bad news: Apple's GB can only export midi to Apple's high-end DAW, Logic. This means that when you outgrow GB, if you upgrade to anything besides Logic, you'll lose access to all the midi you've created in GB. In other words, GB is a "midi trap".

Now the good news: midiO by RetroWare is a free program that allows midi to be streamed from GB (but not exported as a file). It can send GB midi to Reaper for retuning, and Reaper can send it back to GB via IAC. Reaper can also record it and then export it as a midi file. Get midiO here: http://mysite.verizon.net/retroware/. An informative page worth reading: http://macaudioguy.com/midi-and-garageband/. Set midiO's "midi destination" to an IAC bus. Be sure to read the midiO manual to avoid midi feedback!

To export midi from GB as a file, use the free program GB2MIDI: http://www.larskobbe.de/midi-export-in-apples-garageband/. Also see http://scotttroyer.com/2014/05/export-midi-from-garageband/

So GB midi loops are retunable, what about GB instruments?

First the bad news: Unfortunately, GB's instruments "listen" to all midi channels on all midi inputs all the time. If you stream midi from GB to Reaper for retuning and stream it back to GB, the GB instrument will play both the original and the retuned midi at once. Therefore using retuned GB midi loops with GB instruments is a multi-step process. You have to use the file export/import method described above to get the retuned midi loops from Reaper to GB.

Even more bad news: GB instruments are not multi-timbral. By itself that's not so bad, but because these instruments listen to everything, the usual workaround of using multiple instances of an instrument will not work for streamed midi. The only way to retune GB instruments is to export/import separate midi files for each channel of alt-tuner output, and put each midi file on a separate track in GB.

Either record your midi in Reaper, or if you're using GB midi loops, export the midi to a file and import it to Reaper. Apply alt-tuner to the midi item as described above to create a tuned midi take. Export this item from Reaper, then import the file you just created back into Reaper, selecting "single-channel items on multiple tracks" from the dialog box. (If there is no dialog box, go to Reaper/Prefs/Media/MIDI and check the "Import multi-channel MIDI files" setting.) Reaper will create 16 tracks with 16 files. If a channel is unused, the midi file will be empty; delete these tracks. Then re-export all the remaining single-channel files, using the "selected tracks only" and "multi-track MIDI file" options. Drag this file into GB. GB will create new tracks; set them all to the same instrument. Optional: to export the audio from GB, solo these tracks, then GB/share/export song to disc uncompressed. Alternative way to export audio: right-click on the GB file for your song, choose "show package contents", and go to the "Media" folder. Duplicate the aif files and drag them to another folder.

A little good news: monophonic instruments like flute or bass can be retuned live without exporting/importing, but only if the midi isn't being streamed from GB at the same time.

One final piece of bad news: some GB instruments, like the grand piano, don't respond to pitch bending and can't be retuned at all.

To sum up, here's your GB options:

A) use GB midi loops but don't use GB instruments, live sound
B) use GB instruments but don't use GB midi loops, live sound only for monophonic audio
C) use GB midi loops and GB instruments, no live sound
D) use GB drum loops and/or GB drum sounds, no retuning needed, live sound

Logic Pro (OS X only) Untested. ReaJS is not an option, use Reaper and ReWire/IAC. LP can't be a ReWire slave. LP has a feature called Hermode Tuning that claims to do 7-limit adaptive just intonation. However, based on my online reading, it can't handle any comma pumps except for the usual g1 = 81/80 comma pump. Because it is an automatic "hands off" system, you can't input a comma like ggg2 = 128/125 or r1 = 64/63, nor can you distinguish between g3 = 6/5 and b3 = 7/6. The complexities of 7-limit adaptive just intonation go beyond what an automatic system can handle. Only alt-tuner gives you the power to fully control your chords' tuning and your lattice modulations.

MuLab (MuTools) (Windows and OS X, only the Windows version has been tested) is retunable via ReaJS. MuLab filters out all sysex messages, so alt-tuner can't use sysex modes. For keyswitch-retunable, multi-midi-channel or multi-timbral synths, you only need one rack:

slot 1: ReaJS / alt-keyswitcher
slot 2: ReaJS / alt-tuner
slot 3: pianoTeq

For multi-instance synths, set up these racks:

alt-tuner rack, slot 1: ReaJS / alt-keyswitcher
alt-tuner rack, slot 2: ReaJS / alt-tuner
alt-tuner rack, slot 3: send to synth #1 rack
alt-tuner rack, slot 4: send to synth #2 rack
alt-tuner rack, slot 5: send to synth #3 rack
alt-tuner rack, slot 6: send to sends rack

sends rack, slot 1: send to synth #4 rack
sends rack, slot 2: send to synth #5 rack
sends rack, slot 3: send to synth #6 rack

synth #1 rack, slot 1: ReaJS / alt-midiFilter, set to channel 1
synth #1 rack, slot 2: synth

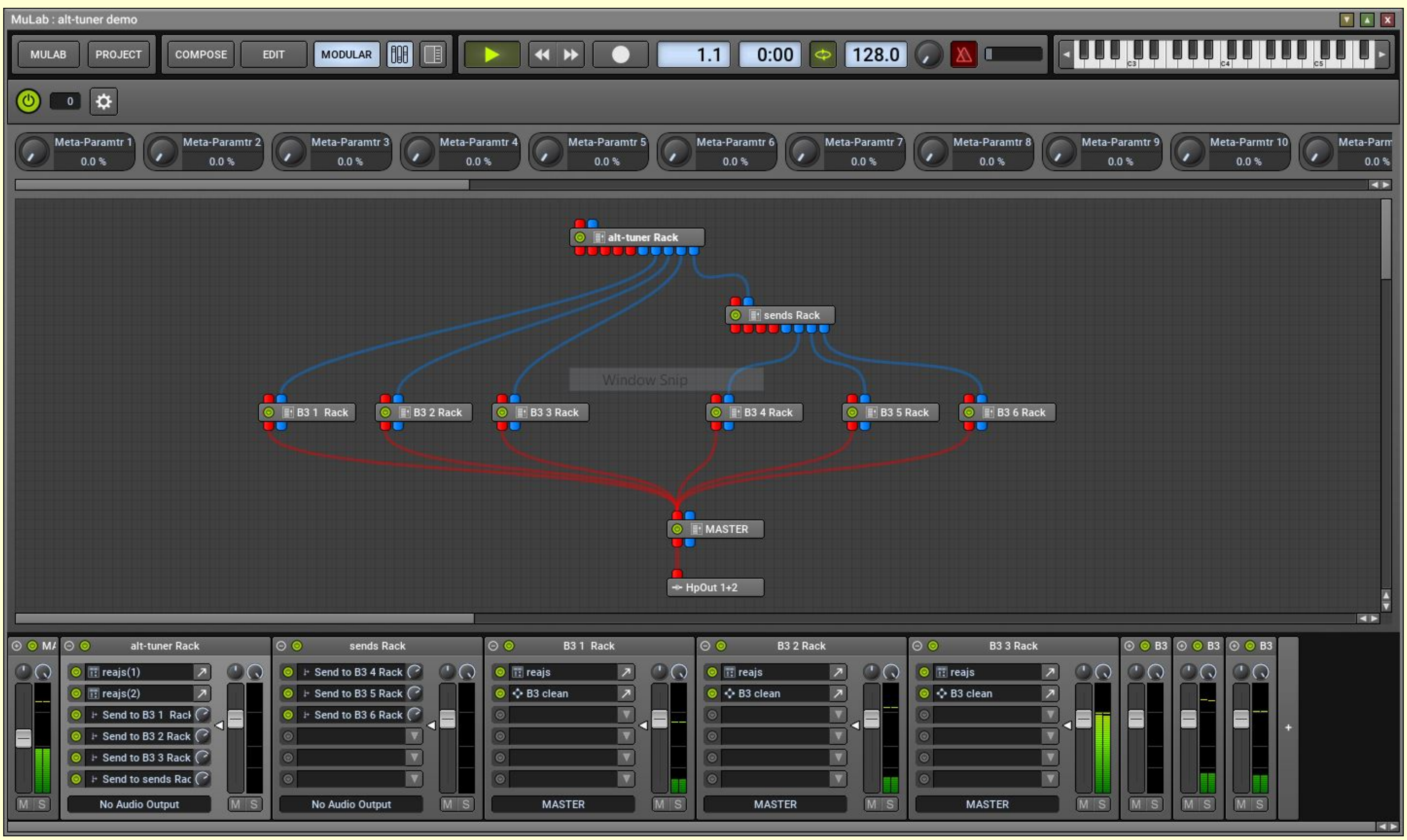
synth #2 rack, slot 1: ReaJS / alt-midiFilter, set to channel 2
synth #2 rack, slot 2: synth

etc.

Alt-midiFilter, a sample MuLab project, and higher-res screenshots are available here:
http://www.tallkite.com/forum/index.php/topic,68.msg298.html#msg298.

There may be an easier way, not using alt-midiFilter, to route only one channel of midi to each rack.

MuLab screenshots:



ProTools is retunable. PT can't be a ReWire slave, and it can't receive midi notes from the slave, so a virtual midi cable is essential. Midi tracks in ProTools can't contain multiple channels of midi data, so ProTools must use multiple tracks to record the output of alt-tuner when it is in octave or non-octave mode.

Slave Reaper to Protools, send the midi from ProTools into Reaper via your VMC's bus #1, send the retuned midi from Reaper to your VMC's bus #2, channels 1-16, then into ProTools, using separate tracks for each channel. Repeat for each instrument.

ReaJS is a possibility for ProTools, if you have a VST wrapper, and if the ReaJS wrapper can be loaded into it. ReaJS inside a VST host like Cantabile Lite is also an option, using a VMC like LoopMIDI.

Hardware: Keyboards, controllers, pedalboards, etc. Alt-tuner expects your keyboard to be in standard 12-ET tuning. Some keyboards have a list of alternate tunings built-in. If you use one of them, your keyboard will be doubly retuned, probably not what you want.

Some keyboards have the ability to be retuned to any scale by custom sysex. All synths (except hardsynths with insufficient timbrality, see below) can be retuned with pitch bends; sysex retuning merely provides another option. Sysex retuning has the advantage of less set-up time and more polyphony, but may limit the size of the octave, the number of keys per octave, etc. Alt-tuner can use either MTS Universal Sysex #82 (real-time single-note tuning change) or Sysex #88 (non-real-time scale/octave dump). Most synths don't respond to any sysexes and must be retuned via pitch bends. Some synths can be retuned via sysex #82 or sysex #88. Some synths don't respond to either sysex but can be retuned by a custom sysex written especially for that specific synth. Such custom sysexes will be available in the future as add-ons to alt-tuner, sold separately. If your synth lets you fine-tune each key individually, it probably can be retuned by sysex. Check the alt-tuner forum for the latest information.

To set up your keyboard for multi-channel modes, for each of your favorite voices, create a separate multi-timbral patch with the first 12 or so parts set to that voice. This lets you quickly switch among your favorite voices. It's important that each part 's settings be absolutely identical. See the troubleshooting section for an easy way to check this by ear.

Polyphony refers to how many voices can play at once, timbrality refers to how many sounds can play at once. Timbrality runs from 1 (mono-timbral) up to a maximum of 16 (fully multi-timbral). A mono-timbral synth can't do splits or layers. In alt-tuner's octave and non-octave modes, the timbrality limits the polyphony. In non-octave mode, the number of timbres limits the number of simultaneous voices. A 16-part multi-timbral synth can only play 16 notes at a time, and playing additional notes will cause dropped notes. In octave mode, the number of timbres limits the number of simultaneous pitch classes. A 4-part multi-timbral synth like the Nord Lead can only play tetrads, not pentads.

If your keyboard is not fully multi-timbral, and does not support MTS sysex retuning, using alt-tuner will reduce your polyphony!

Korg: Retunable. Custom sysex retuning is an additional possibility with certain models. It would have the same restrictions and advantages as Roland sysex88 retuning, except the retuning range is generally +/- 99¢. This is very model-specific, so contact me via the forum for more information.

Nord: Retunable, but with greatly reduced polyphony. These keyboards can't use sysex mode and are not fully multi-timbral. Most are only mono- or bi-timbral. See the timbrality discussion above. However, the Nord Lead 2 and higher are 4-part multi-timbral, and can play tetrads, but not pentads. Set alt-tuner to 4 output channels, and on the Nord, set all 4 slots of a performance to the same program, and don't use layering.

Roland: Retunable. The Fantom, Juno and RS-70, and perhaps other Roland keyboards, can also be retuned via sysex88. This mode restricts the octave to 1200¢ and the number of keys to 12, but only requires one midi channel per instrument. The retuning range for each key is +/- 64¢. Retroactive retuning is not possible in sysex88 mode.

Yamaha: Retunable. The insertion effects can't work on all 16 channels simultaneously. Even on the high-end motifs and s90s, they are limited to 8 channels. With certain sounds, the wet/dry difference is very noticeable. To work around this, either bypass the insertion effects, or else set the number of output channels to 8 or less.

Yamaha model numbers for alt-keyswitcher: Model 0 = motif es 6/7/8 & mo 6/8, 1 = s90 es, 3 = motif xs 6/7/8, 13 = s90 xs & s70 xs, 18 = motif xf6/7/8, 20 = mox6/8. Click to your model and OK it.

Line 6: The Line 6 FBV Express MkII pedalboard works well with alt-tuner. It provides 4 footswitches and a rocker pedal for retuning. The FBV connects to the computer, not the synth, via USB, which also powers the FBV. Footswitches A, B, C & D are set to midi CC, with 4 different CC numbers, port 1, channel 1 & momentary. The rocker pedal's vol & wah are both set to a fifth CC #, port 1, channel 1.

The included "pedalboard fixer" effect corrects a problem with the FBV pedalboard not reacting properly to pressing two footswitches simultaneously. The problem is that the 2nd CC "on" won't be sent, but the 2nd "off" will be sent on release. Pedalboard fixer sends a 2nd "on" CC message when the 2nd footswitch is released.

Guitar-to-midi converters: Untested. Set your converter to 6 channels of mono output (one for each string). Set alt-tuner to mono mode and set the midi input channel to 0 = all. Send all 6 channels of alt-tuner output to your synth. Individual string bends should be possible. Alternatively, it may be possible to set alt-tuner to one of the sysex modes and combine the six channels of sysex output into one with Rechanneler.

Chapter 6.11 – Troubleshooting and Other Considerations

Troubleshooting:

Can't play in sharp keys: To play in, say, C♯ minor, go to D♭, modulate fourthward (leftward) to B, then modulate fifthward to C♯.

Inconsistent sounds in octave or non-octave mode: Make sure that the different instances of your softsynth, or different patches on your hardsynth, have identical settings. See the softsynth section of the last chapter. Hardsynth users: see also the warning in chapter 6.10 about Yamaha insertion effects.

When using multi-timbral or multi-instance synths, or when using hardware synths in multi-channel mode, the multiple instruments/patches/voices should all have identical settings – the same volume, pan, EQ, ADSR envelope, etc. Especially important is the pitch bend range. If you overlook any of these settings, the sound of the instrument will vary in a random way. Once you have set up your instruments, it's a good idea to double-check the settings by ear. Alt-tuner provides an easy way to do this. If you're in octave mode, set the number of midi channels to less than 12 (or more generally, to less than the number of keys on the keyboard screen). In non-octave mode, the number of channels doesn't matter. Now play one note repeatedly. Alt-tuner will send this note to each channel in turn, and each instrument will respond in turn. Verify this by watching the midi channel monitor on the prefs/misc screen. Most DAWs let you use the computer keyboard like a virtual MIDI keyboard. This "musical typing" creates notes of uniform volume, good for testing purposes. As you play, listen carefully to the tone of that note. If you hear any variation, check your settings.

New notes throw earlier notes off key: You may be only using a single instance of a multi-instance softsynth, see chapter 6.1 and 6.10. Or alt-tuner may be in monophonic mode, but your synth is in polyphonic mode; if so, set your synth to mono.

Latency (sounds lag behind key presses): Experiment with your audio buffer size. A buffer size too low causes crackles and pops, and a size too high causes latency. Windows users: try downloading the ASIO4ALL driver from www.asio4all.com and selecting ASIO in your DAW's audio preferences.

Synth is out of tune: Every time you use alt-tuner on a new synth, set alt-tuner to 5-edo and play a chromatic scale, to test that the pitch bend range, midi output mode, midi routing, etc. is correct. The 12 notes should only produce 5 pitches. If they don't, check the pitch bend range of your synth by physically moving the pitch bend wheel as you play to verify the range, and setting alt-keyswitcher accordingly.
If you don't have a physical pitch bend wheel, you can use the "pitch bend sweep" reaper project included with the alt-tuner files. It's designed to test how accurately your synth responds to pitch bends, as described in chapter 6.11, but it will also serve to test the total pitch bend range. Put your synth on the track, set it to a sustaining sound, and hit play. Check that the pitch bends by 2 semitones. If not, set alt-keyswitcher accordingly.

Notes don't end: When playing a midi file with sustain pedal messages, it's possible for the sustain pedal to be stuck in the "on" position, if you stop playback after the pedal-press message but before the pedal-release message. Either press and release your physical sustain pedal, or replay the midi file and end after a sustain pedal release.

Notes are dimmed: This is a normal byproduct of overbending a note with the pitch bend wheel. In alt-keyswitcher/other, set the wheel bend range below the synth bend range. If every note is dimmed, check that synth bend range in alt-keyswitcher isn't zero. Remember to update the midi CC #s in alt-tuner.

Notes are outlined with a red square: Dropped notes. There are not enough output channels to handle all the notes being played. Increase the number of channels, or use octave mode with 12 output channels. Also see next paragraph.

Octave stretch slider doesn't move: This is a limitation of octave-equivalent and sysex88 output modes. In prefs/misc, set the midi output mode to non-octave or mono or sysex82. Also see the previous paragraph.

No pedals work: Experiment with the midi threshold in alt-keyswitcher, and update the CC #s in alt-tuner/prefs/CCs.

Note tails change pitch when playback stops: In Reaper/prefs/audio/midi devices, uncheck "reset pitch".

"can't read from alt-keyswitcher": either alt-keyswitcher isn't loaded, or the register blocks don't match.

"wrong version of alt-keyswitcher": You must use identical versions of alt-tuner and alt-keyswitcher.

Data from alt-keyswitcher is garbled when running other Jesusonic effects: See "register blocks" in chapter 6.5.

Other problems: Try resetting alt-tuner to the factory default, recompiling alt-tuner, or closing the Reaper project and reopening it. If you're doing advanced things, the order of your actions may matter. First set the output mode, then work from right to left: keyboard, then rungs, then linkages, then the other screens. Avoid playing notes while clicking on the rows screen, the rungs screen, the midi channel lines of the misc screen, and the # of keys slider in the keyboard screen.

<u>How retunable is your synth</u>? Unless in a sysex or keyswitch mode, alt-tuner does the actual retuning with midi pitch bend messages. Your synth is set to have a certain pitch bend range, usually +/- 200¢. A full sweep of the pitch bend message would theoretically generate about 16,000 possible bend values. For a bend range of 200¢, that would mean steps of about 1/40 of a cent. In practice, your keyboard's bend wheel may not produce such fine gradations, and more to the point, your synth's tone generator may not either.

Set your synth's pitch bend range to the max, hopefully a huge value like 2 or 3 octaves. Even with such a large range, the steps will theoretically be less than a cent, and not audible. Set your sound to a sustaining tone like organ. Move the wheel while playing a note. Does the pitch glide smoothly up and down? Or does it jump by individual "steps" from note to note? If so, count the steps.

Is the limitation from the wheel or the synth? To check your wheel, record the midi generated by slowly moving the wheel to the max and back to center. In your DAW, examine the midi messages. Are there tens of thousands, or only hundreds? Do the bend values increase slowly from one message to the next, or do they jump sharply?

To check your tone generator, load the "pitch bend sweep.mid" file into your DAW. This file plays a sustained note while slowly sweeping up and down the bend range, using all 16,000 values. Again, set your pitch bend range to the max, set your sound to a sustaining tone, and see if you can hear discrete steps. If so, count the steps. The midi file contains three full sweeps, a quick one, a medium one and a slow one. There are 96,000 midi messages in the file. If your synth can't handle all that midi fast enough, start with the slow sweep and/or slow down the tempo of the DAW session.

With my Yamaha s90 ES, I count 32 steps from center to max. That means that the standard bend range of +/- 200¢ will give me steps of about 6 cents (200/32 = 6.25). Those steps might hit my target interval right on, or it might miss by up to 3 cents. Possibly more, depending on how the tone generator rounds things off. Since an interval may have its lower note sharp and its higher note flat, the maximum out-of-tune-ness of an interval is at least 6 cents. The <u>average</u> out-of-tune-ness works out to be one third of that, about 2 cents.

To get more accuracy, you must decrease the synth's pitch bend range. You need at least a 50¢ bend range to close the gap between keys, yielding 1.5¢ minimum accuracy, and 0.5¢ average accuracy, which is about as accurate as you can hear. Set your synth accordingly, and also go to alt-keyswitcher's "other" screen and set the synth bend range. A smaller synth bend range limits what the physical pitch bend wheel can do, since the wheel bend range cannot be more than the synth bend range.

The total pitch bend for any note is the sum of the cents offset, the calibration frequency difference, the tuning offset for that key's ratio, and any manual bends (via pitch bend wheel or aftertouch). For example, if the cents offset is +20¢ from A-442, playing a red 3rd sends a midi pitch bend of 20¢ + 8¢ + 35¢ = 63¢ (442/440 = 8¢ and r3 = 435¢). If the synth bend range and the wheel bend range are both 200¢, and after playing the note you wheel-bend as sharp as possible, the total bend becomes 263¢. The total bend exceeds the synth bend range, and alt-tuner will not be able to tune the note accurately. The note will be dimmed on the screen as a warning. You can distinguish this dimmed note from an unselected gray note because it will be circled. This is different from dropped notes, which are outlined with a red square. To see this warning in action, with wheel bend range equal to synth bend range, use the wheel to bend almost any note up and down to the max. To avoid overbending, set the wheel bend range to be less than the synth bend range. Only one red square or dimmed note is displayed at a time, no matter how many notes you drop or overbend.

If the initial bend (without manual bends) is above or below the synth bend range, alt-tuner will automatically transpose the note and adjust the pitch bend accordingly. In the previous example, if the synth bend range is 50¢ and the initial bend is 63¢, the midi note-on message will be transposed up one key and the pitch bend message will be -37¢. Repeated modulations or extreme tempering can push the initial bend beyond 200¢. Non-12-tone keyboard settings will push the initial bend far, far beyond. For example, 24 keys per octave requires the C key two octaves above middle-C to be transposed down a full 12 semitones.

<u>What about acoustic music</u>? Acoustic music can be retuned with audio retuners like auto-tune, as long as it's monophonic. Unfortunately it's difficult to control what scale an audio retuner tunes to. Autotune uses only a dozen or so preset scales. Fortunately one of them is a 43-note 11-limit JI scale, and you can bypass individual notes, so JI is possible. Autotune is incapable of tuning audio to most non-JI scales. Waves Tune lets you manually adjust the pitch of each of the 12 notes independently. However, changing the scale while playing is impossible, as is stretched-octave scales or scales with more than 12 notes.

What is sorely needed is an audio tuner that responds to MTS sysex messages or virtual keyswitches. It could be driven by alt-tuner much like a 2nd player's synth can be driven, via tuning CCs. Then a keyboardist/vocalist could simultaneously retune both their keyboard and their vocals with the usual pedals and keyswitches.

It's also possible to build an acoustic string instrument that can be retuned by alt-tuner. It would use mechanical retuning devices such as the sharping levers of the Celtic harps or the foot pedals of the classical harp. These devices would adjust the pitch not by a whole semitone but by a small fraction of one, like the mandals of a Qanun or Kanun (a Turkish zither). These devices in turn would be controlled by midi output from alt-tuner. Such an instrument would be tuned by midi, but it could be played either directly or via midi. For a non-microtonal example of the latter, see Pat Metheny's Orchestrion.

<u>The design philosophy of alt-tuner expressed as a FAQ</u>:

Why doesn't alt-tuner have a pretty background picture of wood grain and burnished steel like my other VSTs do?

Because alt-tuner displays a lot more information than most VSTs do. A pretty picture would just get in the way of the data.

Why doesn't alt-tuner have pretty knobs like my other VSTs do?

Because you can't twist something with a mouse. When you see a knob on the screen, you must guess whether you're supposed to drag it up or sideways or clockwise. With sliders, you don't have to guess, you know exactly what to do.

Why doesn't alt-tuner have tool tips that appear when you hover over something?

For the same reason that clickable text is yellow and non-clickable text is green: so you don't have to guess where to click or hover. (There are a few "hidden" click spots, see the last page of this manual for a list of them.)

Why doesn't alt-tuner have hundreds of built-in presets for various tunings?

Several reasons. Reason #1: that drop-down menu of presets at the top of the screen is where your set list goes. On stage, you don't want to be scrolling through hundreds of tunings just to set up for the next song.

Reason #2: Moving sliders and clicking on number boxes is more fun than hunting for something in a long list, especially if you aren't sure what it's called. Plus sometimes it's much faster, for example moving the EDO slider to 22 vs. finding a 22-EDO preset.

Reason #3: Remember the computer game Myst? It was a big hit because it was one of the first explorable video games. You go to a new place, you manipulate various things, and you see and hear other things as a result. In the process, you learn about this magical new universe you are in.

Alt-tuner is designed to be similarly explorable and educational. For example: you start off playing in JI and you discover the dissonant yellow 5th. You see on the lattice that it has a different shape than the other perfect 5ths. You come to the graph view and you see the P5 line dips down at that yellow 5th. Later on, you play with the white and yellow sliders. Watching the graph, you find you can lessen the dip and improve the yellow 5th at the expense of the white 5ths. Watching the lattice, you find that you can make the yellow 5th resonate with the white 5th. Later still, you get to the linkages screen. You discover that you can link the white and yellow sliders with the green comma. You can move the white slider and explore all possible tunings that have this resonance.

By the time you do all this, you understand exactly what meantone temperament is and why one would use it. If I had included a meantone preset, would you have understood it as well?

Why isn't adaptive just intonation automatic? Why can't alt-tuner just analyze the chords you play and tune them?

Just intonation is simply too complex to automate. For example, the C7 chord can be tuned so that B♭ is a just minor 3rd above G (g7 = 9/5). Or it can be tuned so that B♭ is two just fourths above C (w7 = 16/9). The choice depends on the musical context. For example, if you're playing V - IV - I in F, the C7 is followed by a B♭ chord, and the 2nd tuning will be better because the 1st tuning causes a jarring 22¢ pitch shift for B♭. But if you're in A♭, and coming from or going to an A♭ add 9 chord, the 1st tuning is better because the 2nd causes a similar pitch shift.

The problem is compounded when you use 7-limit just intonation. C7 might have a slightly flat B♭ tuned to the sweet "barbershop 7th" (b7= 7/4). If you follow this chord with a Cm7 chord, the E♭ can be tuned a just 5th below B♭, making a rather narrow minor 3rd (b3 = 7/6). Or it can be tuned to the usual minor 3rd (g3 = 6/5). But this E♭ makes a fifth with B♭ that's almost half a semitone flat (by5 = 35/24 = 653¢), which sounds very dissonant and probably isn't what you want. So you would probably sharpen B♭ by a half semitone, a <u>very</u> noticeable shift! A third option is to tune the C7 chord with the sharper B♭ (g7 = 9/5), making the C7 less sweet but avoiding the pitch shift.

In some situations, you might prefer the narrow minor 3rd, in others you might prefer the shifting minor 7th, in others you might prefer the less sweet C7 chord. There's no one right answer. It's an artistic decision that should be made by you, not your tuning software.

Furthermore, because chords are often arpeggiated, even tuning a single chord can be impossible. For example, suppose you decide C7 should be tuned with 7/4, but Cm7 with 9/5. If the C chord is arpeggiated and the 7th of the chord is played before the 3rd, the tuning of the 7th depends on the not-yet-played 3rd. Software can't predict the future, and it can't read your mind.

Why can't I save my tuning as a .tun file?

Two reasons. The first is because of the limitations of Jesusonic. The second is because .tun files lock you into one tuning. You can't modulate, or EDOtap, or use adaptive tunings. Save your tuning as an alt-tuner preset instead.

Why isn't there a free limited-functionality demo version of alt-tuner? Why isn't there a standalone version of alt-tuner?

Because of the limitations of Jesusonic.

Can alt-tuner really handle any tuning scenario at all?

As far as I know. You may have to resort to switching among many custom tunings. You may also need to switch among many alt-tuner presets, each with their own set of custom tunings, or even switch among many instances of alt-tuner, each with their own presets. (See the last example in chapter 6.9.) It becomes a matter of how many pedals or keyswitches you're willing to use and how much set-up you're willing to do.

Why is it such a hassle using alt-tuner with Ableton Live?

Most VSTi's aren't designed to be retuned microtonally. You have to "trick" them by using several instances and multiple midi channels. But Ableton Live's internal midi routing (which gets midi from alt-tuner to your synth) doesn't allow multiple midi channels.

Will there ever be an iPad or iPhone version of alt-tuner?

Probably not. Apple chose to have iOS not support plug-ins. That means no third-party VSTi's or effects. Every iOS DAW is a "closed box". You can cut and paste audio and midi from one iOS app to another, but not while you're playing. To play live, an iOS alt-tuner would have to be a standalone app with its own built-in sounds.These sounds would necessarily be few in number and far inferior to the vast universe of available VSTi's. (Those who use synth workstations like the Yamaha Motif or the Roland Fantom might prefer having an iPad on stage instead of a laptop. This is the only situation in which a midi-only iOS app makes any sense. So maybe someday...)

Chapter 6.12 – Customizing Alt-tuner

To semi-permanently set various options (like pedal functions or screen colors): from the presets menu, select "Reset to factory default". Set the options as desired. Then save the preset as a default preset, perhaps named "Reset to my default". Now whenever you insert alt-tuner in a track, it will start off with this preset loaded. If you need to reset alt-tuner after doing something complicated, you can reset to either default. You can also modify "Reset to my default" by loading it, changing whatever options you want, and saving it again.

Some customizations are a matter of adding or replacing graphics files. Others require editing the program itself. Click the edit button and open it in an external editor (NotePad, TextEdit, etc.) Only the first few pages of the program are editable; the rest is encoded. Type carefully, it's easy to accidentally render the program unfunctional! Always "save as" under a different name, so the original is unaltered. There are two ways to do this; see the final section of this chapter. After editing alt-tuner, sometimes you may need to quit your DAW and start it again.

To change the summary: When you press the edit button, a summary appears that is from the first lines of the program. You can edit this to reflect your own shorthand, or to add any notes or reminders you may need.

To change the look of the notes or the accidental signs: The notes and accidentals used in alt-tuner are stored as png files in the AltTunerGfx folder. To replace any png file with your own, give the new one the same name as the old one, after first either renaming the old one or moving the old one to a subfolder. You can create your new png files from scratch, or you can edit SourceFile1.odg. It's in Open Office format; get the free software at www.OpenOffice.org, or www.NeoOffice.org for mac users. After editing, save the individual pages of the odg file as png files, and crop if needed. The image should be white on black; the final color of the image is controlled by the prefs/layout screen.

If you want to add completely new note symbols, like letters of the greek alphabet, put your png files in the CustomNoteSymbols folder. They should be named Symbol1.png, Symbol2.png, etc. You can add up to 48 new note symbols this way. If you want to add more than 48, you have to add lines to the filename section of the program:

filename:170,AltTunerGfx/CustomNoteSymbols/Symbol48.png
becomes
filename:171,AltTunerGfx/CustomNoteSymbols/Symbol48.png
filename:172,AltTunerGfx/CustomNoteSymbols/Symbol49.png

Make sure there are no trailing spaces or tabs after the file name! If you want to use different file names or folder names, you can edit the filename section of the program to reflect this:

filename:32,AltTunerGfx/Letters/LetterA.png
becomes
filename:32,AltTunerGfx/Letters/myLetterA.png
or
filename:32,AltTunerGfx/GreekLetters/Alpha.png

There are 8 accidental signs included with alt-tuner: sharp, flat, double-sharp, double-flat, up, down, double-up and double-down. You can replace the png files with your own as described above. You can also add your own custom accidental signs. Put the new accidentals in the Accidentals folder and name them Sharp3.png, Flat3.png, Sharp4.png, Flat4.png, etc., or Up3.png, Down3.png, Up4.png, Down4.png, etc. The Up4 accidental represents a sharpening of the natural note by 4 "semitones", using the word loosely to mean one key in the diagram in the prefs/keyboard screen. The meaning of the Sharp4 accidental is determined by the "1 sharp = [_] keys" box on the keyboard screen. If 1 sharp = 2 keys, Sharp4 would mean the natural note has been sharpened by 8 keys.

There are placeholder filenames for up to Sharp15, Flat15, Up15 and Down15; you can add more filenames as described above. Because filename numbers must run in sequence and not have any gaps, and because the symbol files follow the accidentals, to go beyond Sharp15 or Up15 you have to increase all the symbol file numbers. You must also update firstSymbol to be the first note symbol's file number. If you add sharp/flat accidentals, you must update firstUpSymbol as well. To add two accidental signs:

firstUpSymbol = 32; *becomes* firstUpSymbol = 34;

firstSymbol = 62; *becomes* firstSymbol = 64;

filename:30,AltTunerGfx/Accidentals/sharp15.png
filename:31,AltTunerGfx/Accidentals/flat15.png
filename:32,AltTunerGfx/Accidentals/Up.png
filename:33,AltTunerGfx/Accidentals/Down.png
becomes
filename:30,AltTunerGfx/Accidentals/sharp15.png
filename:31,AltTunerGfx/Accidentals/flat15.png
filename:32,AltTunerGfx/Accidentals/sharp16.png
filename:33,AltTunerGfx/Accidentals/flat16.png
filename:34,AltTunerGfx/Accidentals/Up.png
filename:35,AltTunerGfx/Accidentals/Down.png
etc.

Sliders: There are 64 sliders. Here's what each slider does:

slider1 = tonic slider = key or center note
slider2 = offset slider = cents offset of the key note from A-440 (or A-calibration frequency)
slider3-50 = scale sliders = tap values for up to 48 tappable keys
slider51-59 = tempering sliders = cents values of up to 9 temperable rungs
slider60 = stretch slider = cents value of the octave, aka period or interval of equivalence
slider61 = EDOtap slider
slider62 = EDO slider
slider63 = tempering strength slider
slider64 = reserved for future use

To hide sliders: If you won't be using certain features, for example octave stretching or emerald tempering, you can reduce screen clutter by hiding sliders with a minus sign "-".

slider60: 1200 <1100.0, 1300.0, 0.1> octave stretch relative to middle C
becomes
slider60: 1200 <1100.0, 1300.0, 0.1> -octave stretch relative to middle C

Hiding a tempering slider is not the same as reducing the number of rungs on the misc/rungs screen. The former keeps you from seeing the slider and accessing it directly, and the latter makes certain tempering sliders irrelevant.

You can also unhide the scale sliders 1-14 if desired.

To change the range or resolution of a slider: First, when the 3/2 slider says 702.0, a 3/2 is not really rounded off but is actually 701.955¢ (see chapter 6.8 for details). Secondly, you can type in any number you want in the little box to the right of a slider. Thus 3/2 can be tempered to, say, 600¢, even though you can't drag the slider all the way down to 600¢. Also, you can type in, say, 696.578¢ (the quarter-comma meantone 5th), even though the slider can only be dragged to 696.5¢ or 696.6¢. But if you want to put in such values without any typing, you have to change the dragging range and/or the resolution. Edit the slider line, following this format: sliderXX: DEFAULT <MIN, MAX, STEPSIZE> LABEL. For a narrower or wider range of tempering of 5ths:

slider51: 702.0 <602.0, 802.0, 0.1> white rung tempering (3/2)
becomes
slider51: 702.0 <692.0, 712.0, 0.01> white rung tempering (3/2)
or
slider51: 702 <0, 2400, 1> white rung tempering (3/2)

These two slider examples are already set up for you, ready to be uncommented. Remove the "//" from the beginning of the line. Comment out the original slider51 by adding "//" to the beginning of the line.

For extremely accurate tempering, edit sliders 51-60 to thousandths and also set minTemper to the new stepsize:

slider51: 702.0 <602.0, 802.0, 0.1> white rung tempering (3/2)
becomes
slider51: 701.955 <692, 712, 0.001> white rung tempering (3/2)

minTemper = 0.1; *becomes* minTemper = 0.001;

See chapter 6.8 for an explanation of minTemper. Given the standard 200¢ pitchbend range, alt-tuner's tuning accuracy is theoretically about 1/40 of a cent.

You can also change the default value of these sliders (702¢, 386¢, etc.) from the JI value to something else, if you're working with non-harmonic timbres. This is usually only done for rungs that have been redefined. The little black mark will be moved off center, and double-clicking the slider will send it to this new value. You can also modify the color names and/or the ratios in the label, to change "white rung tempering (3/2)" to something else.

You may want to change the range and labels of slider 1 if you work with non-12 tunings a lot. The range of slider 2 is related to your synth bend range.

If you add or remove ratios from the lattice, you may want to change the scale sliders' range and labels. Otherwise the automation envelopes will either not have a wide enough range, or have misleading labels. Controlling the envelope range and options is actually the only reason to modify any hidden slider. Follow this format: sliderXX: DEFAULT <0, NumberOfOptions-1 , 1 {1stOption, 2ndOption... LastOption}> LABEL. The first option is always silence:

slider4: 2 <0, 4, 1 { silent, blue 2nd, bluish 2nd, green 2nd, reddish aug unison}> -min2
becomes
slider4: 2 <0, 5, 1 { silent, blue 2nd, bluish 2nd, green 2nd, reddish aug unison, amber 2nd}> -min2

If you work with more than 12 keys, you may want to edit sliders 15-50 to add appropriate options and labels:

slider15: 2 <0, 4, 1> -#12
becomes
slider15: 1 <0, 3, 1 { silent, yellow 7th, reddish 7th, red 7th}> -Maj7

If you work a lot with EDOtap, you may want to change the scale sliders' range. 22-EDO is already set up, just uncomment them. The default slider value is the number of EDO-steps in the interval, plus one.

slider3: 1 <0, 22, 1> -Perf1
slider4: 3 <0, 22, 1> -min2
slider5: 5 <0, 22, 1> -Maj2
slider6: 6 <0, 22, 1> -min3
slider7: 8 <0, 22, 1> -Maj3
slider8: 10 <0, 22, 1> -Perf4
slider9: 12 <0, 22, 1> -tritone
slider10: 14 <0, 22, 1> -Perf5
slider11: 16 <0, 22, 1> -min6
slider12: 18 <0, 22, 1> -Maj6
slider13: 19 <0, 22, 1> -min7
slider14: 21 <0, 22, 1> -Maj7

To experiment with fractional EDOs, change slider62: 0 <0, 72, 1> to slider62: 0 <0, 72, 0.1>. If you are in 2.5-EDO, all tempering sliders must be either 480¢ or 960¢ (every other degree of 5-EDO). I'm not sure how useful this is!

You can increase the range of slider 62 to more easily work with EDOs beyond 72-EDO. There's no particular reason to change sliders 61, 63 or 64.

If you add rungs in alt-tuner's prefs/rungs screen , you may want to unhide sliders 56-59:

slider56: 105.0 <5.0, 205.0, 0.1> -17ish rung tempering (17/16)
becomes
slider56: 105.0 <5.0, 205.0, 0.1> 17ish rung tempering (17/16)

<u>To remove sliders</u>: When you click on Reaper's automation envelope button, you'll see all 63 of alt-tuner's sliders. You may want to reduce screen clutter by removing sliders you don't use, especially sliders 15-50 and 56-59. Just comment them out:

slider15: 2 <0, 4, 1> -#12
becomes

```
//slider15: 2 <0, 4, 1> -#12
```

MaxNum limits: These numbers set the basic limits of alt-tuner. You may need to increase them if you're doing something particularly complicated. Decreasing them is possible but not recommended. Default settings:

```
maxNumRungs = 25;         // only the first 10 rungs are temperable/stretchable
maxNumRows = 100;         // rows in the lattice
maxNumRatios = 1000;      // ratios in the lattice
maxNumPresets = 8;        // preset scales to cycle through
maxNumTunings = 30;       // custom tunings to switch among
maxNumSwitchModes = 8;    // modes used to switch among custom tunings
```

See also "presets from customized versions" in chapter 6.8.

There's a relationship between maxNumRungs and the slider maximum on the prefs/rungs screen. Because the default maximum is 99, the highest prime allowed is 97. Because 97 is the 25th prime, a logical maxNumRungs value is 25.

If you increase maxNumRungs, then go to the rungs screen and increase the # of rungs, alt-tuner will automatically create a series of rungs with the appropriate prime numbers, like 101/1, 103/1, etc. However, when you click "nextpage" to view these rungs, if your slider maximum is only 99, these larger ratios will be clipped to 99/1. To avoid this, increase the slider maximum before you click "nextpage".

To add a custom sysex: See "Hardware and Software Issues" and see www.TallKite.com for the latest info.

Naming considerations: Whenever you edit alt-tuner, always "save as" under a different name, so the original is unaltered. There are two ways to save your customized alt-tuner, an easy way and a safe way. The easy way is to rename the original program "alt-tuner original" and save the edited version as "alt-tuner". The safe way is to save the edited alt-tuner under a new name, something like "my-alt-tuner".

If you only want to use the new version, you can use the easy method. If you want to use both the original version and the new version, you should use the safe method.

The safe method creates two problems. First, all your alt-tuner presets will disappear from the new version. To get them back, you have to copy alt-tuner's presets file and rename it to the new name. The file is called "js-MIDI_alt-tuner.ini" (assuming that you installed alt-tuner in the MIDI folder). Run Reaper, and with all effects windows closed, in the menu choose "Options/Show REAPER resource path in explorer/finder", and look in the Presets folder. Copy it and rename the copy to something like "js-MIDI_my-alt-tuner.ini". There is no easy way to merge one group of presets with another, so do this before you create any presets with your new version.

The second problem is that all your older Reaper projects will use the earlier version of alt-tuner. This is only a problem if you want to use the newer version in your older projects. If so, you'll have to manually replace the earlier version with the new version in each project. In the process, you'll lose the alt-tuner settings that are stored in the project. If your project settings are not the same as one of your presets, it can be laborious to recreate the settings manually. It's easier and safer to save the settings for the old alt-tuner as a preset. Then copy and rename the presets file as described above, and use the preset in the new version. This method only works if you haven't created any presets with my-alt-tuner yet. If you have, use this method:

1) Open the old project and save the alt-tuner settings to preset X, if they haven't been saved already.
2) Rename "my-alt-tuner.ini" to "my-alt-tuner backup.ini".
3) Copy "alt-tuner.ini" and rename the copy "my-alt-tuner.ini".
4) Back in the project, remove alt-tuner and add my-alt-tuner. Load preset X into my-alt-tuner.
5) Delete "my-alt-tuner.ini" and rename "my-alt-tuner backup.ini" to "my-alt-tuner.ini".
6) Back in the project, save my-alt-tuner's settings to preset X. Save the project.

If you want to use my-alt-tuner in several projects, you can deal with all the projects at once in steps 1, 4 and 6.

Full alt-tuner support at www.TallKite.com

Happy retuning!

- Kite

Appendix 6.1 – 3000 Ratios

In the rungs screen, the rung ratio sliders go up to 99, creating 99-odd-limit ratios. There are about 3000 such ratios, after removing duplicates and inverses. This appendix lists all 3000, sorted by size.

99 / 98 = 17.58 ¢	53 / 52 = 32.98 ¢	92 / 89 = 57.39 ¢	85 / 81 = 83.45 ¢	81 / 76 = 110.31 ¢
98 / 97 = 17.76 ¢	52 / 51 = 33.62 ¢	61 / 59 = 57.71 ¢	21 / 20 = 84.47 ¢	97 / 91 = 110.54 ¢
97 / 96 = 17.94 ¢	51 / 50 = 34.28 ¢	91 / 88 = 58.04 ¢	83 / 79 = 85.51 ¢	16 / 15 = 111.73 ¢
96 / 95 = 18.13 ¢	50 / 49 = 34.98 ¢	30 / 29 = 58.69 ¢	62 / 59 = 85.86 ¢	95 / 89 = 112.95 ¢
95 / 94 = 18.32 ¢	99 / 97 = 35.33 ¢	89 / 86 = 59.36 ¢	41 / 39 = 86.58 ¢	79 / 74 = 113.19 ¢
94 / 93 = 18.52 ¢	49 / 48 = 35.7 ¢	59 / 57 = 59.7 ¢	61 / 58 = 87.31 ¢	63 / 59 = 113.56 ¢
93 / 92 = 18.72 ¢	97 / 95 = 36.07 ¢	88 / 85 = 60.05 ¢	81 / 77 = 87.68 ¢	47 / 44 = 114.19 ¢
92 / 91 = 18.92 ¢	48 / 47 = 36.45 ¢	29 / 28 = 60.75 ¢	20 / 19 = 88.8 ¢	78 / 73 = 114.69 ¢
91 / 90 = 19.13 ¢	95 / 93 = 36.84 ¢	86 / 83 = 61.47 ¢	99 / 94 = 89.72 ¢	31 / 29 = 115.46 ¢
90 / 89 = 19.34 ¢	47 / 46 = 37.23 ¢	57 / 55 = 61.84 ¢	79 / 75 = 89.95 ¢	77 / 72 = 116.23 ¢
89 / 88 = 19.56 ¢	93 / 91 = 37.64 ¢	85 / 82 = 62.21 ¢	59 / 56 = 90.35 ¢	46 / 43 = 116.76 ¢
88 / 87 = 19.79 ¢	46 / 45 = 38.05 ¢	28 / 27 = 62.96 ¢	98 / 93 = 90.66 ¢	61 / 57 = 117.42 ¢
87 / 86 = 20.01 ¢	91 / 89 = 38.47 ¢	83 / 80 = 63.73 ¢	39 / 37 = 91.14 ¢	76 / 71 = 117.82 ¢
86 / 85 = 20.25 ¢	45 / 44 = 38.91 ¢	55 / 53 = 64.13 ¢	97 / 92 = 91.62 ¢	91 / 85 = 118.08 ¢
85 / 84 = 20.49 ¢	89 / 87 = 39.35 ¢	82 / 79 = 64.53 ¢	58 / 55 = 91.95 ¢	15 / 14 = 119.44 ¢
84 / 83 = 20.73 ¢	44 / 43 = 39.8 ¢	27 / 26 = 65.34 ¢	77 / 73 = 92.35 ¢	89 / 83 = 120.83 ¢
83 / 82 = 20.98 ¢	87 / 85 = 40.26 ¢	80 / 77 = 66.17 ¢	96 / 91 = 92.6 ¢	74 / 69 = 121.11 ¢
82 / 81 = 21.24 ¢	43 / 42 = 40.74 ¢	53 / 51 = 66.59 ¢	19 / 18 = 93.6 ¢	59 / 55 = 121.54 ¢
81 / 80 = 21.51 ¢	85 / 83 = 41.22 ¢	79 / 76 = 67.02 ¢	94 / 89 = 94.63 ¢	44 / 41 = 122.26 ¢
80 / 79 = 21.78 ¢	42 / 41 = 41.72 ¢	26 / 25 = 67.9 ¢	75 / 71 = 94.89 ¢	73 / 68 = 122.83 ¢
79 / 78 = 22.05 ¢	83 / 81 = 42.23 ¢	77 / 74 = 68.8 ¢	56 / 53 = 95.32 ¢	29 / 27 = 123.71 ¢
78 / 77 = 22.34 ¢	41 / 40 = 42.75 ¢	51 / 49 = 69.26 ¢	93 / 88 = 95.67 ¢	72 / 67 = 124.6 ¢
77 / 76 = 22.63 ¢	81 / 79 = 43.28 ¢	76 / 73 = 69.72 ¢	37 / 35 = 96.2 ¢	43 / 40 = 125.2 ¢
76 / 75 = 22.93 ¢	40 / 39 = 43.83 ¢	25 / 24 = 70.67 ¢	92 / 87 = 96.74 ¢	57 / 53 = 125.96 ¢
75 / 74 = 23.24 ¢	79 / 77 = 44.39 ¢	99 / 95 = 71.4 ¢	55 / 52 = 97.1 ¢	71 / 66 = 126.42 ¢
74 / 73 = 23.55 ¢	39 / 38 = 44.97 ¢	74 / 71 = 71.65 ¢	73 / 69 = 97.56 ¢	85 / 79 = 126.73 ¢
73 / 72 = 23.88 ¢	77 / 75 = 45.56 ¢	49 / 47 = 72.15 ¢	91 / 86 = 97.84 ¢	99 / 92 = 126.95 ¢
72 / 71 = 24.21 ¢	38 / 37 = 46.17 ¢	73 / 70 = 72.65 ¢	18 / 17 = 98.95 ¢	14 / 13 = 128.3 ¢
71 / 70 = 24.56 ¢	75 / 73 = 46.79 ¢	97 / 93 = 72.9 ¢	89 / 84 = 100.1 ¢	97 / 90 = 129.67 ¢
70 / 69 = 24.91 ¢	37 / 36 = 47.43 ¢	24 / 23 = 73.68 ¢	71 / 67 = 100.39 ¢	83 / 77 = 129.9 ¢
69 / 68 = 25.27 ¢	73 / 71 = 48.09 ¢	95 / 91 = 74.47 ¢	53 / 50 = 100.88 ¢	69 / 64 = 130.23 ¢
68 / 67 = 25.65 ¢	36 / 35 = 48.77 ¢	71 / 68 = 74.74 ¢	88 / 83 = 101.27 ¢	55 / 51 = 130.72 ¢
67 / 66 = 26.03 ¢	71 / 69 = 49.47 ¢	47 / 45 = 75.28 ¢	35 / 33 = 101.87 ¢	96 / 89 = 131.07 ¢
66 / 65 = 26.43 ¢	35 / 34 = 50.18 ¢	70 / 67 = 75.83 ¢	87 / 82 = 102.47 ¢	41 / 38 = 131.55 ¢
65 / 64 = 26.84 ¢	69 / 67 = 50.92 ¢	93 / 89 = 76.11 ¢	52 / 49 = 102.88 ¢	68 / 63 = 132.22 ¢
64 / 63 = 27.26 ¢	34 / 33 = 51.68 ¢	23 / 22 = 76.96 ¢	69 / 65 = 103.39 ¢	95 / 88 = 132.51 ¢
63 / 62 = 27.7 ¢	67 / 65 = 52.47 ¢	91 / 87 = 77.82 ¢	86 / 81 = 103.7 ¢	27 / 25 = 133.24 ¢
62 / 61 = 28.15 ¢	33 / 32 = 53.27 ¢	68 / 65 = 78.11 ¢	17 / 16 = 104.96 ¢	94 / 87 = 133.97 ¢
61 / 60 = 28.62 ¢	98 / 95 = 53.83 ¢	45 / 43 = 78.71 ¢	84 / 79 = 106.24 ¢	67 / 62 = 134.27 ¢
60 / 59 = 29.1 ¢	65 / 63 = 54.11 ¢	67 / 64 = 79.31 ¢	67 / 63 = 106.57 ¢	40 / 37 = 134.97 ¢
59 / 58 = 29.59 ¢	97 / 94 = 54.39 ¢	89 / 85 = 79.61 ¢	50 / 47 = 107.12 ¢	93 / 86 = 135.47 ¢
58 / 57 = 30.11 ¢	32 / 31 = 54.96 ¢	22 / 21 = 80.54 ¢	83 / 78 = 107.56 ¢	53 / 49 = 135.85 ¢
57 / 56 = 30.64 ¢	95 / 92 = 55.55 ¢	87 / 83 = 81.48 ¢	33 / 31 = 108.24 ¢	66 / 61 = 136.39 ¢
56 / 55 = 31.19 ¢	63 / 61 = 55.85 ¢	65 / 62 = 81.81 ¢	82 / 77 = 108.92 ¢	79 / 73 = 136.75 ¢
55 / 54 = 31.77 ¢	94 / 91 = 56.15 ¢	43 / 41 = 82.46 ¢	49 / 46 = 109.38 ¢	92 / 85 = 137.01 ¢
54 / 53 = 32.36 ¢	31 / 30 = 56.77 ¢	64 / 61 = 83.12 ¢	65 / 61 = 109.96 ¢	13 / 12 = 138.57 ¢

90 / 83 = 140.18¢	94 / 85 = 174.24¢	96 / 85 = 210.69¢	98 / 85 = 246.38¢	86 / 73 = 283.73¢
77 / 71 = 140.45¢	73 / 66 = 174.52¢	61 / 54 = 211.02¢	15 / 13 = 247.74¢	33 / 28 = 284.45¢
64 / 59 = 140.83¢	52 / 47 = 175.02¢	87 / 77 = 211.39¢	97 / 84 = 249.11¢	79 / 67 = 285.23¢
51 / 47 = 141.4¢	83 / 75 = 175.46¢	26 / 23 = 212.25¢	82 / 71 = 249.37¢	46 / 39 = 285.79¢
89 / 82 = 141.82¢	31 / 28 = 176.21¢	95 / 84 = 213.05¢	67 / 58 = 249.73¢	59 / 50 = 286.54¢
38 / 35 = 142.37¢	72 / 65 = 177.07¢	69 / 61 = 213.34¢	52 / 45 = 250.3¢	72 / 61 = 287.03¢
63 / 58 = 143.16¢	41 / 37 = 177.72¢	43 / 38 = 214¢	89 / 77 = 250.74¢	85 / 72 = 287.36¢
88 / 81 = 143.5¢	92 / 83 = 178.23¢	60 / 53 = 214.76¢	37 / 32 = 251.34¢	98 / 83 = 287.6¢
25 / 23 = 144.35¢	51 / 46 = 178.64¢	77 / 68 = 215.19¢	96 / 83 = 251.91¢	13 / 11 = 289.21¢
87 / 80 = 145.22¢	61 / 55 = 179.25¢	94 / 83 = 215.46¢	59 / 51 = 252.26¢	97 / 82 = 290.83¢
62 / 57 = 145.57¢	71 / 64 = 179.7¢	17 / 15 = 216.69¢	81 / 70 = 252.68¢	84 / 71 = 291.08¢
99 / 91 = 145.87¢	81 / 73 = 180.03¢	93 / 82 = 217.93¢	22 / 19 = 253.8¢	71 / 60 = 291.43¢
37 / 34 = 146.39¢	91 / 82 = 180.29¢	76 / 67 = 218.21¢	95 / 82 = 254.76¢	58 / 49 = 291.93¢
86 / 79 = 146.98¢	10 / 9 = 182.4¢	59 / 52 = 218.64¢	73 / 63 = 255.05¢	45 / 38 = 292.71¢
49 / 45 = 147.43¢	99 / 89 = 184.35¢	42 / 37 = 219.44¢	51 / 44 = 255.59¢	77 / 65 = 293.3¢
61 / 56 = 148.06¢	89 / 80 = 184.57¢	67 / 59 = 220.14¢	80 / 69 = 256.08¢	32 / 27 = 294.13¢
73 / 67 = 148.48¢	79 / 71 = 184.84¢	92 / 81 = 220.45¢	29 / 25 = 256.95¢	83 / 70 = 294.91¢
85 / 78 = 148.79¢	69 / 62 = 185.19¢	25 / 22 = 221.31¢	94 / 81 = 257.69¢	51 / 43 = 295.39¢
97 / 89 = 149.02¢	59 / 53 = 185.67¢	83 / 73 = 222.26¢	65 / 56 = 258.02¢	70 / 59 = 295.97¢
12 / 11 = 150.64¢	49 / 44 = 186.33¢	58 / 51 = 222.67¢	36 / 31 = 258.87¢	89 / 75 = 296.3¢
95 / 87 = 152.29¢	88 / 79 = 186.78¢	91 / 80 = 223.04¢	79 / 68 = 259.58¢	19 / 16 = 297.51¢
83 / 76 = 152.53¢	39 / 35 = 187.34¢	33 / 29 = 223.7¢	43 / 37 = 260.17¢	82 / 69 = 298.83¢
71 / 65 = 152.86¢	68 / 61 = 188.07¢	74 / 65 = 224.5¢	93 / 80 = 260.68¢	63 / 53 = 299.23¢
59 / 54 = 153.31¢	97 / 87 = 188.36¢	41 / 36 = 225.15¢	50 / 43 = 261.11¢	44 / 37 = 299.97¢
47 / 43 = 153.99¢	29 / 26 = 189.05¢	90 / 79 = 225.69¢	57 / 49 = 261.82¢	69 / 58 = 300.65¢
82 / 75 = 154.48¢	77 / 69 = 189.91¢	49 / 43 = 226.13¢	64 / 55 = 262.37¢	94 / 79 = 300.97¢
35 / 32 = 155.14¢	48 / 43 = 190.44¢	57 / 50 = 226.84¢	71 / 61 = 262.81¢	25 / 21 = 301.85¢
93 / 85 = 155.72¢	67 / 60 = 191.04¢	65 / 57 = 227.37¢	78 / 67 = 263.18¢	81 / 68 = 302.86¢
58 / 53 = 156.07¢	86 / 77 = 191.37¢	73 / 64 = 227.79¢	85 / 73 = 263.48¢	56 / 47 = 303.32¢
81 / 74 = 156.48¢	19 / 17 = 192.56¢	81 / 71 = 228.12¢	92 / 79 = 263.74¢	87 / 73 = 303.74¢
23 / 21 = 157.49¢	85 / 76 = 193.76¢	89 / 78 = 228.4¢	99 / 85 = 263.96¢	31 / 26 = 304.51¢
80 / 73 = 158.52¢	66 / 59 = 194.1¢	97 / 85 = 228.63¢	7 / 6 = 266.87¢	99 / 83 = 305.18¢
57 / 52 = 158.94¢	47 / 42 = 194.73¢	8 / 7 = 231.17¢	97 / 83 = 269.85¢	68 / 57 = 305.49¢
91 / 83 = 159.31¢	75 / 67 = 195.28¢	95 / 83 = 233.78¢	90 / 77 = 270.08¢	37 / 31 = 306.31¢
34 / 31 = 159.92¢	28 / 25 = 196.2¢	87 / 76 = 234.02¢	83 / 71 = 270.35¢	80 / 67 = 307.01¢
79 / 72 = 160.63¢	93 / 83 = 196.94¢	79 / 69 = 234.31¢	76 / 65 = 270.67¢	43 / 36 = 307.61¢
45 / 41 = 161.16¢	65 / 58 = 197.26¢	71 / 62 = 234.66¢	69 / 59 = 271.06¢	92 / 77 = 308.13¢
56 / 51 = 161.92¢	37 / 33 = 198.07¢	63 / 55 = 235.1¢	62 / 53 = 271.53¢	49 / 41 = 308.59¢
67 / 61 = 162.42¢	83 / 74 = 198.7¢	55 / 48 = 235.68¢	55 / 47 = 272.13¢	55 / 46 = 309.36¢
78 / 71 = 162.79¢	46 / 41 = 199.21¢	47 / 41 = 236.44¢	48 / 41 = 272.89¢	61 / 51 = 309.97¢
89 / 81 = 163.06¢	55 / 49 = 199.98¢	86 / 75 = 236.94¢	89 / 76 = 273.37¢	67 / 56 = 310.48¢
11 / 10 = 165¢	64 / 57 = 200.53¢	39 / 34 = 237.53¢	41 / 35 = 273.92¢	73 / 61 = 310.9¢
98 / 89 = 166.77¢	73 / 65 = 200.95¢	70 / 61 = 238.25¢	75 / 64 = 274.58¢	79 / 66 = 311.26¢
87 / 79 = 167¢	82 / 73 = 201.27¢	31 / 27 = 239.17¢	34 / 29 = 275.38¢	85 / 71 = 311.57¢
76 / 69 = 167.28¢	91 / 81 = 201.53¢	85 / 74 = 239.93¢	95 / 81 = 276.01¢	91 / 76 = 311.84¢
65 / 59 = 167.67¢	9 / 8 = 203.91¢	54 / 47 = 240.36¢	61 / 52 = 276.36¢	97 / 81 = 312.08¢
54 / 49 = 168.21¢	98 / 87 = 206.12¢	77 / 67 = 240.84¢	88 / 75 = 276.74¢	6 / 5 = 315.64¢
97 / 88 = 168.58¢	89 / 79 = 206.34¢	23 / 20 = 241.96¢	27 / 23 = 277.59¢	95 / 79 = 319.29¢
43 / 39 = 169.04¢	80 / 71 = 206.62¢	84 / 73 = 242.99¢	74 / 63 = 278.61¢	89 / 74 = 319.54¢
75 / 68 = 169.63¢	71 / 63 = 206.96¢	61 / 53 = 243.38¢	47 / 40 = 279.19¢	83 / 69 = 319.82¢
32 / 29 = 170.42¢	62 / 55 = 207.4¢	99 / 86 = 243.71¢	67 / 57 = 279.84¢	77 / 64 = 320.14¢
85 / 77 = 171.13¢	53 / 47 = 208¢	38 / 33 = 244.24¢	87 / 74 = 280.19¢	71 / 59 = 320.52¢
53 / 48 = 171.55¢	97 / 86 = 208.38¢	91 / 79 = 244.82¢	20 / 17 = 281.36¢	65 / 54 = 320.98¢
74 / 67 = 172.04¢	44 / 39 = 208.84¢	53 / 46 = 245.23¢	93 / 79 = 282.45¢	59 / 49 = 321.52¢
95 / 86 = 172.31¢	79 / 70 = 209.4¢	68 / 59 = 245.78¢	73 / 62 = 282.75¢	53 / 44 = 322.19¢
21 / 19 = 173.27¢	35 / 31 = 210.1¢	83 / 72 = 246.14¢	53 / 45 = 283.28¢	47 / 39 = 323.02¢

88 / 73 = 323.53¢	53 / 43 = 361.99¢	29 / 23 = 401.3¢	84 / 65 = 443.94¢	86 / 65 = 484.68¢
41 / 34 = 324.11¢	90 / 73 = 362.43¢	82 / 65 = 402.22¢	53 / 41 = 444.44¢	45 / 34 = 485.27¢
76 / 63 = 324.78¢	37 / 30 = 363.08¢	53 / 42 = 402.72¢	75 / 58 = 445.01¢	94 / 71 = 485.81¢
35 / 29 = 325.56¢	95 / 77 = 363.68¢	77 / 61 = 403.26¢	97 / 75 = 445.31¢	49 / 37 = 486.31¢
99 / 82 = 326.17¢	58 / 47 = 364.07¢	24 / 19 = 404.44¢	22 / 17 = 446.36¢	53 / 40 = 487.19¢
64 / 53 = 326.5¢	79 / 64 = 364.54¢	91 / 72 = 405.44¢	79 / 61 = 447.65¢	57 / 43 = 487.95¢
93 / 77 = 326.85¢	21 / 17 = 365.83¢	67 / 53 = 405.8¢	57 / 44 = 448.15¢	61 / 46 = 488.61¢
29 / 24 = 327.62¢	89 / 72 = 366.97¢	43 / 34 = 406.56¢	92 / 71 = 448.58¢	65 / 49 = 489.19¢
81 / 67 = 328.51¢	68 / 55 = 367.32¢	62 / 49 = 407.38¢	35 / 27 = 449.27¢	69 / 52 = 489.7¢
52 / 43 = 329.01¢	47 / 38 = 367.99¢	81 / 64 = 407.82¢	83 / 64 = 450.05¢	73 / 55 = 490.16¢
75 / 62 = 329.55¢	73 / 59 = 368.62¢	19 / 15 = 409.24¢	48 / 37 = 450.61¢	77 / 58 = 490.57¢
98 / 81 = 329.83¢	99 / 80 = 368.91¢	90 / 71 = 410.53¢	61 / 47 = 451.38¢	81 / 61 = 490.94¢
23 / 19 = 330.76¢	26 / 21 = 369.75¢	71 / 56 = 410.87¢	74 / 57 = 451.88¢	85 / 64 = 491.27¢
86 / 71 = 331.82¢	83 / 67 = 370.74¢	52 / 41 = 411.47¢	87 / 67 = 452.23¢	89 / 67 = 491.57¢
63 / 52 = 332.21¢	57 / 46 = 371.19¢	85 / 67 = 411.96¢	13 / 10 = 454.21¢	93 / 70 = 491.85¢
40 / 33 = 333.04¢	88 / 71 = 371.62¢	33 / 26 = 412.75¢	95 / 73 = 456.04¢	97 / 73 = 492.11¢
97 / 80 = 333.58¢	31 / 25 = 372.41¢	80 / 63 = 413.58¢	82 / 63 = 456.33¢	4 / 3 = 498.04¢
57 / 47 = 333.96¢	98 / 79 = 373.11¢	47 / 37 = 414.16¢	69 / 53 = 456.72¢	99 / 74 = 503.88¢
74 / 61 = 334.46¢	67 / 54 = 373.44¢	61 / 48 = 414.93¢	56 / 43 = 457.31¢	95 / 71 = 504.13¢
91 / 75 = 334.77¢	36 / 29 = 374.33¢	75 / 59 = 415.41¢	99 / 76 = 457.71¢	91 / 68 = 504.4¢
17 / 14 = 336.13¢	77 / 62 = 375.11¢	89 / 70 = 415.74¢	43 / 33 = 458.24¢	87 / 65 = 504.69¢
96 / 79 = 337.42¢	41 / 33 = 375.79¢	14 / 11 = 417.51¢	73 / 56 = 458.96¢	83 / 62 = 505.01¢
79 / 65 = 337.7¢	87 / 70 = 376.39¢	93 / 73 = 419.2¢	30 / 23 = 459.99¢	79 / 59 = 505.37¢
62 / 51 = 338.13¢	46 / 37 = 376.93¢	79 / 62 = 419.5¢	77 / 59 = 460.97¢	75 / 56 = 505.76¢
45 / 37 = 338.88¢	97 / 78 = 377.41¢	65 / 51 = 419.93¢	47 / 36 = 461.6¢	71 / 53 = 506.19¢
73 / 60 = 339.52¢	51 / 41 = 377.85¢	51 / 40 = 420.6¢	64 / 49 = 462.35¢	67 / 50 = 506.68¢
28 / 23 = 340.55¢	56 / 45 = 378.6¢	88 / 69 = 421.09¢	81 / 62 = 462.78¢	63 / 47 = 507.23¢
95 / 78 = 341.34¢	61 / 49 = 379.23¢	37 / 29 = 421.77¢	98 / 75 = 463.07¢	59 / 44 = 507.85¢
67 / 55 = 341.68¢	66 / 53 = 379.77¢	97 / 76 = 422.38¢	17 / 13 = 464.43¢	55 / 41 = 508.57¢
39 / 32 = 342.48¢	71 / 57 = 380.23¢	60 / 47 = 422.76¢	89 / 68 = 465.92¢	51 / 38 = 509.4¢
89 / 73 = 343.09¢	76 / 61 = 380.63¢	83 / 65 = 423.21¢	72 / 55 = 466.28¢	98 / 73 = 509.86¢
50 / 41 = 343.57¢	81 / 65 = 380.98¢	23 / 18 = 424.36¢	55 / 42 = 466.85¢	47 / 35 = 510.37¢
61 / 50 = 344.26¢	86 / 69 = 381.29¢	78 / 61 = 425.6¢	93 / 71 = 467.29¢	90 / 67 = 510.92¢
72 / 59 = 344.74¢	91 / 73 = 381.56¢	55 / 43 = 426.11¢	38 / 29 = 467.94¢	43 / 32 = 511.52¢
83 / 68 = 345.09¢	96 / 77 = 381.81¢	87 / 68 = 426.58¢	97 / 74 = 468.55¢	82 / 61 = 512.18¢
94 / 77 = 345.36¢	5 / 4 = 386.31¢	32 / 25 = 427.37¢	59 / 45 = 468.95¢	39 / 29 = 512.91¢
11 / 9 = 347.41¢	99 / 79 = 390.69¢	73 / 57 = 428.32¢	80 / 61 = 469.43¢	74 / 55 = 513.71¢
93 / 76 = 349.48¢	94 / 75 = 390.92¢	41 / 32 = 429.06¢	21 / 16 = 470.78¢	35 / 26 = 514.61¢
82 / 67 = 349.76¢	89 / 71 = 391.18¢	91 / 71 = 429.66¢	88 / 67 = 472.01¢	66 / 49 = 515.62¢
71 / 58 = 350.12¢	84 / 67 = 391.47¢	50 / 39 = 430.14¢	67 / 51 = 472.4¢	97 / 72 = 515.99¢
60 / 49 = 350.62¢	79 / 63 = 391.8¢	59 / 46 = 430.9¢	46 / 35 = 473.13¢	31 / 23 = 516.76¢
49 / 40 = 351.34¢	74 / 59 = 392.17¢	68 / 53 = 431.45¢	71 / 54 = 473.83¢	89 / 66 = 517.61¢
87 / 71 = 351.84¢	69 / 55 = 392.6¢	77 / 60 = 431.88¢	96 / 73 = 474.17¢	58 / 43 = 518.06¢
38 / 31 = 352.48¢	64 / 51 = 393.09¢	86 / 67 = 432.21¢	25 / 19 = 475.11¢	85 / 63 = 518.53¢
65 / 53 = 353.34¢	59 / 47 = 393.67¢	95 / 74 = 432.48¢	79 / 60 = 476.27¢	27 / 20 = 519.55¢
92 / 75 = 353.69¢	54 / 43 = 394.35¢	9 / 7 = 435.08¢	54 / 41 = 476.8¢	77 / 57 = 520.68¢
27 / 22 = 354.55¢	49 / 39 = 395.17¢	94 / 73 = 437.72¢	83 / 63 = 477.31¢	50 / 37 = 521.28¢
97 / 79 = 355.36¢	93 / 74 = 395.65¢	85 / 66 = 438¢	29 / 22 = 478.26¢	73 / 54 = 521.92¢
70 / 57 = 355.67¢	44 / 35 = 396.18¢	76 / 59 = 438.34¢	91 / 69 = 479.12¢	96 / 71 = 522.26¢
43 / 35 = 356.38¢	83 / 66 = 396.77¢	67 / 52 = 438.78¢	62 / 47 = 479.53¢	23 / 17 = 523.32¢
59 / 48 = 357.22¢	39 / 31 = 397.45¢	58 / 45 = 439.35¢	95 / 72 = 479.92¢	88 / 65 = 524.48¢
75 / 61 = 357.7¢	73 / 58 = 398.21¢	49 / 38 = 440.14¢	33 / 25 = 480.65¢	65 / 48 = 524.89¢
91 / 74 = 358.01¢	34 / 27 = 399.09¢	89 / 69 = 440.65¢	70 / 53 = 481.64¢	42 / 31 = 525.75¢
16 / 13 = 359.47¢	97 / 77 = 399.75¢	40 / 31 = 441.28¢	37 / 28 = 482.52¢	61 / 45 = 526.66¢
85 / 69 = 361.04¢	63 / 50 = 400.11¢	71 / 55 = 442.06¢	78 / 59 = 483.31¢	80 / 59 = 527.14¢
69 / 56 = 361.4¢	92 / 73 = 400.48¢	31 / 24 = 443.08¢	41 / 31 = 484.03¢	99 / 73 = 527.44¢

19 / 14 = 528.69¢	71 / 51 = 572.79¢	93 / 65 = 620.15¢	69 / 47 = 664.72¢	62 / 41 = 715.97¢
91 / 67 = 530.05¢	39 / 28 = 573.66¢	83 / 58 = 620.47¢	47 / 32 = 665.51¢	59 / 39 = 716.69¢
72 / 53 = 530.41¢	85 / 61 = 574.38¢	73 / 51 = 620.88¢	72 / 49 = 666.26¢	56 / 37 = 717.48¢
53 / 39 = 531.02¢	46 / 33 = 575¢	63 / 44 = 621.42¢	97 / 66 = 666.62¢	53 / 35 = 718.36¢
87 / 64 = 531.53¢	99 / 71 = 575.53¢	53 / 37 = 622.16¢	25 / 17 = 667.67¢	50 / 33 = 719.35¢
34 / 25 = 532.33¢	53 / 38 = 575.99¢	96 / 67 = 622.65¢	78 / 53 = 668.98¢	97 / 64 = 719.9¢
83 / 61 = 533.16¢	60 / 43 = 576.75¢	43 / 30 = 623.25¢	53 / 36 = 669.59¢	47 / 31 = 720.47¢
49 / 36 = 533.74¢	67 / 48 = 577.35¢	76 / 53 = 624.01¢	81 / 55 = 670.19¢	91 / 60 = 721.08¢
64 / 47 = 534.49¢	74 / 53 = 577.84¢	33 / 23 = 625¢	28 / 19 = 671.31¢	44 / 29 = 721.74¢
79 / 58 = 534.96¢	81 / 58 = 578.24¢	89 / 62 = 625.84¢	87 / 59 = 672.36¢	85 / 56 = 722.44¢
94 / 69 = 535.28¢	88 / 63 = 578.58¢	56 / 39 = 626.34¢	59 / 40 = 672.86¢	41 / 27 = 723.2¢
15 / 11 = 536.95¢	95 / 68 = 578.87¢	79 / 55 = 626.91¢	90 / 61 = 673.34¢	79 / 52 = 724.01¢
86 / 63 = 538.78¢	7 / 5 = 582.51¢	23 / 16 = 628.27¢	31 / 21 = 674.25¢	38 / 25 = 724.89¢
71 / 52 = 539.17¢	94 / 67 = 586.2¢	82 / 57 = 629.59¢	96 / 65 = 675.11¢	73 / 48 = 725.83¢
56 / 41 = 539.76¢	87 / 62 = 586.5¢	59 / 41 = 630.11¢	65 / 44 = 675.52¢	35 / 23 = 726.87¢
97 / 71 = 540.2¢	80 / 57 = 586.85¢	95 / 66 = 630.55¢	99 / 67 = 675.92¢	67 / 44 = 727.99¢
41 / 30 = 540.79¢	73 / 52 = 587.26¢	36 / 25 = 631.28¢	34 / 23 = 676.68¢	99 / 65 = 728.39¢
67 / 49 = 541.66¢	66 / 47 = 587.77¢	85 / 59 = 632.1¢	71 / 48 = 677.74¢	32 / 21 = 729.22¢
93 / 68 = 542.04¢	59 / 42 = 588.39¢	49 / 34 = 632.7¢	37 / 25 = 678.72¢	93 / 61 = 730.11¢
26 / 19 = 543.01¢	52 / 37 = 589.18¢	62 / 43 = 633.52¢	77 / 52 = 679.62¢	61 / 40 = 730.57¢
89 / 65 = 544.04¢	97 / 69 = 589.67¢	75 / 52 = 634.05¢	40 / 27 = 680.45¢	90 / 59 = 731.05¢
63 / 46 = 544.46¢	45 / 32 = 590.22¢	88 / 61 = 634.43¢	83 / 56 = 681.22¢	29 / 19 = 732.06¢
37 / 27 = 545.48¢	83 / 59 = 590.88¢	13 / 9 = 636.62¢	43 / 29 = 681.94¢	84 / 55 = 733.15¢
85 / 62 = 546.23¢	38 / 27 = 591.65¢	94 / 65 = 638.67¢	89 / 60 = 682.61¢	55 / 36 = 733.72¢
48 / 35 = 546.82¢	69 / 49 = 592.58¢	81 / 56 = 638.99¢	46 / 31 = 683.24¢	81 / 53 = 734.32¢
59 / 43 = 547.65¢	31 / 22 = 593.72¢	68 / 47 = 639.45¢	95 / 64 = 683.83¢	26 / 17 = 735.57¢
70 / 51 = 548.23¢	86 / 61 = 594.63¢	55 / 38 = 640.12¢	49 / 33 = 684.38¢	75 / 49 = 736.93¢
81 / 59 = 548.65¢	55 / 39 = 595.15¢	97 / 67 = 640.59¢	52 / 35 = 685.39¢	49 / 32 = 737.65¢
92 / 67 = 548.97¢	79 / 56 = 595.71¢	42 / 29 = 641.2¢	55 / 37 = 686.29¢	72 / 47 = 738.4¢
11 / 8 = 551.32¢	24 / 17 = 597¢	71 / 49 = 642.04¢	58 / 39 = 687.09¢	95 / 62 = 738.79¢
95 / 69 = 553.6¢	89 / 63 = 598.14¢	29 / 20 = 643.26¢	61 / 41 = 687.82¢	23 / 15 = 740.01¢
84 / 61 = 553.9¢	65 / 46 = 598.57¢	74 / 51 = 644.43¢	64 / 43 = 688.48¢	89 / 58 = 741.3¢
73 / 53 = 554.28¢	41 / 29 = 599.49¢	45 / 31 = 645.19¢	67 / 45 = 689.08¢	66 / 43 = 741.76¢
62 / 45 = 554.81¢	99 / 70 = 600.09¢	61 / 42 = 646.1¢	70 / 47 = 689.63¢	43 / 28 = 742.69¢
51 / 37 = 555.57¢	58 / 41 = 600.51¢	77 / 53 = 646.64¢	73 / 49 = 690.14¢	63 / 41 = 743.67¢
91 / 66 = 556.08¢	75 / 53 = 601.08¢	93 / 64 = 646.99¢	76 / 51 = 690.6¢	83 / 54 = 744.18¢
40 / 29 = 556.74¢	92 / 65 = 601.43¢	16 / 11 = 648.68¢	79 / 53 = 691.03¢	20 / 13 = 745.79¢
69 / 50 = 557.6¢	17 / 12 = 603¢	99 / 68 = 650.27¢	82 / 55 = 691.43¢	97 / 63 = 747.16¢
98 / 71 = 557.96¢	95 / 67 = 604.52¢	83 / 57 = 650.58¢	85 / 57 = 691.8¢	77 / 50 = 747.52¢
29 / 21 = 558.8¢	78 / 55 = 604.85¢	67 / 46 = 651.03¢	88 / 59 = 692.15¢	57 / 37 = 748.12¢
76 / 55 = 559.88¢	61 / 43 = 605.37¢	51 / 35 = 651.77¢	91 / 61 = 692.47¢	94 / 61 = 748.62¢
47 / 34 = 560.55¢	44 / 31 = 606.28¢	86 / 59 = 652.35¢	94 / 63 = 692.77¢	37 / 24 = 749.39¢
65 / 47 = 561.33¢	71 / 50 = 607.07¢	35 / 24 = 653.18¢	97 / 65 = 693.05¢	91 / 59 = 750.18¢
83 / 60 = 561.78¢	98 / 69 = 607.42¢	89 / 61 = 654¢	**3 / 2 = 701.96¢**	54 / 35 = 750.73¢
18 / 13 = 563.38¢	27 / 19 = 608.35¢	54 / 37 = 654.52¢	98 / 65 = 710.81¢	71 / 46 = 751.42¢
97 / 70 = 564.76¢	91 / 64 = 609.35¢	73 / 50 = 655.16¢	95 / 63 = 711.09¢	88 / 57 = 751.85¢
79 / 57 = 565.07¢	64 / 45 = 609.78¢	92 / 63 = 655.54¢	92 / 61 = 711.39¢	17 / 11 = 753.64¢
61 / 44 = 565.57¢	37 / 26 = 610.82¢	19 / 13 = 656.99¢	89 / 59 = 711.71¢	99 / 64 = 755.23¢
43 / 31 = 566.48¢	84 / 59 = 611.61¢	98 / 67 = 658.34¢	86 / 57 = 712.05¢	82 / 53 = 755.56¢
68 / 49 = 567.3¢	47 / 33 = 612.23¢	79 / 54 = 658.67¢	83 / 55 = 712.42¢	65 / 42 = 756.06¢
93 / 67 = 567.68¢	57 / 40 = 613.15¢	60 / 41 = 659.21¢	80 / 53 = 712.81¢	48 / 31 = 756.92¢
25 / 18 = 568.72¢	67 / 47 = 613.8¢	41 / 28 = 660.24¢	77 / 51 = 713.23¢	79 / 51 = 757.63¢
82 / 59 = 569.89¢	77 / 54 = 614.28¢	63 / 43 = 661.22¢	74 / 49 = 713.69¢	31 / 20 = 758.72¢
57 / 41 = 570.41¢	87 / 61 = 614.65¢	85 / 58 = 661.69¢	71 / 47 = 714.19¢	76 / 49 = 759.86¢
89 / 64 = 570.88¢	97 / 68 = 614.94¢	22 / 15 = 663.05¢	68 / 45 = 714.73¢	45 / 29 = 760.65¢
32 / 23 = 571.73¢	10 / 7 = 617.49¢	91 / 62 = 664.32¢	65 / 43 = 715.32¢	59 / 38 = 761.66¢

73 / 47 = 762.28 ¢	8 / 5 = 813.69 ¢	61 / 37 = 865.54 ¢	80 / 47 = 920.81 ¢	95 / 54 = 977.96 ¢
87 / 56 = 762.71 ¢	93 / 58 = 817.41 ¢	94 / 57 = 866.04 ¢	63 / 37 = 921.39 ¢	44 / 25 = 978.69 ¢
14 / 9 = 764.92 ¢	85 / 53 = 817.76 ¢	33 / 20 = 866.96 ¢	46 / 27 = 922.41 ¢	81 / 46 = 979.55 ¢
95 / 61 = 766.94 ¢	77 / 48 = 818.19 ¢	71 / 43 = 868.18 ¢	75 / 44 = 923.26 ¢	37 / 21 = 980.56 ¢
81 / 52 = 767.29 ¢	69 / 43 = 818.71 ¢	38 / 23 = 869.24 ¢	29 / 17 = 924.62 ¢	67 / 38 = 981.79 ¢
67 / 43 = 767.79 ¢	61 / 38 = 819.37 ¢	81 / 49 = 870.17 ¢	99 / 58 = 925.65 ¢	97 / 55 = 982.26 ¢
53 / 34 = 768.55 ¢	53 / 33 = 820.23 ¢	43 / 26 = 870.99 ¢	70 / 41 = 926.08 ¢	30 / 17 = 983.31 ¢
92 / 59 = 769.1 ¢	98 / 61 = 820.77 ¢	91 / 55 = 871.72 ¢	41 / 24 = 927.11 ¢	83 / 47 = 984.54 ¢
39 / 25 = 769.86 ¢	45 / 28 = 821.4 ¢	48 / 29 = 872.38 ¢	94 / 55 = 927.87 ¢	53 / 30 = 985.24 ¢
64 / 41 = 770.94 ¢	82 / 51 = 822.15 ¢	53 / 32 = 873.5 ¢	53 / 31 = 928.47 ¢	76 / 43 = 986 ¢
89 / 57 = 771.41 ¢	37 / 23 = 823.07 ¢	58 / 35 = 874.44 ¢	65 / 38 = 929.33 ¢	99 / 56 = 986.4 ¢
25 / 16 = 772.63 ¢	66 / 41 = 824.21 ¢	63 / 38 = 875.22 ¢	77 / 45 = 929.92 ¢	23 / 13 = 987.75 ¢
86 / 55 = 773.89 ¢	95 / 59 = 824.66 ¢	68 / 41 = 875.89 ¢	89 / 52 = 930.35 ¢	85 / 48 = 989.31 ¢
61 / 39 = 774.4 ¢	29 / 18 = 825.67 ¢	73 / 44 = 876.47 ¢	12 / 7 = 933.13 ¢	62 / 35 = 989.9 ¢
97 / 62 = 774.86 ¢	79 / 49 = 826.89 ¢	78 / 47 = 876.98 ¢	91 / 53 = 935.85 ¢	39 / 22 = 991.16 ¢
36 / 23 = 775.64 ¢	50 / 31 = 827.59 ¢	83 / 50 = 877.42 ¢	79 / 46 = 936.26 ¢	94 / 53 = 992 ¢
83 / 53 = 776.54 ¢	71 / 44 = 828.38 ¢	88 / 53 = 877.81 ¢	67 / 39 = 936.82 ¢	55 / 31 = 992.6 ¢
47 / 30 = 777.24 ¢	92 / 57 = 828.81 ¢	93 / 56 = 878.16 ¢	55 / 32 = 937.63 ¢	71 / 40 = 993.38 ¢
58 / 37 = 778.23 ¢	21 / 13 = 830.25 ¢	98 / 59 = 878.48 ¢	98 / 57 = 938.18 ¢	87 / 49 = 993.88 ¢
69 / 44 = 778.91 ¢	97 / 60 = 831.63 ¢	5 / 3 = 884.36 ¢	43 / 25 = 938.89 ¢	16 / 9 = 996.09 ¢
80 / 51 = 779.4 ¢	76 / 47 = 832.01 ¢	97 / 58 = 890.32 ¢	74 / 43 = 939.83 ¢	89 / 50 = 998.25 ¢
91 / 58 = 779.78 ¢	55 / 34 = 832.68 ¢	92 / 55 = 890.64 ¢	31 / 18 = 941.13 ¢	73 / 41 = 998.73 ¢
11 / 7 = 782.49 ¢	89 / 55 = 833.25 ¢	87 / 52 = 891 ¢	81 / 47 = 942.31 ¢	57 / 32 = 999.47 ¢
96 / 61 = 785.07 ¢	34 / 21 = 834.17 ¢	82 / 49 = 891.41 ¢	50 / 29 = 943.05 ¢	98 / 55 = 1000.02 ¢
85 / 54 = 785.4 ¢	81 / 50 = 835.19 ¢	77 / 46 = 891.87 ¢	69 / 40 = 943.92 ¢	41 / 23 = 1000.79 ¢
74 / 47 = 785.84 ¢	47 / 29 = 835.93 ¢	72 / 43 = 892.39 ¢	88 / 51 = 944.41 ¢	66 / 37 = 1001.93 ¢
63 / 40 = 786.42 ¢	60 / 37 = 836.92 ¢	67 / 40 = 892.99 ¢	19 / 11 = 946.2 ¢	91 / 51 = 1002.44 ¢
52 / 33 = 787.25 ¢	73 / 45 = 837.57 ¢	62 / 37 = 893.69 ¢	83 / 48 = 948.09 ¢	25 / 14 = 1003.8 ¢
93 / 59 = 787.82 ¢	86 / 53 = 838.01 ¢	57 / 34 = 894.51 ¢	64 / 37 = 948.66 ¢	84 / 47 = 1005.27 ¢
41 / 26 = 788.53 ¢	99 / 61 = 838.34 ¢	52 / 31 = 895.49 ¢	45 / 26 = 949.7 ¢	59 / 33 = 1005.9 ¢
71 / 45 = 789.47 ¢	13 / 8 = 840.53 ¢	99 / 59 = 896.06 ¢	71 / 41 = 950.63 ¢	93 / 52 = 1006.46 ¢
30 / 19 = 790.76 ¢	96 / 59 = 842.78 ¢	47 / 28 = 896.68 ¢	97 / 56 = 951.07 ¢	34 / 19 = 1007.44 ¢
79 / 50 = 791.91 ¢	83 / 51 = 843.14 ¢	89 / 53 = 897.38 ¢	26 / 15 = 952.26 ¢	77 / 43 = 1008.63 ¢
49 / 31 = 792.62 ¢	70 / 43 = 843.62 ¢	42 / 25 = 898.15 ¢	85 / 49 = 953.62 ¢	43 / 24 = 1009.56 ¢
68 / 43 = 793.44 ¢	57 / 35 = 844.33 ¢	79 / 47 = 899.03 ¢	59 / 34 = 954.22 ¢	95 / 53 = 1010.32 ¢
87 / 55 = 793.9 ¢	44 / 27 = 845.45 ¢	37 / 22 = 900.03 ¢	92 / 53 = 954.77 ¢	52 / 29 = 1010.95 ¢
19 / 12 = 795.56 ¢	75 / 46 = 846.31 ¢	69 / 41 = 901.17 ¢	33 / 19 = 955.76 ¢	61 / 34 = 1011.93 ¢
84 / 53 = 797.28 ¢	31 / 19 = 847.52 ¢	32 / 19 = 902.49 ¢	73 / 42 = 957.01 ¢	70 / 39 = 1012.66 ¢
65 / 41 = 797.78 ¢	80 / 49 = 848.66 ¢	91 / 54 = 903.49 ¢	40 / 23 = 958.04 ¢	79 / 44 = 1013.22 ¢
46 / 29 = 798.7 ¢	49 / 30 = 849.38 ¢	59 / 35 = 904.03 ¢	87 / 50 = 958.9 ¢	88 / 49 = 1013.67 ¢
73 / 46 = 799.52 ¢	67 / 41 = 850.24 ¢	86 / 51 = 904.61 ¢	47 / 27 = 959.64 ¢	97 / 54 = 1014.03 ¢
27 / 17 = 800.91 ¢	85 / 52 = 850.74 ¢	27 / 16 = 905.87 ¢	54 / 31 = 960.83 ¢	9 / 5 = 1017.6 ¢
89 / 56 = 802.05 ¢	18 / 11 = 852.59 ¢	76 / 45 = 907.29 ¢	61 / 35 = 961.75 ¢	92 / 51 = 1021.36 ¢
62 / 39 = 802.55 ¢	95 / 58 = 854.25 ¢	49 / 29 = 908.07 ¢	68 / 39 = 962.47 ¢	83 / 46 = 1021.77 ¢
97 / 61 = 803.01 ¢	77 / 47 = 854.64 ¢	71 / 42 = 908.92 ¢	75 / 43 = 963.06 ¢	74 / 41 = 1022.28 ¢
35 / 22 = 803.82 ¢	59 / 36 = 855.26 ¢	93 / 55 = 909.36 ¢	82 / 47 = 963.56 ¢	65 / 36 = 1022.93 ¢
78 / 49 = 804.83 ¢	41 / 25 = 856.43 ¢	22 / 13 = 910.79 ¢	89 / 51 = 963.97 ¢	56 / 31 = 1023.79 ¢
43 / 27 = 805.65 ¢	64 / 39 = 857.52 ¢	83 / 49 = 912.4 ¢	96 / 55 = 964.32 ¢	47 / 26 = 1024.98 ¢
94 / 59 = 806.33 ¢	87 / 53 = 858.03 ¢	61 / 36 = 912.97 ¢	7 / 4 = 968.83 ¢	85 / 47 = 1025.76 ¢
51 / 32 = 806.91 ¢	23 / 14 = 859.45 ¢	39 / 23 = 914.21 ¢	93 / 53 = 973.49 ¢	38 / 21 = 1026.73 ¢
59 / 37 = 807.83 ¢	97 / 59 = 860.72 ¢	95 / 56 = 915 ¢	86 / 49 = 973.87 ¢	67 / 37 = 1027.96 ¢
67 / 42 = 808.53 ¢	74 / 45 = 861.12 ¢	56 / 33 = 915.55 ¢	79 / 45 = 974.31 ¢	96 / 53 = 1028.45 ¢
75 / 47 = 809.08 ¢	51 / 31 = 861.87 ¢	73 / 43 = 916.27 ¢	72 / 41 = 974.85 ¢	29 / 16 = 1029.58 ¢
83 / 52 = 809.52 ¢	79 / 48 = 862.58 ¢	90 / 53 = 916.72 ¢	65 / 37 = 975.5 ¢	78 / 43 = 1030.96 ¢
91 / 57 = 809.89 ¢	28 / 17 = 863.87 ¢	17 / 10 = 918.64 ¢	58 / 33 = 976.3 ¢	49 / 27 = 1031.79 ¢
99 / 62 = 810.19 ¢	89 / 54 = 865.02 ¢	97 / 57 = 920.43 ¢	51 / 29 = 977.33 ¢	69 / 38 = 1032.72 ¢

89 / 49	=	1033.23 ¢	47 / 25	=	1092.88 ¢	72 / 37	=	1152.57 ¢	67 / 33	=	1226.03 ¢	40 / 19	=	1288.8 ¢
20 / 11	=	1035 ¢	79 / 42	=	1093.76 ¢	37 / 19	=	1153.83 ¢	65 / 32	=	1226.84 ¢	99 / 47	=	1289.72 ¢
91 / 50	=	1036.73 ¢	32 / 17	=	1095.04 ¢	76 / 39	=	1155.03 ¢	63 / 31	=	1227.7 ¢	59 / 28	=	1290.35 ¢
71 / 39	=	1037.21 ¢	81 / 43	=	1096.3 ¢	39 / 20	=	1156.17 ¢	61 / 30	=	1228.62 ¢	78 / 37	=	1291.14 ¢
51 / 28	=	1038.08 ¢	49 / 26	=	1097.12 ¢	80 / 41	=	1157.25 ¢	59 / 29	=	1229.59 ¢	97 / 46	=	1291.62 ¢
82 / 45	=	1038.84 ¢	66 / 35	=	1098.13 ¢	41 / 21	=	1158.28 ¢	57 / 28	=	1230.64 ¢	19 / 9	=	1293.6 ¢
31 / 17	=	1040.08 ¢	83 / 44	=	1098.73 ¢	84 / 43	=	1159.26 ¢	55 / 27	=	1231.77 ¢	93 / 44	=	1295.67 ¢
73 / 40	=	1041.48 ¢	17 / 9	=	1101.05 ¢	43 / 22	=	1160.2 ¢	53 / 26	=	1232.98 ¢	74 / 35	=	1296.2 ¢
42 / 23	=	1042.51 ¢	87 / 46	=	1103.26 ¢	88 / 45	=	1161.09 ¢	51 / 25	=	1234.28 ¢	55 / 26	=	1297.1 ¢
95 / 52	=	1043.3 ¢	70 / 37	=	1103.8 ¢	45 / 23	=	1161.95 ¢	49 / 24	=	1235.7 ¢	91 / 43	=	1297.84 ¢
53 / 29	=	1043.93 ¢	53 / 28	=	1104.68 ¢	92 / 47	=	1162.77 ¢	96 / 47	=	1236.45 ¢	36 / 17	=	1298.95 ¢
64 / 35	=	1044.86 ¢	89 / 47	=	1105.37 ¢	47 / 24	=	1163.55 ¢	47 / 23	=	1237.23 ¢	89 / 42	=	1300.1 ¢
75 / 41	=	1045.52 ¢	36 / 19	=	1106.4 ¢	96 / 49	=	1164.3 ¢	92 / 45	=	1238.05 ¢	53 / 25	=	1300.88 ¢
86 / 47	=	1046.01 ¢	91 / 48	=	1107.4 ¢	49 / 25	=	1165.02 ¢	45 / 22	=	1238.91 ¢	70 / 33	=	1301.87 ¢
97 / 53	=	1046.39 ¢	55 / 29	=	1108.05 ¢	51 / 26	=	1166.38 ¢	88 / 43	=	1239.8 ¢	87 / 41	=	1302.47 ¢
11 / 6	=	1049.36 ¢	74 / 39	=	1108.86 ¢	53 / 27	=	1167.64 ¢	43 / 21	=	1240.74 ¢	17 / 8	=	1304.96 ¢
90 / 49	=	1052.57 ¢	93 / 49	=	1109.34 ¢	55 / 28	=	1168.81 ¢	84 / 41	=	1241.72 ¢	83 / 39	=	1307.56 ¢
79 / 43	=	1053.02 ¢	19 / 10	=	1111.2 ¢	57 / 29	=	1169.89 ¢	41 / 20	=	1242.75 ¢	66 / 31	=	1308.24 ¢
68 / 37	=	1053.61 ¢	97 / 51	=	1112.99 ¢	59 / 30	=	1170.9 ¢	80 / 39	=	1243.83 ¢	49 / 23	=	1309.38 ¢
57 / 31	=	1054.43 ¢	78 / 41	=	1113.42 ¢	61 / 31	=	1171.85 ¢	39 / 19	=	1244.97 ¢	81 / 38	=	1310.31 ¢
46 / 25	=	1055.65 ¢	59 / 31	=	1114.14 ¢	63 / 32	=	1172.74 ¢	76 / 37	=	1246.17 ¢	32 / 15	=	1311.73 ¢
81 / 44	=	1056.5 ¢	99 / 52	=	1114.7 ¢	65 / 33	=	1173.57 ¢	37 / 18	=	1247.43 ¢	79 / 37	=	1313.19 ¢
35 / 19	=	1057.63 ¢	40 / 21	=	1115.53 ¢	67 / 34	=	1174.35 ¢	72 / 35	=	1248.77 ¢	47 / 22	=	1314.19 ¢
94 / 51	=	1058.6 ¢	61 / 32	=	1116.88 ¢	69 / 35	=	1175.09 ¢	35 / 17	=	1250.18 ¢	62 / 29	=	1315.46 ¢
59 / 32	=	1059.17 ¢	82 / 43	=	1117.54 ¢	71 / 36	=	1175.79 ¢	68 / 33	=	1251.68 ¢	77 / 36	=	1316.23 ¢
83 / 45	=	1059.82 ¢	21 / 11	=	1119.46 ¢	73 / 37	=	1176.45 ¢	33 / 16	=	1253.27 ¢	92 / 43	=	1316.76 ¢
24 / 13	=	1061.43 ¢	86 / 45	=	1121.29 ¢	75 / 38	=	1177.07 ¢	97 / 47	=	1254.39 ¢	15 / 7	=	1319.44 ¢
85 / 46	=	1062.99 ¢	65 / 34	=	1121.89 ¢	77 / 39	=	1177.66 ¢	64 / 31	=	1254.96 ¢	88 / 41	=	1322.26 ¢
61 / 33	=	1063.61 ¢	44 / 23	=	1123.04 ¢	79 / 40	=	1178.22 ¢	95 / 46	=	1255.55 ¢	73 / 34	=	1322.83 ¢
98 / 53	=	1064.15 ¢	67 / 35	=	1124.17 ¢	81 / 41	=	1178.76 ¢	31 / 15	=	1256.77 ¢	58 / 27	=	1323.71 ¢
37 / 20	=	1065.03 ¢	90 / 47	=	1124.72 ¢	83 / 42	=	1179.27 ¢	91 / 44	=	1258.04 ¢	43 / 20	=	1325.2 ¢
87 / 47	=	1066.03 ¢	23 / 12	=	1126.32 ¢	85 / 43	=	1179.75 ¢	60 / 29	=	1258.69 ¢	71 / 33	=	1326.42 ¢
50 / 27	=	1066.76 ¢	94 / 49	=	1127.85 ¢	87 / 44	=	1180.21 ¢	89 / 43	=	1259.36 ¢	99 / 46	=	1326.95 ¢
63 / 34	=	1067.78 ¢	71 / 37	=	1128.35 ¢	89 / 45	=	1180.66 ¢	29 / 14	=	1260.75 ¢	28 / 13	=	1328.3 ¢
76 / 41	=	1068.45 ¢	48 / 25	=	1129.33 ¢	91 / 46	=	1181.08 ¢	85 / 41	=	1262.21 ¢	97 / 45	=	1329.67 ¢
89 / 48	=	1068.93 ¢	73 / 38	=	1130.28 ¢	93 / 47	=	1181.48 ¢	56 / 27	=	1262.96 ¢	69 / 32	=	1330.23 ¢
13 / 7	=	1071.7 ¢	98 / 51	=	1130.74 ¢	95 / 48	=	1181.87 ¢	83 / 40	=	1263.73 ¢	41 / 19	=	1331.55 ¢
93 / 50	=	1074.36 ¢	25 / 13	=	1132.1 ¢	97 / 49	=	1182.24 ¢	27 / 13	=	1265.34 ¢	95 / 44	=	1332.51 ¢
80 / 43	=	1074.8 ¢	77 / 40	=	1133.83 ¢	99 / 50	=	1182.6 ¢	79 / 38	=	1267.02 ¢	54 / 25	=	1333.24 ¢
67 / 36	=	1075.4 ¢	52 / 27	=	1134.66 ¢	**2 / 1**	**=**	**1200 ¢**	52 / 25	=	1267.9 ¢	67 / 31	=	1334.27 ¢
54 / 29	=	1076.29 ¢	79 / 41	=	1135.47 ¢	99 / 49	=	1217.58 ¢	77 / 37	=	1268.8 ¢	80 / 37	=	1334.97 ¢
95 / 51	=	1076.92 ¢	27 / 14	=	1137.04 ¢	97 / 48	=	1217.94 ¢	25 / 12	=	1270.67 ¢	93 / 43	=	1335.47 ¢
41 / 22	=	1077.74 ¢	83 / 43	=	1138.53 ¢	95 / 47	=	1218.32 ¢	98 / 47	=	1272.15 ¢	13 / 6	=	1338.57 ¢
69 / 37	=	1078.89 ¢	56 / 29	=	1139.25 ¢	93 / 46	=	1218.72 ¢	73 / 35	=	1272.65 ¢	89 / 41	=	1341.82 ¢
97 / 52	=	1079.37 ¢	85 / 44	=	1139.95 ¢	91 / 45	=	1219.13 ¢	48 / 23	=	1273.68 ¢	76 / 35	=	1342.37 ¢
28 / 15	=	1080.56 ¢	29 / 15	=	1141.31 ¢	89 / 44	=	1219.56 ¢	71 / 34	=	1274.74 ¢	63 / 29	=	1343.16 ¢
99 / 53	=	1081.72 ¢	89 / 46	=	1142.61 ¢	87 / 43	=	1220.01 ¢	94 / 45	=	1275.28 ¢	50 / 23	=	1344.35 ¢
71 / 38	=	1082.18 ¢	60 / 31	=	1143.23 ¢	85 / 42	=	1220.49 ¢	23 / 11	=	1276.96 ¢	87 / 40	=	1345.22 ¢
43 / 23	=	1083.24 ¢	91 / 47	=	1143.85 ¢	83 / 41	=	1220.98 ¢	90 / 43	=	1278.71 ¢	37 / 17	=	1346.39 ¢
58 / 31	=	1084.54 ¢	31 / 16	=	1145.04 ¢	81 / 40	=	1221.51 ¢	67 / 32	=	1279.31 ¢	98 / 45	=	1347.43 ¢
73 / 39	=	1085.31 ¢	95 / 49	=	1146.17 ¢	79 / 39	=	1222.05 ¢	44 / 21	=	1280.54 ¢	61 / 28	=	1348.06 ¢
88 / 47	=	1085.81 ¢	64 / 33	=	1146.73 ¢	77 / 38	=	1222.63 ¢	65 / 31	=	1281.81 ¢	85 / 39	=	1348.79 ¢
15 / 8	=	1088.27 ¢	97 / 50	=	1147.27 ¢	75 / 37	=	1223.24 ¢	86 / 41	=	1282.46 ¢	24 / 11	=	1350.64 ¢
92 / 49	=	1090.62 ¢	33 / 17	=	1148.32 ¢	73 / 36	=	1223.88 ¢	21 / 10	=	1284.47 ¢	83 / 38	=	1352.53 ¢
77 / 41	=	1091.08 ¢	68 / 35	=	1149.82 ¢	71 / 35	=	1224.56 ¢	82 / 39	=	1286.58 ¢	59 / 27	=	1353.31 ¢
62 / 33	=	1091.76 ¢	35 / 18	=	1151.23 ¢	69 / 34	=	1225.27 ¢	61 / 29	=	1287.31 ¢	94 / 43	=	1353.99 ¢

35 / 16	=	1355.14 ¢	57 / 25	=	1426.84 ¢	81 / 34	=	1502.86 ¢	87 / 35	=	1576.39 ¢	89 / 34	=	1665.92 ¢
81 / 37	=	1356.48 ¢	73 / 32	=	1427.79 ¢	31 / 13	=	1504.51 ¢	92 / 37	=	1576.93 ¢	55 / 21	=	1666.85 ¢
46 / 21	=	1357.49 ¢	89 / 39	=	1428.4 ¢	74 / 31	=	1506.31 ¢	97 / 39	=	1577.41 ¢	76 / 29	=	1667.94 ¢
57 / 26	=	1358.94 ¢	16 / 7	=	1431.17 ¢	43 / 18	=	1507.61 ¢	5 / 2	=	1586.31 ¢	97 / 37	=	1668.55 ¢
68 / 31	=	1359.92 ¢	87 / 38	=	1434.02 ¢	98 / 41	=	1508.59 ¢	98 / 39	=	1595.17 ¢	21 / 8	=	1670.78 ¢
79 / 36	=	1360.63 ¢	71 / 31	=	1434.66 ¢	55 / 23	=	1509.36 ¢	93 / 37	=	1595.65 ¢	92 / 35	=	1673.13 ¢
90 / 41	=	1361.16 ¢	55 / 24	=	1435.68 ¢	67 / 28	=	1510.48 ¢	88 / 35	=	1596.18 ¢	71 / 27	=	1673.83 ¢
11 / 5	=	1365 ¢	94 / 41	=	1436.44 ¢	79 / 33	=	1511.26 ¢	83 / 33	=	1596.77 ¢	50 / 19	=	1675.11 ¢
97 / 44	=	1368.58 ¢	39 / 17	=	1437.53 ¢	91 / 38	=	1511.84 ¢	78 / 31	=	1597.45 ¢	79 / 30	=	1676.27 ¢
86 / 39	=	1369.04 ¢	62 / 27	=	1439.17 ¢	12 / 5	=	1515.64 ¢	73 / 29	=	1598.21 ¢	29 / 11	=	1678.26 ¢
75 / 34	=	1369.63 ¢	85 / 37	=	1439.93 ¢	89 / 37	=	1519.54 ¢	68 / 27	=	1599.09 ¢	95 / 36	=	1679.92 ¢
64 / 29	=	1370.42 ¢	23 / 10	=	1441.96 ¢	77 / 32	=	1520.14 ¢	63 / 25	=	1600.11 ¢	66 / 25	=	1680.65 ¢
53 / 24	=	1371.55 ¢	99 / 43	=	1443.71 ¢	65 / 27	=	1520.98 ¢	58 / 23	=	1601.3 ¢	37 / 14	=	1682.52 ¢
95 / 43	=	1372.31 ¢	76 / 33	=	1444.24 ¢	53 / 22	=	1522.19 ¢	53 / 21	=	1602.72 ¢	82 / 31	=	1684.03 ¢
42 / 19	=	1373.27 ¢	53 / 23	=	1445.23 ¢	94 / 39	=	1523.02 ¢	48 / 19	=	1604.44 ¢	45 / 17	=	1685.27 ¢
73 / 33	=	1374.52 ¢	83 / 36	=	1446.14 ¢	41 / 17	=	1524.11 ¢	91 / 36	=	1605.44 ¢	98 / 37	=	1686.31 ¢
31 / 14	=	1376.21 ¢	30 / 13	=	1447.74 ¢	70 / 29	=	1525.56 ¢	43 / 17	=	1606.56 ¢	53 / 20	=	1687.19 ¢
82 / 37	=	1377.72 ¢	97 / 42	=	1449.11 ¢	99 / 41	=	1526.17 ¢	81 / 32	=	1607.82 ¢	61 / 23	=	1688.61 ¢
51 / 23	=	1378.64 ¢	67 / 29	=	1449.73 ¢	29 / 12	=	1527.62 ¢	38 / 15	=	1609.24 ¢	69 / 26	=	1689.7 ¢
71 / 32	=	1379.7 ¢	37 / 16	=	1451.34 ¢	75 / 31	=	1529.55 ¢	71 / 28	=	1610.87 ¢	77 / 29	=	1690.57 ¢
91 / 41	=	1380.29 ¢	81 / 35	=	1452.68 ¢	46 / 19	=	1530.76 ¢	33 / 13	=	1612.75 ¢	85 / 32	=	1691.27 ¢
20 / 9	=	1382.4 ¢	44 / 19	=	1453.8 ¢	63 / 26	=	1532.21 ¢	94 / 37	=	1614.16 ¢	93 / 35	=	1691.85 ¢
89 / 40	=	1384.57 ¢	95 / 41	=	1454.76 ¢	80 / 33	=	1533.04 ¢	61 / 24	=	1614.93 ¢	8 / 3	=	1698.04 ¢
69 / 31	=	1385.19 ¢	51 / 22	=	1455.59 ¢	97 / 40	=	1533.58 ¢	89 / 35	=	1615.74 ¢	99 / 37	=	1703.88 ¢
49 / 22	=	1386.33 ¢	58 / 25	=	1456.95 ¢	17 / 7	=	1536.13 ¢	28 / 11	=	1617.51 ¢	91 / 34	=	1704.4 ¢
78 / 35	=	1387.34 ¢	65 / 28	=	1458.02 ¢	90 / 37	=	1538.88 ¢	79 / 31	=	1619.5 ¢	83 / 31	=	1705.01 ¢
29 / 13	=	1389.05 ¢	72 / 31	=	1458.87 ¢	73 / 30	=	1539.52 ¢	51 / 20	=	1620.6 ¢	75 / 28	=	1705.76 ¢
96 / 43	=	1390.44 ¢	79 / 34	=	1459.58 ¢	56 / 23	=	1540.55 ¢	74 / 29	=	1621.77 ¢	67 / 25	=	1706.68 ¢
67 / 30	=	1391.04 ¢	86 / 37	=	1460.17 ¢	95 / 39	=	1541.34 ¢	97 / 38	=	1622.38 ¢	59 / 22	=	1707.85 ¢
38 / 17	=	1392.56 ¢	93 / 40	=	1460.68 ¢	39 / 16	=	1542.48 ¢	23 / 9	=	1624.36 ¢	51 / 19	=	1709.4 ¢
85 / 38	=	1393.76 ¢	7 / 3	=	1466.87 ¢	61 / 25	=	1544.26 ¢	87 / 34	=	1626.58 ¢	94 / 35	=	1710.37 ¢
47 / 21	=	1394.73 ¢	96 / 41	=	1472.89 ¢	83 / 34	=	1545.09 ¢	64 / 25	=	1627.37 ¢	43 / 16	=	1711.52 ¢
56 / 25	=	1396.2 ¢	89 / 38	=	1473.37 ¢	22 / 9	=	1547.41 ¢	41 / 16	=	1629.06 ¢	78 / 29	=	1712.91 ¢
65 / 29	=	1397.26 ¢	82 / 35	=	1473.92 ¢	93 / 38	=	1549.48 ¢	59 / 23	=	1630.9 ¢	35 / 13	=	1714.61 ¢
74 / 33	=	1398.07 ¢	75 / 32	=	1474.58 ¢	71 / 29	=	1550.12 ¢	77 / 30	=	1631.88 ¢	97 / 36	=	1715.99 ¢
83 / 37	=	1398.7 ¢	68 / 29	=	1475.38 ¢	49 / 20	=	1551.34 ¢	95 / 37	=	1632.48 ¢	62 / 23	=	1716.76 ¢
92 / 41	=	1399.21 ¢	61 / 26	=	1476.36 ¢	76 / 31	=	1552.48 ¢	18 / 7	=	1635.08 ¢	89 / 33	=	1717.61 ¢
9 / 4	=	1403.91 ¢	54 / 23	=	1477.59 ¢	27 / 11	=	1554.55 ¢	85 / 33	=	1638 ¢	27 / 10	=	1719.55 ¢
97 / 43	=	1408.38 ¢	47 / 20	=	1479.19 ¢	86 / 35	=	1556.38 ¢	67 / 26	=	1638.78 ¢	73 / 27	=	1721.92 ¢
88 / 39	=	1408.84 ¢	87 / 37	=	1480.19 ¢	59 / 24	=	1557.22 ¢	49 / 19	=	1640.14 ¢	46 / 17	=	1723.32 ¢
79 / 35	=	1409.4 ¢	40 / 17	=	1481.36 ¢	91 / 37	=	1558.01 ¢	80 / 31	=	1641.28 ¢	65 / 24	=	1724.89 ¢
70 / 31	=	1410.1 ¢	73 / 31	=	1482.75 ¢	32 / 13	=	1559.47 ¢	31 / 12	=	1643.08 ¢	84 / 31	=	1725.75 ¢
61 / 27	=	1411.02 ¢	33 / 14	=	1484.45 ¢	69 / 28	=	1561.4 ¢	75 / 29	=	1645.01 ¢	19 / 7	=	1728.69 ¢
52 / 23	=	1412.25 ¢	92 / 39	=	1485.79 ¢	37 / 15	=	1563.08 ¢	44 / 17	=	1646.36 ¢	87 / 32	=	1731.53 ¢
95 / 42	=	1413.05 ¢	59 / 25	=	1486.54 ¢	79 / 32	=	1564.54 ¢	57 / 22	=	1648.15 ¢	68 / 25	=	1732.33 ¢
43 / 19	=	1414 ¢	85 / 36	=	1487.36 ¢	42 / 17	=	1565.83 ¢	70 / 27	=	1649.27 ¢	49 / 18	=	1733.74 ¢
77 / 34	=	1415.19 ¢	26 / 11	=	1489.21 ¢	89 / 36	=	1566.97 ¢	83 / 32	=	1650.05 ¢	79 / 29	=	1734.96 ¢
34 / 15	=	1416.69 ¢	97 / 41	=	1490.83 ¢	47 / 19	=	1567.99 ¢	96 / 37	=	1650.61 ¢	30 / 11	=	1736.95 ¢
93 / 41	=	1417.93 ¢	71 / 30	=	1491.43 ¢	99 / 40	=	1568.91 ¢	13 / 5	=	1654.21 ¢	71 / 26	=	1739.17 ¢
59 / 26	=	1418.64 ¢	45 / 19	=	1492.71 ¢	52 / 21	=	1569.75 ¢	99 / 38	=	1657.71 ¢	41 / 15	=	1740.79 ¢
84 / 37	=	1419.44 ¢	64 / 27	=	1494.13 ¢	57 / 23	=	1571.19 ¢	86 / 33	=	1658.24 ¢	93 / 34	=	1742.04 ¢
25 / 11	=	1421.31 ¢	83 / 35	=	1494.91 ¢	62 / 25	=	1572.41 ¢	73 / 28	=	1658.96 ¢	52 / 19	=	1743.01 ¢
91 / 40	=	1423.04 ¢	19 / 8	=	1497.51 ¢	67 / 27	=	1573.44 ¢	60 / 23	=	1659.99 ¢	63 / 23	=	1744.46 ¢
66 / 29	=	1423.7 ¢	88 / 37	=	1499.97 ¢	72 / 29	=	1574.33 ¢	47 / 18	=	1661.6 ¢	74 / 27	=	1745.48 ¢
41 / 18	=	1425.15 ¢	69 / 29	=	1500.65 ¢	77 / 31	=	1575.11 ¢	81 / 31	=	1662.78 ¢	85 / 31	=	1746.23 ¢
98 / 43	=	1426.13 ¢	50 / 21	=	1501.85 ¢	82 / 33	=	1575.79 ¢	34 / 13	=	1664.43 ¢	96 / 35	=	1746.82 ¢

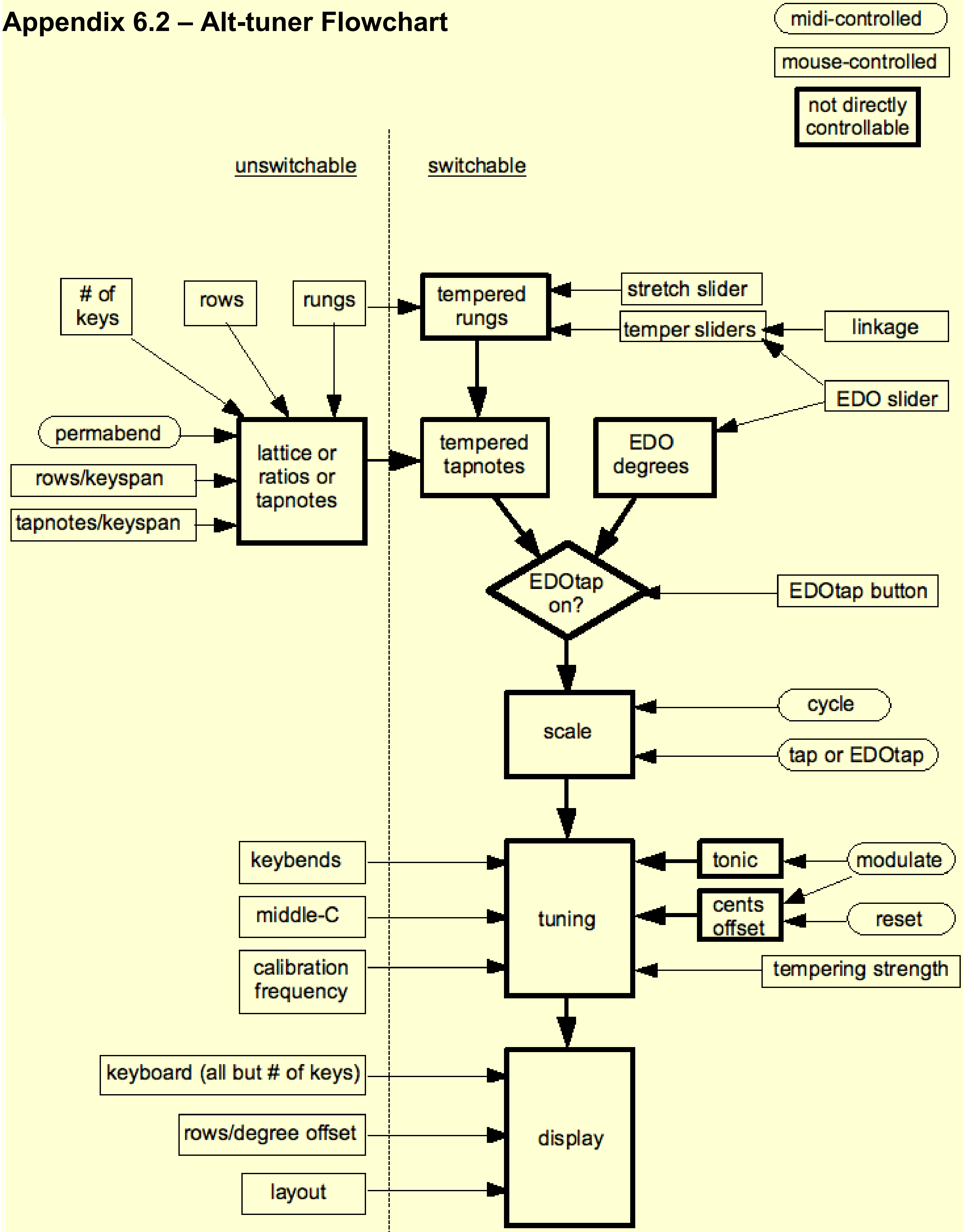
11 / 4	=	1751.32 ¢	84 / 29	=	1841.2 ¢	43 / 14	=	1942.69 ¢	13 / 4	=	2040.53 ¢	83 / 24	=	2148.09 ¢
91 / 33	=	1756.08 ¢	29 / 10	=	1843.26 ¢	83 / 27	=	1944.18 ¢	88 / 27	=	2045.45 ¢	45 / 13	=	2149.7 ¢
80 / 29	=	1756.74 ¢	90 / 31	=	1845.19 ¢	40 / 13	=	1945.79 ¢	75 / 23	=	2046.31 ¢	97 / 28	=	2151.07 ¢
69 / 25	=	1757.6 ¢	61 / 21	=	1846.1 ¢	77 / 25	=	1947.52 ¢	62 / 19	=	2047.52 ¢	52 / 15	=	2152.26 ¢
58 / 21	=	1758.8 ¢	93 / 32	=	1846.99 ¢	37 / 12	=	1949.39 ¢	49 / 15	=	2049.38 ¢	59 / 17	=	2154.22 ¢
47 / 17	=	1760.55 ¢	32 / 11	=	1848.68 ¢	71 / 23	=	1951.42 ¢	85 / 26	=	2050.74 ¢	66 / 19	=	2155.76 ¢
83 / 30	=	1761.78 ¢	99 / 34	=	1850.27 ¢	34 / 11	=	1953.64 ¢	36 / 11	=	2052.59 ¢	73 / 21	=	2157.01 ¢
36 / 13	=	1763.38 ¢	67 / 23	=	1851.03 ¢	99 / 32	=	1955.23 ¢	95 / 29	=	2054.25 ¢	80 / 23	=	2158.04 ¢
97 / 35	=	1764.76 ¢	35 / 12	=	1853.18 ¢	65 / 21	=	1956.06 ¢	59 / 18	=	2055.26 ¢	87 / 25	=	2158.9 ¢
61 / 22	=	1765.57 ¢	73 / 25	=	1855.16 ¢	96 / 31	=	1956.92 ¢	82 / 25	=	2056.43 ¢	94 / 27	=	2159.64 ¢
86 / 31	=	1766.48 ¢	38 / 13	=	1856.99 ¢	31 / 10	=	1958.72 ¢	23 / 7	=	2059.45 ¢	7 / 2	=	2168.83 ¢
25 / 9	=	1768.72 ¢	79 / 27	=	1858.67 ¢	90 / 29	=	1960.65 ¢	79 / 24	=	2062.58 ¢	95 / 27	=	2177.96 ¢
89 / 32	=	1770.88 ¢	41 / 14	=	1860.24 ¢	59 / 19	=	1961.66 ¢	56 / 17	=	2063.87 ¢	88 / 25	=	2178.69 ¢
64 / 23	=	1771.73 ¢	85 / 29	=	1861.69 ¢	87 / 28	=	1962.71 ¢	89 / 27	=	2065.02 ¢	81 / 23	=	2179.55 ¢
39 / 14	=	1773.66 ¢	44 / 15	=	1863.05 ¢	28 / 9	=	1964.92 ¢	33 / 10	=	2066.96 ¢	74 / 21	=	2180.56 ¢
92 / 33	=	1775 ¢	91 / 31	=	1864.32 ¢	81 / 26	=	1967.29 ¢	76 / 23	=	2069.24 ¢	67 / 19	=	2181.79 ¢
53 / 19	=	1775.99 ¢	47 / 16	=	1865.51 ¢	53 / 17	=	1968.55 ¢	43 / 13	=	2070.99 ¢	60 / 17	=	2183.31 ¢
67 / 24	=	1777.35 ¢	97 / 33	=	1866.62 ¢	78 / 25	=	1969.86 ¢	96 / 29	=	2072.38 ¢	53 / 15	=	2185.24 ¢
81 / 29	=	1778.24 ¢	50 / 17	=	1867.67 ¢	25 / 8	=	1972.63 ¢	53 / 16	=	2073.5 ¢	99 / 28	=	2186.4 ¢
95 / 34	=	1778.87 ¢	53 / 18	=	1869.59 ¢	97 / 31	=	1974.86 ¢	63 / 19	=	2075.22 ¢	46 / 13	=	2187.75 ¢
14 / 5	=	1782.51 ¢	56 / 19	=	1871.31 ¢	72 / 23	=	1975.64 ¢	73 / 22	=	2076.47 ¢	85 / 24	=	2189.31 ¢
87 / 31	=	1786.5 ¢	59 / 20	=	1872.86 ¢	47 / 15	=	1977.24 ¢	83 / 25	=	2077.42 ¢	39 / 11	=	2191.16 ¢
73 / 26	=	1787.26 ¢	62 / 21	=	1874.25 ¢	69 / 22	=	1978.91 ¢	93 / 28	=	2078.16 ¢	71 / 20	=	2193.38 ¢
59 / 21	=	1788.39 ¢	65 / 22	=	1875.52 ¢	91 / 29	=	1979.78 ¢	10 / 3	=	2084.36 ¢	32 / 9	=	2196.09 ¢
45 / 16	=	1790.22 ¢	68 / 23	=	1876.68 ¢	22 / 7	=	1982.49 ¢	97 / 29	=	2090.32 ¢	89 / 25	=	2198.25 ¢
76 / 27	=	1791.65 ¢	71 / 24	=	1877.74 ¢	85 / 27	=	1985.4 ¢	87 / 26	=	2091 ¢	57 / 16	=	2199.47 ¢
31 / 11	=	1793.72 ¢	74 / 25	=	1878.72 ¢	63 / 20	=	1986.42 ¢	77 / 23	=	2091.87 ¢	82 / 23	=	2200.79 ¢
79 / 28	=	1795.71 ¢	77 / 26	=	1879.62 ¢	41 / 13	=	1988.53 ¢	67 / 20	=	2092.99 ¢	25 / 7	=	2203.8 ¢
48 / 17	=	1797 ¢	80 / 27	=	1880.45 ¢	60 / 19	=	1990.76 ¢	57 / 17	=	2094.51 ¢	93 / 26	=	2206.46 ¢
65 / 23	=	1798.57 ¢	83 / 28	=	1881.22 ¢	79 / 25	=	1991.91 ¢	47 / 14	=	2096.68 ¢	68 / 19	=	2207.44 ¢
82 / 29	=	1799.49 ¢	86 / 29	=	1881.94 ¢	98 / 31	=	1992.62 ¢	84 / 25	=	2098.15 ¢	43 / 12	=	2209.56 ¢
99 / 35	=	1800.09 ¢	89 / 30	=	1882.61 ¢	19 / 6	=	1995.56 ¢	37 / 11	=	2100.03 ¢	61 / 17	=	2211.93 ¢
17 / 6	=	1803 ¢	92 / 31	=	1883.24 ¢	92 / 29	=	1998.7 ¢	64 / 19	=	2102.49 ¢	79 / 22	=	2213.22 ¢
88 / 31	=	1806.28 ¢	95 / 32	=	1883.83 ¢	73 / 23	=	1999.52 ¢	91 / 27	=	2103.49 ¢	97 / 27	=	2214.03 ¢
71 / 25	=	1807.07 ¢	98 / 33	=	1884.38 ¢	54 / 17	=	2000.91 ¢	27 / 8	=	2105.87 ¢	18 / 5	=	2217.6 ¢
54 / 19	=	1808.35 ¢	**3 / 1**	**=**	**1901.96 ¢**	89 / 28	=	2002.05 ¢	98 / 29	=	2108.07 ¢	83 / 23	=	2221.77 ¢
91 / 32	=	1809.35 ¢	97 / 32	=	1919.9 ¢	35 / 11	=	2003.82 ¢	71 / 21	=	2108.92 ¢	65 / 18	=	2222.93 ¢
37 / 13	=	1810.82 ¢	94 / 31	=	1920.47 ¢	86 / 27	=	2005.65 ¢	44 / 13	=	2110.79 ¢	47 / 13	=	2224.98 ¢
94 / 33	=	1812.23 ¢	91 / 30	=	1921.08 ¢	51 / 16	=	2006.91 ¢	61 / 18	=	2112.97 ¢	76 / 21	=	2226.73 ¢
57 / 20	=	1813.15 ¢	88 / 29	=	1921.74 ¢	67 / 21	=	2008.53 ¢	78 / 23	=	2114.21 ¢	29 / 8	=	2229.58 ¢
77 / 27	=	1814.28 ¢	85 / 28	=	1922.44 ¢	83 / 26	=	2009.52 ¢	95 / 28	=	2115 ¢	98 / 27	=	2231.79 ¢
97 / 34	=	1814.94 ¢	82 / 27	=	1923.2 ¢	99 / 31	=	2010.19 ¢	17 / 5	=	2118.64 ¢	69 / 19	=	2232.72 ¢
20 / 7	=	1817.49 ¢	79 / 26	=	1924.01 ¢	16 / 5	=	2013.69 ¢	92 / 27	=	2122.41 ¢	40 / 11	=	2235 ¢
83 / 29	=	1820.47 ¢	76 / 25	=	1924.89 ¢	93 / 29	=	2017.41 ¢	75 / 22	=	2123.26 ¢	91 / 25	=	2236.73 ¢
63 / 22	=	1821.42 ¢	73 / 24	=	1925.83 ¢	77 / 24	=	2018.19 ¢	58 / 17	=	2124.62 ¢	51 / 14	=	2238.08 ¢
43 / 15	=	1823.25 ¢	70 / 23	=	1926.87 ¢	61 / 19	=	2019.37 ¢	99 / 29	=	2125.65 ¢	62 / 17	=	2240.08 ¢
66 / 23	=	1825 ¢	67 / 22	=	1927.99 ¢	45 / 14	=	2021.4 ¢	41 / 12	=	2127.11 ¢	73 / 20	=	2241.48 ¢
89 / 31	=	1825.84 ¢	64 / 21	=	1929.22 ¢	74 / 23	=	2023.07 ¢	65 / 19	=	2129.33 ¢	84 / 23	=	2242.51 ¢
23 / 8	=	1828.27 ¢	61 / 20	=	1930.57 ¢	29 / 9	=	2025.67 ¢	89 / 26	=	2130.35 ¢	95 / 26	=	2243.3 ¢
95 / 33	=	1830.55 ¢	58 / 19	=	1932.06 ¢	71 / 22	=	2028.38 ¢	24 / 7	=	2133.13 ¢	11 / 3	=	2249.36 ¢
72 / 25	=	1831.28 ¢	55 / 18	=	1933.72 ¢	42 / 13	=	2030.25 ¢	79 / 23	=	2136.26 ¢	92 / 25	=	2255.65 ¢
49 / 17	=	1832.7 ¢	52 / 17	=	1935.57 ¢	97 / 30	=	2031.63 ¢	55 / 16	=	2137.63 ¢	81 / 22	=	2256.5 ¢
75 / 26	=	1834.05 ¢	49 / 16	=	1937.65 ¢	55 / 17	=	2032.68 ¢	86 / 25	=	2138.89 ¢	70 / 19	=	2257.63 ¢
26 / 9	=	1836.62 ¢	95 / 31	=	1938.79 ¢	68 / 21	=	2034.17 ¢	31 / 9	=	2141.13 ¢	59 / 16	=	2259.17 ¢
81 / 28	=	1838.99 ¢	46 / 15	=	1940.01 ¢	81 / 25	=	2035.19 ¢	69 / 20	=	2143.92 ¢	48 / 13	=	2261.43 ¢
55 / 19	=	1840.12 ¢	89 / 29	=	1941.3 ¢	94 / 29	=	2035.93 ¢	38 / 11	=	2146.2 ¢	85 / 23	=	2262.99 ¢

37 / 10 =	2265.03¢	79 / 20 =	2378.22¢	30 / 7 =	2519.44¢	93 / 20 =	2660.68¢	87 / 17 =	2826.58¢
63 / 17 =	2267.78¢	83 / 21 =	2379.27¢	73 / 17 =	2522.83¢	14 / 3 =	2666.87¢	41 / 8 =	2829.06¢
89 / 24 =	2268.93¢	87 / 22 =	2380.21¢	43 / 10 =	2525.2¢	89 / 19 =	2673.37¢	77 / 15 =	2831.88¢
26 / 7 =	2271.7¢	91 / 23 =	2381.08¢	99 / 23 =	2526.95¢	75 / 16 =	2674.58¢	36 / 7 =	2835.08¢
93 / 25 =	2274.36¢	95 / 24 =	2381.87¢	56 / 13 =	2528.3¢	61 / 13 =	2676.36¢	67 / 13 =	2838.78¢
67 / 18 =	2275.4¢	99 / 25 =	2382.6¢	69 / 16 =	2530.23¢	47 / 10 =	2679.19¢	98 / 19 =	2840.14¢
41 / 11 =	2277.74¢	**4 / 1 =**	**2400¢**	82 / 19 =	2531.55¢	80 / 17 =	2681.36¢	31 / 6 =	2843.08¢
97 / 26 =	2279.37¢	97 / 24 =	2417.94¢	95 / 22 =	2532.51¢	33 / 7 =	2684.45¢	88 / 17 =	2846.36¢
56 / 15 =	2280.56¢	93 / 23 =	2418.72¢	13 / 3 =	2538.57¢	85 / 18 =	2687.36¢	57 / 11 =	2848.15¢
71 / 19 =	2282.18¢	89 / 22 =	2419.56¢	87 / 20 =	2545.22¢	52 / 11 =	2689.21¢	83 / 16 =	2850.05¢
86 / 23 =	2283.24¢	85 / 21 =	2420.49¢	74 / 17 =	2546.39¢	71 / 15 =	2691.43¢	26 / 5 =	2854.21¢
15 / 4 =	2288.27¢	81 / 20 =	2421.51¢	61 / 14 =	2548.06¢	90 / 19 =	2692.71¢	99 / 19 =	2857.71¢
94 / 25 =	2292.88¢	77 / 19 =	2422.63¢	48 / 11 =	2550.64¢	19 / 4 =	2697.51¢	73 / 14 =	2858.96¢
79 / 21 =	2293.76¢	73 / 18 =	2423.88¢	83 / 19 =	2552.53¢	81 / 17 =	2702.86¢	47 / 9 =	2861.6¢
64 / 17 =	2295.04¢	69 / 17 =	2425.27¢	35 / 8 =	2555.14¢	62 / 13 =	2704.51¢	68 / 13 =	2864.43¢
49 / 13 =	2297.12¢	65 / 16 =	2426.84¢	92 / 21 =	2557.49¢	43 / 9 =	2707.61¢	89 / 17 =	2865.92¢
83 / 22 =	2298.73¢	61 / 15 =	2428.62¢	57 / 13 =	2558.94¢	67 / 14 =	2710.48¢	21 / 4 =	2870.78¢
34 / 9 =	2301.05¢	57 / 14 =	2430.64¢	79 / 18 =	2560.63¢	91 / 19 =	2711.84¢	79 / 15 =	2876.27¢
87 / 23 =	2303.26¢	53 / 13 =	2432.98¢	22 / 5 =	2565¢	24 / 5 =	2715.64¢	58 / 11 =	2878.26¢
53 / 14 =	2304.68¢	49 / 12 =	2435.7¢	97 / 22 =	2568.58¢	77 / 16 =	2720.14¢	95 / 18 =	2879.92¢
72 / 19 =	2306.4¢	94 / 23 =	2437.23¢	75 / 17 =	2569.63¢	53 / 11 =	2722.19¢	37 / 7 =	2882.52¢
91 / 24 =	2307.4¢	45 / 11 =	2438.91¢	53 / 12 =	2571.55¢	82 / 17 =	2724.11¢	90 / 17 =	2885.27¢
19 / 5 =	2311.2¢	86 / 21 =	2440.74¢	84 / 19 =	2573.27¢	29 / 6 =	2727.62¢	53 / 10 =	2887.19¢
99 / 26 =	2314.7¢	41 / 10 =	2442.75¢	31 / 7 =	2576.21¢	92 / 19 =	2730.76¢	69 / 13 =	2889.7¢
80 / 21 =	2315.53¢	78 / 19 =	2444.97¢	71 / 16 =	2579.7¢	63 / 13 =	2732.21¢	85 / 16 =	2891.27¢
61 / 16 =	2316.88¢	37 / 9 =	2447.43¢	40 / 9 =	2582.4¢	97 / 20 =	2733.58¢	16 / 3 =	2898.04¢
42 / 11 =	2319.46¢	70 / 17 =	2450.18¢	89 / 20 =	2584.57¢	34 / 7 =	2736.13¢	91 / 17 =	2904.4¢
65 / 17 =	2321.89¢	33 / 8 =	2453.27¢	49 / 11 =	2586.33¢	73 / 15 =	2739.52¢	75 / 14 =	2905.76¢
88 / 23 =	2323.04¢	95 / 23 =	2455.55¢	58 / 13 =	2589.05¢	39 / 8 =	2742.48¢	59 / 11 =	2907.85¢
23 / 6 =	2326.32¢	62 / 15 =	2456.77¢	67 / 15 =	2591.04¢	83 / 17 =	2745.09¢	43 / 8 =	2911.52¢
96 / 25 =	2329.33¢	91 / 22 =	2458.04¢	76 / 17 =	2592.56¢	44 / 9 =	2747.41¢	70 / 13 =	2914.61¢
73 / 19 =	2330.28¢	29 / 7 =	2460.75¢	85 / 19 =	2593.76¢	93 / 19 =	2749.48¢	97 / 18 =	2915.99¢
50 / 13 =	2332.1¢	83 / 20 =	2463.73¢	94 / 21 =	2594.73¢	49 / 10 =	2751.34¢	27 / 5 =	2919.55¢
77 / 20 =	2333.83¢	54 / 13 =	2465.34¢	9 / 2 =	2603.91¢	54 / 11 =	2754.55¢	92 / 17 =	2923.32¢
27 / 7 =	2337.04¢	79 / 19 =	2467.02¢	95 / 21 =	2613.05¢	59 / 12 =	2757.22¢	65 / 12 =	2924.89¢
85 / 22 =	2339.95¢	25 / 6 =	2470.67¢	86 / 19 =	2614¢	64 / 13 =	2759.47¢	38 / 7 =	2928.69¢
58 / 15 =	2341.31¢	96 / 23 =	2473.68¢	77 / 17 =	2615.19¢	69 / 14 =	2761.4¢	87 / 16 =	2931.53¢
89 / 23 =	2342.61¢	71 / 17 =	2474.74¢	68 / 15 =	2616.69¢	74 / 15 =	2763.08¢	49 / 9 =	2933.74¢
31 / 8 =	2345.04¢	46 / 11 =	2476.96¢	59 / 13 =	2618.64¢	79 / 16 =	2764.54¢	60 / 11 =	2936.95¢
97 / 25 =	2347.27¢	67 / 16 =	2479.31¢	50 / 11 =	2621.31¢	84 / 17 =	2765.83¢	71 / 13 =	2939.17¢
66 / 17 =	2348.32¢	88 / 21 =	2480.54¢	91 / 20 =	2623.04¢	89 / 18 =	2766.97¢	82 / 15 =	2940.79¢
35 / 9 =	2351.23¢	21 / 5 =	2484.47¢	41 / 9 =	2625.15¢	94 / 19 =	2767.99¢	93 / 17 =	2942.04¢
74 / 19 =	2353.83¢	80 / 19 =	2488.8¢	73 / 16 =	2627.79¢	99 / 20 =	2768.91¢	11 / 2 =	2951.32¢
39 / 10 =	2356.17¢	59 / 14 =	2490.35¢	32 / 7 =	2631.17¢	5 / 1 =	2786.31¢	94 / 17 =	2960.55¢
82 / 21 =	2358.28¢	97 / 23 =	2491.62¢	87 / 19 =	2634.02¢	96 / 19 =	2804.44¢	83 / 15 =	2961.78¢
43 / 11 =	2360.2¢	38 / 9 =	2493.6¢	55 / 12 =	2635.68¢	91 / 18 =	2805.44¢	72 / 13 =	2963.38¢
90 / 23 =	2361.95¢	93 / 22 =	2495.67¢	78 / 17 =	2637.53¢	86 / 17 =	2806.56¢	61 / 11 =	2965.57¢
47 / 12 =	2363.55¢	55 / 13 =	2497.1¢	23 / 5 =	2641.96¢	81 / 16 =	2807.82¢	50 / 9 =	2968.72¢
98 / 25 =	2365.02¢	72 / 17 =	2498.95¢	83 / 18 =	2646.14¢	76 / 15 =	2809.24¢	89 / 16 =	2970.88¢
51 / 13 =	2366.38¢	89 / 21 =	2500.1¢	60 / 13 =	2647.74¢	71 / 14 =	2810.87¢	39 / 7 =	2973.66¢
55 / 14 =	2368.81¢	17 / 4 =	2504.96¢	97 / 21 =	2649.11¢	66 / 13 =	2812.75¢	67 / 12 =	2977.35¢
59 / 15 =	2370.9¢	98 / 23 =	2509.38¢	37 / 8 =	2651.34¢	61 / 12 =	2814.93¢	95 / 17 =	2978.87¢
63 / 16 =	2372.74¢	81 / 19 =	2510.31¢	88 / 19 =	2653.8¢	56 / 11 =	2817.51¢	28 / 5 =	2982.51¢
67 / 17 =	2374.35¢	64 / 15 =	2511.73¢	51 / 11 =	2655.59¢	51 / 10 =	2820.6¢	73 / 13 =	2987.26¢
71 / 18 =	2375.79¢	47 / 11 =	2514.19¢	65 / 14 =	2658.02¢	97 / 19 =	2822.38¢	45 / 8 =	2990.22¢
75 / 19 =	2377.07¢	77 / 18 =	2516.23¢	79 / 17 =	2659.58¢	46 / 9 =	2824.36¢	62 / 11 =	2993.72¢

79 / 14	=	2995.71 ¢	63 / 10	=	3186.42 ¢	93 / 13	=	3406.46 ¢	74 / 9	=	3647.43 ¢	68 / 7	=	3936.13 ¢
96 / 17	=	2997 ¢	82 / 13	=	3188.53 ¢	43 / 6	=	3409.56 ¢	33 / 4	=	3653.27 ¢	39 / 4	=	3942.48 ¢
17 / 3	=	3003 ¢	19 / 3	=	3195.56 ¢	79 / 11	=	3413.22 ¢	91 / 11	=	3658.04 ¢	88 / 9	=	3947.41 ¢
91 / 16	=	3009.35 ¢	89 / 14	=	3202.05 ¢	36 / 5	=	3417.6 ¢	58 / 7	=	3660.75 ¢	49 / 5	=	3951.34 ¢
74 / 13	=	3010.82 ¢	70 / 11	=	3203.82 ¢	65 / 9	=	3422.93 ¢	83 / 10	=	3663.73 ¢	59 / 6	=	3957.22 ¢
57 / 10	=	3013.15 ¢	51 / 8	=	3206.91 ¢	94 / 13	=	3424.98 ¢	25 / 3	=	3670.67 ¢	69 / 7	=	3961.4 ¢
97 / 17	=	3014.94 ¢	83 / 13	=	3209.52 ¢	29 / 4	=	3429.58 ¢	92 / 11	=	3676.96 ¢	79 / 8	=	3964.54 ¢
40 / 7	=	3017.49 ¢	32 / 5	=	3213.69 ¢	80 / 11	=	3435 ¢	67 / 8	=	3679.31 ¢	89 / 9	=	3966.97 ¢
63 / 11	=	3021.42 ¢	77 / 12	=	3218.19 ¢	51 / 7	=	3438.08 ¢	42 / 5	=	3684.47 ¢	99 / 10	=	3968.91 ¢
86 / 15	=	3023.25 ¢	45 / 7	=	3221.4 ¢	73 / 10	=	3441.48 ¢	59 / 7	=	3690.35 ¢	10 / 1	=	3986.31 ¢
23 / 4	=	3028.27 ¢	58 / 9	=	3225.67 ¢	95 / 13	=	3443.3 ¢	76 / 9	=	3693.6 ¢	91 / 9	=	4005.44 ¢
98 / 17	=	3032.7 ¢	71 / 11	=	3228.38 ¢	22 / 3	=	3449.36 ¢	93 / 11	=	3695.67 ¢	81 / 8	=	4007.82 ¢
75 / 13	=	3034.05 ¢	84 / 13	=	3230.25 ¢	81 / 11	=	3456.5 ¢	17 / 2	=	3704.96 ¢	71 / 7	=	4010.87 ¢
52 / 9	=	3036.62 ¢	97 / 15	=	3231.63 ¢	59 / 8	=	3459.17 ¢	94 / 11	=	3714.19 ¢	61 / 6	=	4014.93 ¢
81 / 14	=	3038.99 ¢	13 / 2	=	3240.53 ¢	96 / 13	=	3461.43 ¢	77 / 9	=	3716.23 ¢	51 / 5	=	4020.6 ¢
29 / 5	=	3043.26 ¢	98 / 15	=	3249.38 ¢	37 / 5	=	3465.03 ¢	60 / 7	=	3719.44 ¢	92 / 9	=	4024.36 ¢
93 / 16	=	3046.99 ¢	85 / 13	=	3250.74 ¢	89 / 12	=	3468.93 ¢	43 / 5	=	3725.2 ¢	41 / 4	=	4029.06 ¢
64 / 11	=	3048.68 ¢	72 / 11	=	3252.59 ¢	52 / 7	=	3471.7 ¢	69 / 8	=	3730.23 ¢	72 / 7	=	4035.08 ¢
99 / 17	=	3050.27 ¢	59 / 9	=	3255.26 ¢	67 / 9	=	3475.4 ¢	95 / 11	=	3732.51 ¢	31 / 3	=	4043.08 ¢
35 / 6	=	3053.18 ¢	46 / 7	=	3259.45 ¢	82 / 11	=	3477.74 ¢	26 / 3	=	3738.57 ¢	83 / 8	=	4050.05 ¢
76 / 13	=	3056.99 ¢	79 / 12	=	3262.58 ¢	97 / 13	=	3479.37 ¢	87 / 10	=	3745.22 ¢	52 / 5	=	4054.21 ¢
41 / 7	=	3060.24 ¢	33 / 5	=	3266.96 ¢	15 / 2	=	3488.27 ¢	61 / 7	=	3748.06 ¢	73 / 7	=	4058.96 ¢
88 / 15	=	3063.05 ¢	86 / 13	=	3270.99 ¢	98 / 13	=	3497.12 ¢	96 / 11	=	3750.64 ¢	94 / 9	=	4061.6 ¢
47 / 8	=	3065.51 ¢	53 / 8	=	3273.5 ¢	83 / 11	=	3498.73 ¢	35 / 4	=	3755.14 ¢	21 / 2	=	4070.78 ¢
53 / 9	=	3069.59 ¢	73 / 11	=	3276.47 ¢	68 / 9	=	3501.05 ¢	79 / 9	=	3760.63 ¢	95 / 9	=	4079.92 ¢
59 / 10	=	3072.86 ¢	93 / 14	=	3278.16 ¢	53 / 7	=	3504.68 ¢	44 / 5	=	3765 ¢	74 / 7	=	4082.52 ¢
65 / 11	=	3075.52 ¢	20 / 3	=	3284.36 ¢	91 / 12	=	3507.4 ¢	97 / 11	=	3768.58 ¢	53 / 5	=	4087.19 ¢
71 / 12	=	3077.74 ¢	87 / 13	=	3291 ¢	38 / 5	=	3511.2 ¢	53 / 6	=	3771.55 ¢	85 / 8	=	4091.27 ¢
77 / 13	=	3079.62 ¢	67 / 10	=	3292.99 ¢	99 / 13	=	3514.7 ¢	62 / 7	=	3776.21 ¢	32 / 3	=	4098.04 ¢
83 / 14	=	3081.22 ¢	47 / 7	=	3296.68 ¢	61 / 8	=	3516.88 ¢	71 / 8	=	3779.7 ¢	75 / 7	=	4105.76 ¢
89 / 15	=	3082.61 ¢	74 / 11	=	3300.03 ¢	84 / 11	=	3519.46 ¢	80 / 9	=	3782.4 ¢	43 / 4	=	4111.52 ¢
95 / 16	=	3083.83 ¢	27 / 4	=	3305.87 ¢	23 / 3	=	3526.32 ¢	89 / 10	=	3784.57 ¢	97 / 9	=	4115.99 ¢
6 / 1	=	3101.96 ¢	88 / 13	=	3310.79 ¢	77 / 10	=	3533.83 ¢	98 / 11	=	3786.33 ¢	54 / 5	=	4119.55 ¢
97 / 16	=	3119.9 ¢	61 / 9	=	3312.97 ¢	54 / 7	=	3537.04 ¢	9 / 1	=	3803.91 ¢	65 / 6	=	4124.89 ¢
91 / 15	=	3121.08 ¢	95 / 14	=	3315 ¢	85 / 11	=	3539.95 ¢	91 / 10	=	3823.04 ¢	76 / 7	=	4128.69 ¢
85 / 14	=	3122.44 ¢	34 / 5	=	3318.64 ¢	31 / 4	=	3545.04 ¢	82 / 9	=	3825.15 ¢	87 / 8	=	4131.53 ¢
79 / 13	=	3124.01 ¢	75 / 11	=	3323.26 ¢	70 / 9	=	3551.23 ¢	73 / 8	=	3827.79 ¢	98 / 9	=	4133.74 ¢
73 / 12	=	3125.83 ¢	41 / 6	=	3327.11 ¢	39 / 5	=	3556.17 ¢	64 / 7	=	3831.17 ¢	11 / 1	=	4151.32 ¢
67 / 11	=	3127.99 ¢	89 / 13	=	3330.35 ¢	86 / 11	=	3560.2 ¢	55 / 6	=	3835.68 ¢	89 / 8	=	4170.88 ¢
61 / 10	=	3130.57 ¢	48 / 7	=	3333.13 ¢	47 / 6	=	3563.55 ¢	46 / 5	=	3841.96 ¢	78 / 7	=	4173.66 ¢
55 / 9	=	3133.72 ¢	55 / 8	=	3337.63 ¢	55 / 7	=	3568.81 ¢	83 / 9	=	3846.14 ¢	67 / 6	=	4177.35 ¢
49 / 8	=	3137.65 ¢	62 / 9	=	3341.13 ¢	63 / 8	=	3572.74 ¢	37 / 4	=	3851.34 ¢	56 / 5	=	4182.51 ¢
92 / 15	=	3140.01 ¢	69 / 10	=	3343.92 ¢	71 / 9	=	3575.79 ¢	65 / 7	=	3858.02 ¢	45 / 4	=	4190.22 ¢
43 / 7	=	3142.69 ¢	76 / 11	=	3346.2 ¢	79 / 10	=	3578.22 ¢	93 / 10	=	3860.68 ¢	79 / 7	=	4195.71 ¢
80 / 13	=	3145.79 ¢	83 / 12	=	3348.09 ¢	87 / 11	=	3580.21 ¢	28 / 3	=	3866.87 ¢	34 / 3	=	4203 ¢
37 / 6	=	3149.39 ¢	90 / 13	=	3349.7 ¢	95 / 12	=	3581.87 ¢	75 / 8	=	3874.58 ¢	91 / 8	=	4209.35 ¢
68 / 11	=	3153.64 ¢	97 / 14	=	3351.07 ¢	**8 / 1**	**=**	**3600 ¢**	47 / 5	=	3879.19 ¢	57 / 5	=	4213.15 ¢
99 / 16	=	3155.23 ¢	7 / 1	=	3368.83 ¢	97 / 12	=	3617.94 ¢	66 / 7	=	3884.45 ¢	80 / 7	=	4217.49 ¢
31 / 5	=	3158.72 ¢	99 / 14	=	3386.4 ¢	89 / 11	=	3619.56 ¢	85 / 9	=	3887.36 ¢	23 / 2	=	4228.27 ¢
87 / 14	=	3162.71 ¢	92 / 13	=	3387.75 ¢	81 / 10	=	3621.51 ¢	19 / 2	=	3897.51 ¢	81 / 7	=	4238.99 ¢
56 / 9	=	3164.92 ¢	85 / 12	=	3389.31 ¢	73 / 9	=	3623.88 ¢	86 / 9	=	3907.61 ¢	58 / 5	=	4243.26 ¢
81 / 13	=	3167.29 ¢	78 / 11	=	3391.16 ¢	65 / 8	=	3626.84 ¢	67 / 7	=	3910.48 ¢	93 / 8	=	4246.99 ¢
25 / 4	=	3172.63 ¢	71 / 10	=	3393.38 ¢	57 / 7	=	3630.64 ¢	48 / 5	=	3915.64 ¢	35 / 3	=	4253.18 ¢
94 / 15	=	3177.24 ¢	64 / 9	=	3396.09 ¢	49 / 6	=	3635.7 ¢	77 / 8	=	3920.14 ¢	82 / 7	=	4260.24 ¢
69 / 11	=	3178.91 ¢	57 / 8	=	3399.47 ¢	90 / 11	=	3638.91 ¢	29 / 3	=	3927.62 ¢	47 / 4	=	4265.51 ¢
44 / 7	=	3182.49 ¢	50 / 7	=	3403.8 ¢	41 / 5	=	3642.75 ¢	97 / 10	=	3933.58 ¢	59 / 5	=	4272.86 ¢

71 / 6	= 4277.74 ¢	59 / 4	= 4659.17 ¢	39 / 2	= 5142.48 ¢	29 / 1	= 5829.58 ¢	51 / 1	= 6806.91
83 / 7	= 4281.22 ¢	74 / 5	= 4665.03 ¢	98 / 5	= 5151.34 ¢	88 / 3	= 5849.36 ¢	52 / 1	= 6840.53
95 / 8	= 4283.83 ¢	89 / 6	= 4668.93 ¢	59 / 3	= 5157.22 ¢	59 / 2	= 5859.17 ¢	53 / 1	= 6873.5
12 / 1	= 4301.96 ¢	15 / 1	= 4688.27 ¢	79 / 4	= 5164.54 ¢	89 / 3	= 5868.93 ¢	54 / 1	= 6905.87
97 / 8	= 4319.9 ¢	91 / 6	= 4707.4 ¢	99 / 5	= 5168.91 ¢	30 / 1	= 5888.27 ¢	55 / 1	= 6937.63
85 / 7	= 4322.44 ¢	76 / 5	= 4711.2 ¢	20 / 1	= 5186.31 ¢	91 / 3	= 5907.4 ¢	56 / 1	= 6968.83
73 / 6	= 4325.83 ¢	61 / 4	= 4716.88 ¢	81 / 4	= 5207.82 ¢	61 / 2	= 5916.88 ¢	57 / 1	= 6999.47
61 / 5	= 4330.57 ¢	46 / 3	= 4726.32 ¢	61 / 3	= 5214.93 ¢	92 / 3	= 5926.32 ¢	58 / 1	= 7029.58
49 / 4	= 4337.65 ¢	77 / 5	= 4733.83 ¢	41 / 2	= 5229.06 ¢	31 / 1	= 5945.04 ¢	59 / 1	= 7059.17
86 / 7	= 4342.69 ¢	31 / 2	= 4745.04 ¢	62 / 3	= 5243.08 ¢	94 / 3	= 5963.55 ¢	60 / 1	= 7088.27
37 / 3	= 4349.39 ¢	78 / 5	= 4756.17 ¢	83 / 4	= 5250.05 ¢	63 / 2	= 5972.74 ¢	61 / 1	= 7116.88
99 / 8	= 4355.23 ¢	47 / 3	= 4763.55 ¢	21 / 1	= 5270.78 ¢	95 / 3	= 5981.87 ¢	62 / 1	= 7145.04
62 / 5	= 4358.72 ¢	63 / 4	= 4772.74 ¢	85 / 4	= 5291.27 ¢	**32 / 1**	**= 6000 ¢**	63 / 1	= 7172.74
87 / 7	= 4362.71 ¢	79 / 5	= 4778.22 ¢	64 / 3	= 5298.04 ¢	97 / 3	= 6017.94 ¢	**64 / 1**	**= 7200**
25 / 2	= 4372.63 ¢	95 / 6	= 4781.87 ¢	43 / 2	= 5311.52 ¢	65 / 2	= 6026.84 ¢	65 / 1	= 7226.84
88 / 7	= 4382.49 ¢	**16 / 1**	**= 4800 ¢**	65 / 3	= 5324.89 ¢	98 / 3	= 6035.7 ¢	66 / 1	= 7253.27
63 / 5	= 4386.42 ¢	97 / 6	= 4817.94 ¢	87 / 4	= 5331.53 ¢	33 / 1	= 6053.27 ¢	67 / 1	= 7279.31
38 / 3	= 4395.56 ¢	81 / 5	= 4821.51 ¢	22 / 1	= 5351.32 ¢	67 / 2	= 6079.31 ¢	68 / 1	= 7304.96
89 / 7	= 4402.05 ¢	65 / 4	= 4826.84 ¢	89 / 4	= 5370.88 ¢	34 / 1	= 6104.96 ¢	69 / 1	= 7330.23
51 / 4	= 4406.91 ¢	49 / 3	= 4835.7 ¢	67 / 3	= 5377.35 ¢	69 / 2	= 6130.23 ¢	70 / 1	= 7355.14
64 / 5	= 4413.69 ¢	82 / 5	= 4842.75 ¢	45 / 2	= 5390.22 ¢	35 / 1	= 6155.14 ¢	71 / 1	= 7379.7
77 / 6	= 4418.19 ¢	33 / 2	= 4853.27 ¢	68 / 3	= 5403 ¢	71 / 2	= 6179.7 ¢	72 / 1	= 7403.91
90 / 7	= 4421.4 ¢	83 / 5	= 4863.73 ¢	91 / 4	= 5409.35 ¢	36 / 1	= 6203.91 ¢	73 / 1	= 7427.79
13 / 1	= 4440.53 ¢	50 / 3	= 4870.67 ¢	23 / 1	= 5428.27 ¢	73 / 2	= 6227.79 ¢	74 / 1	= 7451.34
92 / 7	= 4459.45 ¢	67 / 4	= 4879.31 ¢	93 / 4	= 5446.99 ¢	37 / 1	= 6251.34 ¢	75 / 1	= 7474.58
79 / 6	= 4462.58 ¢	84 / 5	= 4884.47 ¢	70 / 3	= 5453.18 ¢	75 / 2	= 6274.58 ¢	76 / 1	= 7497.51
66 / 5	= 4466.96 ¢	17 / 1	= 4904.96 ¢	47 / 2	= 5465.51 ¢	38 / 1	= 6297.51 ¢	77 / 1	= 7520.14
53 / 4	= 4473.5 ¢	86 / 5	= 4925.2 ¢	71 / 3	= 5477.74 ¢	77 / 2	= 6320.14 ¢	78 / 1	= 7542.48
93 / 7	= 4478.16 ¢	69 / 4	= 4930.23 ¢	95 / 4	= 5483.83 ¢	39 / 1	= 6342.48 ¢	79 / 1	= 7564.54
40 / 3	= 4484.36 ¢	52 / 3	= 4938.57 ¢	24 / 1	= 5501.96 ¢	79 / 2	= 6364.54 ¢	80 / 1	= 7586.31
67 / 5	= 4492.99 ¢	87 / 5	= 4945.22 ¢	97 / 4	= 5519.9 ¢	40 / 1	= 6386.31 ¢	81 / 1	= 7607.82
94 / 7	= 4496.68 ¢	35 / 2	= 4955.14 ¢	73 / 3	= 5525.83 ¢	81 / 2	= 6407.82 ¢	82 / 1	= 7629.06
27 / 2	= 4505.87 ¢	88 / 5	= 4965 ¢	49 / 2	= 5537.65 ¢	41 / 1	= 6429.06 ¢	83 / 1	= 7650.05
95 / 7	= 4515 ¢	53 / 3	= 4971.55 ¢	74 / 3	= 5549.39 ¢	83 / 2	= 6450.05 ¢	84 / 1	= 7670.78
68 / 5	= 4518.64 ¢	71 / 4	= 4979.7 ¢	99 / 4	= 5555.23 ¢	42 / 1	= 6470.78 ¢	85 / 1	= 7691.27
41 / 3	= 4527.11 ¢	89 / 5	= 4984.57 ¢	25 / 1	= 5572.63 ¢	85 / 2	= 6491.27 ¢	86 / 1	= 7711.52
96 / 7	= 4533.13 ¢	18 / 1	= 5003.91 ¢	76 / 3	= 5595.56 ¢	43 / 1	= 6511.52 ¢	87 / 1	= 7731.53
55 / 4	= 4537.63 ¢	91 / 5	= 5023.04 ¢	51 / 2	= 5606.91 ¢	87 / 2	= 6531.53 ¢	88 / 1	= 7751.32
69 / 5	= 4543.92 ¢	73 / 4	= 5027.79 ¢	77 / 3	= 5618.19 ¢	44 / 1	= 6551.32 ¢	89 / 1	= 7770.88
83 / 6	= 4548.09 ¢	55 / 3	= 5035.68 ¢	26 / 1	= 5640.53 ¢	89 / 2	= 6570.88 ¢	90 / 1	= 7790.22
97 / 7	= 4551.07 ¢	92 / 5	= 5041.96 ¢	79 / 3	= 5662.58 ¢	45 / 1	= 6590.22 ¢	91 / 1	= 7809.35
14 / 1	= 4568.83 ¢	37 / 2	= 5051.34 ¢	53 / 2	= 5673.5 ¢	91 / 2	= 6609.35 ¢	92 / 1	= 7828.27
99 / 7	= 4586.4 ¢	93 / 5	= 5060.68 ¢	80 / 3	= 5684.36 ¢	46 / 1	= 6628.27 ¢	93 / 1	= 7846.99
85 / 6	= 4589.31 ¢	56 / 3	= 5066.87 ¢	27 / 1	= 5705.87 ¢	93 / 2	= 6646.99 ¢	94 / 1	= 7865.51
71 / 5	= 4593.38 ¢	75 / 4	= 5074.58 ¢	82 / 3	= 5727.11 ¢	47 / 1	= 6665.51 ¢	95 / 1	= 7883.83
57 / 4	= 4599.47 ¢	94 / 5	= 5079.19 ¢	55 / 2	= 5737.63 ¢	95 / 2	= 6683.83 ¢	96 / 1	= 7901.96
43 / 3	= 4609.56 ¢	19 / 1	= 5097.51 ¢	83 / 3	= 5748.09 ¢	48 / 1	= 6701.96 ¢	97 / 1	= 7919.9
72 / 5	= 4617.6 ¢	96 / 5	= 5115.64 ¢	28 / 1	= 5768.83 ¢	97 / 2	= 6719.9 ¢	98 / 1	= 7937.65
29 / 2	= 4629.58 ¢	77 / 4	= 5120.14 ¢	85 / 3	= 5789.31 ¢	49 / 1	= 6737.65 ¢	99 / 1	= 7955.23
73 / 5	= 4641.48 ¢	58 / 3	= 5127.62 ¢	57 / 2	= 5799.47 ¢	99 / 2	= 6755.23 ¢		
44 / 3	= 4649.36 ¢	97 / 5	= 5133.58 ¢	86 / 3	= 5809.56 ¢	50 / 1	= 6772.63 ¢		

Appendix 6.2 – Alt-tuner Flowchart



Appendix 6.3 – List of Clickable Items

Standard Jesusonic sliders in the top half of the screen:

Double-click a slider to reset it. Control-drag (command-drag on a mac) for more fine control. Right-drag or right-click to temporarily move a slider; it will snap back to its former position when you release the mouse button. Type any number into the number box, even one out of range, and the slider will take on that value.

Graphics area in the bottom half of the screen:

In general, yellow items are clickable, green ones aren't. Alt-clicking, control-clicking, shift-clicking, etc. are all equivalent to right-clicking. "Double-modified" clicking means either clicking with at least two modifier keys (alt, control or shift), right-clicking with at least one modifier key, or clicking with both mouse buttons simultaneously.

Top line of lattice, graph and table screens:

"lattice"/"graph"/"table"	click to cycle through the views forwards, right-click to cycle backwards
"EDOtap"	click to enter/leave EDOtap mode (only appears when the EDO slider is > 1)
custom tuning numbers	click a yellow number to switch to it, right-click it to copy it to the current tuning
when # of tunings > 12:	click an arrow to move the window, right-click it to move the window all the way
cents offset	click to increase, right-click to decrease, double-modified-click to speed-scroll
"A-440"	click to reset cents offset and send all-sound-offs, right-click to send all-sound-offs
"12ch O" or "12ch N"	right-click to switch between octave mode and non-octave mode

Lattice screen:

all notes (C♯, E♭, etc.)	click to tap from/to, right-click to modulate to, double-modified-click to shift/unshift to
center note	click to cycle through presets if center tap is off, right-click to cycle through presets
12-ET circle: "12-ET"	click or right-click to cycle to the next preset
12-ET circle: all notes	click or right-click to cycle to the next preset with this note as the center note

Graph screen:

graph	click to cycle through zoomed-in views
all notes below	click to tap from/to, right-click to modulate to, double-modified-click to shift/unshift to
leftmost note below	click to cycle through presets if center tap is off, right-click to cycle through presets

Table screen:

table	click to cycle through zoomed-in views if the table is too large for the screen
all notes below	click to tap from/to, right-click to modulate to, double-modified-click to shift/unshift to
leftmost note below	click to cycle through presets if center tap is off, right-click to cycle through presets

Prefs screens:

submenus on the top line	click or right-click to enter the submenu, click again to return to the previous submenu
sliders	click on the slider or drag the fader to set, right-click to reset
yellow number boxes	click/right-click to increase/decrease, double-mod-click speed-scrolls, hold to autorepeat
yellow text boxes	click to either select the option or turn the option on and off
all notes	click to tap from/to, right-click to modulate to, double-modified-click to shift/unshift to
tapnotes leftmost note	click to cycle through presets if center tap is off, right-click to cycle through presets
keybend graph	hover to see the bend, click to set the bend, click and drag to draw the graph,
"	right-click to reset the bend, click on the black and white squares to fine-tune the bend
linkages "OK" button	right-click to reset all the rung factors to zero
keyboard multi-fader slider	click and drag a fader to position a white key, right-click to drag all the keys at once
keyboard diagram	click on a white key to choose a higher letter, right-click to choose a lower letter

Alt-keyswitcher screens:

submenus on the top line	click or right-click to enter the submenu
yellow number boxes	click to increase, right-click to decrease, hold down the mouse button to autorepeat
yellow text boxes	click to turn them off and on
keyboard diagram	not clickable